2. Tools of statistical mechanics of disordered systems (Replica method)

The replica and cavity methods are somewhat equivalent (although some time to prove they indeed are can be cumbersome for some models) and sometimes which method you use is matter of style or personal preferences. In other occassions you realize that it is very easy to tackle one problem directly with one method instead of the other (even though they are equivalent)

From the point of view of physical intuition I'd say cavity method wins hands down to replicas, as it allows you to understand the problem physically in a deeper way

Sometimes however one lacks some

physical intuition and the replica method should be a method of choice. Does this mean that replica method is devoid of physics? Not at all, you can also learn a lot on how this mathematical framework is able to very beautifully captured the physics of spin glasses.

Consider again our model on a given graph G. Now this graph belongs to an ensemble of graphs $\mathcal{E}$ sharing some features of interest

Given a graph, you will have a Hamiltonian

$$\mathcal{H}_G(\vec{\sigma}) = -\sum_{(ij)\in G} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i$$

and assumed that the graph is a weighted graph with weights J_{ij}, that is the couplings constants are a part of the information of the graph

Suppose that a have an observable of interest, say $\mathcal{O}(\vec{\sigma})$. Its expectation value is

$$\mathcal{O}_G = \langle \mathcal{O}(\vec{\sigma}) \rangle_G = \sum_{\vec{\sigma}} P_G(\vec{\sigma}) \mathcal{O}(\vec{\sigma})$$

where

$$P_G(\vec{\sigma}) = \frac{1}{Z_G} e^{-\beta \mathcal{H}_G(\vec{\sigma})}$$

Sometimes it might be better to use notation of conditional probability

$$P(\vec{\sigma}|G) = \frac{1}{Z_G} e^{-\beta \mathcal{H}_G(\vec{\sigma})}$$

Suppose now that a have a sequence of graphs $\{G_\alpha\}_{\alpha=1}^{N}$ with $G_\alpha \in \mathcal{E}$

and on each graph the ferromagnet will have some typical thermal properties

What is the typical behaviour of the typical thermal properties with respect to the sequences of graph?

$$\overline{O_G} = \frac{1}{N}\sum_{\alpha=1}^{N} O_{G_\alpha}$$

$$= \frac{1}{N}\sum_{\alpha=1}^{N} \langle O \rangle_{G_\alpha}$$

$$\underset{N\to\infty}{=} \int dG\, \rho(G) \langle O \rangle_G$$

or

$$\boxed{\overline{O_G} = \int dG\, \rho(G) \sum_{\vec{\sigma}} P(\vec{\sigma}|G)\, O(\vec{\sigma})}$$

In this problem now I have two types of random variables

* $\vec{\sigma}$ → dynamical variables, coupled to the thermal bath
* G → the graph on which $\vec{\sigma}$ is coupled to the bath

→ this is called quenched disordered

⇒ in the above expression

$$\overline{\langle \mathcal{O}(\vec{\sigma}) \rangle}_G = \int dG\, g(G) \sum_{\vec{\sigma}} P(\vec{\sigma}|G)\, \mathcal{O}(\vec{\sigma})$$

thermal average

quenched average → this is difficult to do. why? Because of the following

thermal average → this is easy to do

$$\langle \sigma(\vec{r}) \rangle_G = \int dG\, \rho(G) \frac{1}{Z_G} \sum_{\vec{\sigma}} e^{-\beta \mathcal{H}_G(\vec{\sigma})} \sigma(\vec{r})$$

Its position in the denominator in the partition function makes this quenched average some mathematical derivation somewhat difficult (impossible?) to do

How do we solve this? (Nb. we could use replica method here directly but let me take some different avenue).

