Mapping I

Consider an ensemble $\mathcal{E}$ of $N \times N$ real symmetric matrices, and let $A \in \mathcal{E}$. Denote as $\vec{\lambda}^A = (\lambda_1^A, \dots, \lambda_N^A)$ the spectrum of A. Define the empirical spectral density, given A, as

$$\rho_A(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i^A)$$

$\rho_A(\lambda)$ can be related to a local observable related to a Hamiltonian for which the matrix entries of A play the role of exchange couplings.

$$\rho_A(\lambda) = \frac{1}{N\pi} \lim_{\eta \to 0^+} \mathrm{Im} \sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i^A - i\eta}$$

$$= \frac{1}{N\pi} \lim_{\eta \to 0^+} \mathrm{Im} \frac{\partial}{\partial z} \sum_{i=1}^{N} \log(z - \lambda_i^A) \Big|_{z = \lambda - i\eta}$$

$$= \frac{1}{N\pi} \lim_{\eta \to 0^+} \mathrm{Im} \frac{\partial}{\partial z} \log\left[\prod_{i=1}^{N} (z - \lambda_i^A)\right] \Big|_{z = \lambda - i\eta}$$

$$= \frac{1}{N\pi} \lim_{\eta\to 0^+} \mathrm{Im} \frac{\partial}{\partial z} \log \det(\mathbb{1} z - A) \Big|_{z=\lambda-i\eta}$$

$$= -\frac{2}{N\pi} \lim_{\eta\to 0^+} \mathrm{Im} \frac{\partial}{\partial z} \log \frac{1}{\sqrt{\det(\mathbb{1} z - A)}} \Big|_{z=\lambda-i\eta}$$

Introduce the "partition function"

$$\mathcal{Z}_A(z) = \int d^N\vec{x}\, \exp\left[-\frac{1}{2}\vec{x}^T(\mathbb{1}z - A)\vec{x}\right]$$

with $\vec{x} = (x_1, \dots, x_N)$

Then

$$\boxed{\rho_A(\lambda) = -\frac{2}{N\pi} \lim_{\eta\to 0^+} \mathrm{Im} \frac{\partial}{\partial z} \log \mathcal{Z}_A(z) \Big|_{z=\lambda-i\eta}}$$

Notice that from the point of view of stat. mech, we have that

$$\rho_A(\lambda) = \frac{1}{\pi N} \lim_{\eta\to 0^+} \mathrm{Im} \sum_{i=1}^{N} \langle x_i^2 \rangle_{|A,z} \Big|_{z=\lambda-i\eta}$$

where $\langle \cdots \rangle$ corresponds to the average with respect to the Gibbs measure

$$P_A(\vec{x};z) = \frac{e^{-\mathcal{H}_A(\vec{x};z)}}{Z_A(z)}$$

$$\langle \cdots \rangle_{|A,z} = \int d\vec{x}\, P_A(\vec{x};z)\,(\cdots)$$

and $\mathcal{H}_A(\vec{x};z) = \frac{1}{2}\sum_{i,j=1}^{N} x_i (\mathbb{1}z - A)_{ij}\, x_j$

Notice that to calculate $\rho_A(\lambda)$ one needs to evaluate the expectation value $\langle x_i^2 \rangle_{|A,z}$. For this we only need single-site marginals $P_i(x_i)$ of $P(\vec{x})$. One can use the cavity method to try to obtain simple closed equations for single site marginales.