Recall that the $\ln Z$ is the generator of thermodynamic observables of interest.

$$\mathcal{H}_G(\vec{\sigma}) \longrightarrow \mathcal{H}_G(\vec{\sigma}) - \lambda\, \sigma(\vec{r})$$

$$Z_G(\lambda) = \sum_{\vec{\sigma}} e^{-\beta \mathcal{H}_G(\vec{\sigma}) + \beta \lambda \sigma(\vec{r})}$$

$$\frac{\partial}{\partial \lambda} \ln Z_G(\lambda)\Big|_{\lambda=0} = \beta \langle B(\vec{\sigma}) \rangle_G$$

from here

$$\frac{\partial}{\partial \lambda} \overline{\ln Z_G(\lambda)}\Big|_{\lambda=0} = \beta \overline{\langle B(\vec{\sigma}) \rangle_G}$$

the fundamental quantity to perform the quenched average over is the free energy. So this means that my object of desire is

$$\overline{\ln Z_G} = \int dG\, \rho(G) \ln Z_G$$

$$= \int dG\, \rho(G) \ln \sum_{\vec{\sigma}} e^{-\beta \mathcal{H}_G(\vec{\sigma})}$$

This is as difficult as the previous expression but it's a nicer starting point to introduce the replica method. So let's start by noting the following: for an integer value n, the following average is, in principle, easy to do

$$\int dG\, g(G)\, Z_G^n = \overline{Z_G^n}$$

$Z_G^n \equiv$ this is equivalent to consider the thermodynamical properties of n copies of your system, all sharing the same graph structure

from another point of view I know that

$$Z_G^n = 1 + n \log Z_G + O(n^2)$$

$$\hookrightarrow \overline{Z_G^n} = 1 + n\, \overline{\log Z_G} + O(n^2)$$

and

$$\log \overline{Z_G^n} = n \, \overline{\log Z_G} + O(n^2)$$

Therefore

$$\overline{\log Z_G} = \lim_{n \to 0} \frac{1}{n} \log \overline{Z_G^n}$$

This is the (first step) of the replica method

The mathematical framework of the replica method is the following:

step 1 consider n copies of $\mathcal{H}_G(\vec{\sigma})$ and do

$$\overline{Z_G^n}$$

step 2 Introduce a hypothesis that says something above the properties of the copies of the system under permutation

Step 3. Step 2 should allow you have a recipe to analytically continue the n from integers to reals

Step 4. Perform the replica limit

$$\lim_{n \to 0} \frac{1}{n} \ln \overline{Z_G^n}$$

(Nb. somewhere one also has to assume that replica limit and thermodynamic limit can be interchanged)

(Nb. Step 2 is the one that took 30 years or so to formally understand)

Good! so let's calculate $\overline{\ln Z_G}$ for the ferromagnetic model when the ensemble of graphs $\mathcal{E}$ has the following probabilistic recipe (for simplicity let's take $h_i = h$ and $J_{ij} = J \; \forall i,j$). If $\underline{\underline{C}}$ is the adjacency matrix of a given G then

$$\rho(\underline{\underline{C}}) = \left\{ \prod_{i<j} \left[\frac{c}{N} \delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right) \delta_{c_{ij},0} \right] \times \delta_{c_{ij},c_{ji}} \right\} \prod_{i=1}^{N} \delta_{c_{ii},0}$$

$c \equiv$ average connectivity

→ ensemble of Poissonian random graphs or Erdős–Renyi graph

This recipe tells you:

* the graph is directed (symmetric)
* the bunch of independent c_{ij} are i.i.d Bernoulli RVs

$$\text{Prob}[C_{ij}=1] = \frac{c}{N}$$

Using C' one can write the Hamiltonian as follows

$$\mathcal{H}_G(\vec{\sigma}) = -J\sum_{i<j} C_{ij}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i$$

$$\Updownarrow$$

$$\mathcal{H}_C(\vec{\sigma})$$

So we have to do the following derivation

$$\overline{\ln Z_C} = \lim_{n\to 0}\frac{1}{n}\ln\int dC\, g(C)\, Z_C^n$$

Let's do it

Step I Assume n integer

$$Z_C^n = \left(\sum e^{-\beta\mathcal{H}_C(\vec{\sigma})}\right)^n$$

$$= \sum_{\vec{\sigma}_1} e^{-\beta \mathcal{H}_C(\vec{\sigma}_1)} \times \cdots \times \sum_{\vec{\sigma}_n} e^{-\beta \mathcal{H}_C(\vec{\sigma}_n)}$$

$$= \sum_{\vec{\sigma}_1} \sum_{\vec{\sigma}_2} \cdots \sum_{\vec{\sigma}_n} e^{-\beta \sum_{\alpha=1}^{n} \mathcal{H}_C(\vec{\sigma}_\alpha)}$$

sum over all configurations of copy/replica #1 ...

sum over all configurations of copy/replica #n

$$Z_C^n = \sum_{\vec{\sigma}_1} \cdots \sum_{\vec{\sigma}_n} e^{\beta J \sum_{\alpha=1}^{n} \sum_{i<j} C_{ij} \sigma_{i\alpha} \sigma_{j\alpha}} e^{\beta h \sum_{\alpha=1}^{n} \sum_{i=1}^{N} \sigma_{i\alpha}}$$

Now I have to do the quenched disorder average of this

$$\overline{Z_C^n} = \sum_{\vec{\sigma}_1} \cdots \sum_{\vec{\sigma}_n} e^{\beta h \sum_{\alpha=1}^{n} \sum_{i=1}^{N} \sigma_{i\alpha}}$$

$$\times \int dc\, \rho(c)\, e^{\beta J \sum_{\alpha=1}^{n} \sum_{i<j} c_{ij} \sigma_{i\alpha} \sigma_{j\alpha}}$$

$$\left(\int dc\, \rho(c) \prod_{i<j} e^{\beta J \sum_{\alpha=1}^{n} c_{ij} \sigma_{i\alpha} \sigma_{j\alpha}} \right.$$

$$\underset{\text{iid}}{=} \prod_{i<j} \int dc_{ij}\, \rho(c_{ij})\, e^{\beta J c_{ij} \sum_{\alpha=1}^{n} \sigma_{i\alpha} \sigma_{j\alpha}}$$

$$= \prod_{i<j} \left[\frac{c}{N} e^{\beta J \sum_{\alpha=1}^{n} \sigma_{i\alpha} \sigma_{j\alpha}} + \left(1 - \frac{c}{N}\right) \right]$$

$$\underset{\text{large } N}{=} \exp\left[\frac{c}{2N} \sum_{i,j=1}^{N} \left(e^{\beta J \sum_{\alpha=1}^{n} \sigma_{i\alpha} \sigma_{j\alpha}} - 1 \right) \right]$$

All in all

$$\overline{Z_C^n} = \sum_{\vec{\sigma}_1} \cdots \sum_{\vec{\sigma}_n} \exp\Big[\beta h \sum_{\alpha=1}^{n} \sum_{i=1}^{N} \sigma_{i\alpha} + \frac{c}{2N} \sum_{i,j=1}^{N} \Big(e^{\beta J \sum_{\alpha=1}^{n} \sigma_{i\alpha}\sigma_{j\alpha}} - 1\Big)\Big]$$

Next, we want to linearize the sums on the argument of the exponential. To achieve this we introduce the following order parameters

$$P(\sigma) = \frac{1}{N} \sum_{i=1}^{N} \prod_{\alpha=1}^{n} \delta(\sigma_\alpha - \sigma_{i\alpha})$$
$$= \frac{1}{N} \sum_{i=1}^{N} \delta(\underline{\sigma} - \underline{\sigma}_i)$$

with notation $\underline{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\underline{\sigma}_i = (\sigma_{i1}, \dots, \sigma_{in})$

$$\overline{Z_C^n} = \int [dP] \sum_{\sigma_1} \cdots \sum_{\sigma_n} \exp\Big[\beta h \sum_{\alpha=1}^{n} \sum_{i=1}^{N} \sigma_{i\alpha}$$