Recall that

$$P_i(x_i) = \int d^N \vec{x}_{\setminus i}\, P(\vec{x}) = \frac{1}{Z_A(z)} \int d^N \vec{x}_{\setminus i}\, e^{-\mathcal{H}_A(\vec{x};z)}$$

Now

$$\mathcal{H}_A(\vec{x};z) = \frac{(z-A_{ii})}{2} x_i^2 - x_i \sum_{\ell \in \partial i} A_{i\ell} x_\ell$$

$$+ \mathcal{H}_A^{(i)}(\vec{x},z)$$

so that

$$P_i(x_i) = \frac{1}{Z_A(z)} e^{-\frac{(z-A_{ii})}{2} x_i^2} \int dx_{\partial i}\, e^{x_i \sum_{\ell \in \partial i} A_{i\ell} x_\ell}$$

$$\times \int d^N \vec{x}_{\setminus \{i \cup \partial i\}}\, e^{-\mathcal{H}_A^{(i)}(\vec{x},z)}$$

$$= \frac{1}{Z_i} e^{-\frac{(z-A_{ii})}{2} x_i^2} \int dx_{\partial i}\, e^{x_i \sum_{\ell \in \partial i} A_{i\ell} x_\ell} P_{\partial i}^{(i)}(x_{\partial i})$$

with

$$Z_i = \frac{Z_A(z)}{Z_A^{(i)}(z)}$$

Of course, we could have started with a system with a node $j \in \partial i$, this means rather straightforwardly that

$$P_i^{(j)}(x_i) = \frac{e^{-\frac{(z-A_{ii})x_i^2}{2}}}{Z_i^{(j)}} \int dx_{\partial i/j}\, e^{x_i \sum_{\ell\in\partial i/j} A_{i\ell} x_\ell}\, P^{(i,j)}_{\partial i/j}(x_{\partial i/j})$$

Assume that the underlying graph structure of the matrix A is a tree, or some other underlying structure that, graph, does not contain short loops. Then

$$P^{(i)}_{\partial i}(x_{\partial i}) = \prod_{\ell\in\partial i} P^{(i)}_\ell(x_\ell)$$

similarly

$$P^{(i,j)}_{\partial i/j}(x_{\partial i/j}) = \prod_{\ell\in\partial i/j} P^{(i,j)}_\ell(x_\ell)$$

$$= \prod_{\ell\in\partial i/j} P^{(i)}_\ell(x_\ell)$$

Which is the so-called Bethe-Peierls approxi-mation. Then exact for certain types of matrices or with this approximation we can write close equations for single-site marginals

$$P_i(x_i) = \frac{e^{-\frac{(z-A_{ii})x_i^2}{2}}}{Z_i} \prod_{l\in\partial i} \int dx_l \, e^{x_i A_{il} x_l} P_l^{(i)}(x_l)$$

$$P_i^{(j)}(x_i) = \frac{e^{-\frac{(z-A_{ii})x_i^2}{2}}}{Z_i^{(j)}} \prod_{l\in\partial i/j} \int dx_l \, e^{x_i A_{il} x_l} P_l^{(i)}(x_l)$$

$\rightsquigarrow$ cavity equations

$$\begin{cases} i = 1, \dots, N \\ j \in \partial i \end{cases}$$

Notice that, in this case, the set of cavity equations involve single-site marginals with continuous variables. So, in principle, you need an infinite number of parameters, to fully characterize them. In this case, however, one realizes that the cavity marginals have a Gaussian form, this form is preserved under the cavity equations. Let us then write

$$P_i^{(j)}(x_i) = \frac{1}{\sqrt{2\pi\Delta_i^{(j)}}} e^{-x_i^2/2\Delta_i^{(j)}}$$

and similarly

$$P_i(x_i) = \frac{1}{\sqrt{2\pi\Delta_i}} e^{-x_i^2/2\Delta_i}$$

$\Delta_i \in \mathbb{C}$
$\Delta_i^{(j)} \in \mathbb{C}$

Then the previous set of cavity equations become:

$$\boxed{\Delta_i^{(j)} = \frac{1}{z - A_{ii} - \sum_{\ell \in \partial i \setminus j} A_{i\ell}^2 \Delta_\ell^{(i)}} \qquad i = 1, \dots, N \quad j \in \partial i}$$

cavity equations for the parameters $\{\Delta_i^{(j)}\}_{\substack{i=1,\dots,N \\ j \in \partial i}}$