$$+\frac{cN}{2}\sum_{\underline{\sigma}}\sum_{\underline{\tau}} P(\underline{\sigma})P(\underline{\tau})\left(e^{+\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)\Big]$$

$$\delta_{(F)}\left[P(\underline{\sigma})-\frac{1}{N}\sum_{i=1}^{N}\delta(\underline{\sigma}-\underline{\sigma}_i)\right]$$

$$=\int[dPd\hat{P}]\sum_{\sigma_1}\cdots\sum_{\sigma_n}\exp\Big[\beta h\sum_{\alpha=1}^{n}\sum_{i=1}^{N}\sigma_{i\alpha}$$

$$+\frac{cN}{2}\sum_{\underline{\sigma}}\sum_{\underline{\tau}}P(\underline{\sigma})P(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)$$

$$+iN\sum_{\underline{\sigma}}\hat{P}(\underline{\sigma})\left(P(\underline{\sigma})-\frac{1}{N}\sum_{i=1}^{N}\delta(\underline{\sigma}-\underline{\sigma}_i)\right)\Big]$$

$$\Rightarrow \overline{Z_c^n}=\int[dPd\hat{P}]\,e^{-N S_n[P,\hat{P}]}$$

with

$$S_n[P,\hat{P}]=-\frac{c}{2}\sum_{\underline{\sigma}}\sum_{\underline{\tau}}P(\underline{\sigma})P(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)$$

$$-i\sum_{\underline{\sigma}}\hat{P}(\underline{\sigma})P(\underline{\sigma})$$

$$-\log\sum_{\underline{\sigma}}e^{\beta h\sum_{\alpha=1}^{n}\sigma_\alpha-i\hat{P}(\underline{\sigma})}$$

This implies that

$$\beta f = -\lim_{N\to\infty} \frac{1}{N} \overline{\log Z_G}$$

$$= -\lim_{N\to\infty} \lim_{n\to 0} \frac{1}{Nn} \log \overline{Z_G^n}$$

$$= -\lim_{n\to 0} \lim_{N\to\infty} \frac{1}{Nn} \log \int [dP\, d\hat{P}]\, e^{-N S_n[P,\hat{P}]}$$

$$= \lim_{n\to 0} \frac{1}{n} S_n[P_0, \hat{P}_0]$$

$$\Rightarrow \beta f = \lim_{n\to 0} \frac{1}{n} S_n[P_0, \hat{P}_0]$$

where $P_0(\underline{\sigma})$ and $\hat{P}_0(\underline{\sigma})$ obey to saddle point equations

$$\frac{\delta S_n[P,\hat{P}]}{\delta P(\underline{\sigma})} = \frac{\delta S_n[P,\hat{P}]}{\delta \hat{P}(\underline{\sigma})} = 0$$

<u>This is step I</u>

Now, notice the following

$$\frac{1}{N}\frac{\partial}{\partial \beta h} \overline{\log Z_\sigma} = \overline{\frac{1}{N}\left\langle \sum_{i=1}^{N} \sigma_i \right\rangle_{|G}} = M(T)$$

thermodynamic magnetization

$$\frac{\partial}{\partial \beta h} \frac{1}{N} \overline{\log Z_\sigma} =$$

$$\frac{\partial}{\partial \beta h} \lim_{n \to 0} \frac{1}{nN} \log \overline{Z_\sigma^n}, \quad \overline{Z_\sigma^n} = \int [dP d\hat{P}] e^{-N S_n[P,\hat{P}]}$$

think about also doing $N \to \infty$

$$\lim_{n \to 0} \frac{1}{nN} \left(\frac{\partial}{\partial \beta h}\right) \log \overline{Z_\sigma^n}$$

$$= \lim_{n \to 0} \frac{1}{nN} (-N) \frac{\int [dP d\hat{P}] e^{-N S_n[P,\hat{P}]} \frac{\partial S_n[P,\hat{P}]}{\partial \beta h}}{\int [dP d\hat{P}] e^{-N S_n[P,\hat{P}]}}$$

$$= \lim_{n \to 0} \left(-\frac{\partial}{\partial \beta h} S_n[P_0, \hat{P}_0]\right)$$

$\uparrow$
$N \to \infty$

But $S_n[P,\hat{P}] = -i \sum_{\underline{\sigma}} P(\underline{\sigma}) \hat{P}(\underline{\sigma})$

$$- \frac{d}{2} \sum_{\underline{\sigma}} \sum_{\underline{\tau}} P(\underline{\sigma}) P(\underline{\tau}) \left(e^{\beta J \underline{\sigma} \cdot \underline{\tau}} - 1 \right)$$

$$- \log \sum_{\underline{\sigma}} e^{\beta h \sum_{\alpha=1}^{n} \sigma_\alpha - i \hat{P}(\underline{\sigma})}$$

$$- \frac{\partial}{\partial \beta h} S_n[P,\hat{P}] \Big|_0 = \sum_{\underline{\sigma}} P(\underline{\sigma}) \sum_{\alpha=1}^{n} \sigma_\alpha$$

Therefore

$$M(T) = \lim_{n \to 0} \frac{1}{n} \sum_{\underline{\sigma}} P(\underline{\sigma}) \sum_{\alpha=1}^{n} \sigma_\alpha$$

To understand the idea of step 2 (ansatz/hypothesis to do the limit $n \to 0$) we will focus on this expression above

Consider the following <u>hypotesis</u>

Since copies are identical when they are introduced ($Z_G^n = Z_G \cdots Z_G$) it were to interchange the replica labels nothing should changed ("the result should be invariant under permutations in replica space)

↳ This is called **replica-symmetric ansatz**

which is the most general form that $P(\underline{\sigma})$ can have under RS ansatz?

$$P_0^{RS}(\underline{\sigma}) = \int dh\, W(h) \prod_{\alpha=1}^{n} q(\sigma_\alpha | h) \qquad \sum_\sigma q(\sigma|h)=1 \quad \int dh\, W(h)=1$$

Let's see how that this expression allows us to perform the replica limit $n \to 0$ in the expression for the magnetization

$$M^{RS}(t) = \lim_{n\to 0} \frac{1}{n} \sum_{\underline{\sigma}} P_0^{RS}(\underline{\sigma}) \sum_{\alpha=1}^{n} \sigma_\alpha$$

$$= \lim_{n\to 0} \frac{1}{n} \sum_{\underline{\sigma}} \int dh\, W(h) \left[\prod_{\alpha=1}^{n} q(\sigma_\alpha | h)\right] \sum_{\alpha=1}^{n} \sigma_\alpha$$

$$= \lim_{n\to 0} \frac{1}{n} \int dh\, \omega(h) \sum_{\sigma_1} q(\sigma_1|h) \cdots \sum_{\sigma_n} q(\sigma_n|h) \times \sum_{\alpha=1}^{n} \sigma_\alpha$$

$$= \lim_{n\to 0} \frac{1}{n} \int dh\, \omega(h)\, n \sum_{\sigma} q(\sigma|h)\sigma$$

$$= \int dh\, \omega(h) \sum_{\sigma} q(\sigma|h)\sigma$$

Take $q(\sigma|h) = \dfrac{e^{\beta h \sigma}}{2\cosh(\beta h)}$

$$\Rightarrow \boxed{M^{RS}(T) = \int dh\, \omega(h) \tanh(\beta h)}$$

from the saddle-point of $P_B(\Gamma)$ we can write the corresponding saddle-point for $W(h)$

$$W(h) = \sum_{n\geq 0} \frac{e^{-c} c^n}{n!} \int \left[\prod_{k=1}^{n} dh_k W(h_k)\right] \times$$

$$\delta\left(h - \sum_{k=1}^{n} u(J, h)\right)\Big]$$

Exercise

self-consistent integral equation for $W(h)$ that can be solved numerically by using population dynamics