$\hookrightarrow$ Once you have the solution to this set of equations, numerically or otherwise, we have obtain the values of $\{\Delta_i\}_{i=1,\dots,N}$ from the other equation:

$$\Delta_i = \frac{1}{z - A_{ii} - \sum_{l \in \partial i} A_{il}^2 \Delta_l^{(i)}}$$

Once we have the values of $\{\Delta_i\}_{i=1,\dots,N}$ then the spectral density can be evaluated as follows. First notice that

$$\langle x_i^2 \rangle = \int dx_i \, P_i(x_i) \, x_i^2$$

$$= \int \frac{dx_i}{\sqrt{2\pi\Delta_i}} e^{-x_i^2/2\Delta_i} x_i^2 = \Delta_i$$

Thus $\boxed{\rho_A(\lambda) = \frac{1}{\pi N} \lim_{\epsilon \to 0^+} \mathrm{Im} \sum_{i=1}^{N} \Delta_i}$

So the algorithm to estimate the spectral density $\rho_A(\lambda)$ of a given matrix A do the following

1 Initialize $\{\Delta_i^{(j)}\}_{i=1,\dots,N}^{j\in\partial i}$ $\Delta_i^{(j)} \in \mathbb{C}$ to

e.g. random values

2. Take a value of $\lambda \in \mathbb{R}$ and small value of η and introduce $z = \lambda + i\eta$

3. Iterate the following set of equations until convergence

$$\Delta_i^{(j)} = \frac{1}{z - A_{ii} - \sum_{l\in\partial i \setminus j} A_{il}^2 \Delta_l^{(i)}} \qquad i=1,\dots,N \quad l\in\partial i$$

4. obtain the values of $\{\Delta_i\}$ using

$$\Delta_i = \frac{1}{z - A_{ii} - \sum_{l\in\partial i} A_{il}^2 \Delta_l^{(i)}}$$

5. Do the operation

$$\rho_A(\lambda) \leftarrow \frac{1}{\pi N} \sum_{i=1}^{N} \mathrm{Im}\, \Delta_i$$

6. repeat the process changing the value of

d.

This will given either an approximation to $\rho_A(\lambda)$ or the exact expression for a certain ensembles $\mathcal{E}$ of matrices

In some cases the set of cavity equations can be solved explicitly. Indeed, consider the ensemble of homogeneous random regular graphs. These are graphs where the degree of each node is the same, and the links between nodes have the same value.

Suppose then homogeneous random regular graphs with weights

$$A_{ii} = a_0 \ , \ A_{ij} = a_1 \quad i \neq j$$

and degree K. Since these graphs are "isotropic" and homogeneous then

$$\Delta_i^{(j)} = \Delta^{cav} \qquad \forall i = 1, \dots, N, \ j \in \partial i$$

$$\Delta_i = \Delta \qquad \forall\, i = 1, \dots, N$$

then

$$\Delta_i^{(j)} = \frac{1}{z - A_{ii} - \sum_{\ell \in \partial i / j} A_{i\ell}^2 \Delta_\ell^{(i)}} \Longrightarrow$$

$$\begin{cases} \Delta^{cav} = \dfrac{1}{z - a_0 - (\bar{k}-1) a_1^2 \Delta^{cav}} \\[2ex] \Delta = \dfrac{1}{z - a_0 - \bar{k} a_1^2 \Delta^{cav}} \end{cases}$$

similarly for this case

$$\rho_A(\lambda) = \lim_{\eta \to 0^+} \frac{1}{\pi N} \operatorname{Im} \sum_{i=1}^{N} \Delta_i$$

$$= \lim_{\eta \to 0^+} \frac{1}{\pi} \operatorname{Im} \Delta$$

homogeneous KRGs

Exercise: find the expression for $\rho_A(\lambda)$, in this case.

[illegible]