Thermodynamics of information

5. Thermodynamic cost of measurement and erasure

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1. Second law and information : two ways to go
2. Memories
3. Measurement and erasure

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Many slides are taken or inspired from the ones of Juan Parrondo

5. Definitions

Gibbs: $$S_{\mathrm{G}}(\rho) = -k \sum_x \rho(x) \ln \rho(x)$$

Shannon: $$S(X) = -\sum_x \rho(x) \log \rho(x)$$

Mutual information: $$I(X, M) = \sum_{x,m} \rho(x,m) \log \frac{\rho(x,m)}{\rho(x)\rho(m)}$$

$$= S(X) + S(M) - S(X, M)$$

$$= S(X) - S(X|M) = S(M) - S(M|X)$$

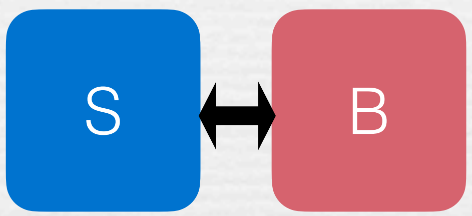
Kullback-Leibler divergence: $$D(p||q) \equiv \sum_i p_i \ln \frac{p_i}{q_i}$$

Non-equilibrium free energy: $$\mathcal{F}(\rho; H) \equiv \langle H \rangle_\rho - kTS(\rho)$$

5.1 Second law and Information

Without information $\Delta S^S + \frac{Q^B}{T} \geq 0$ $\qquad W \geq \Delta F$

With information

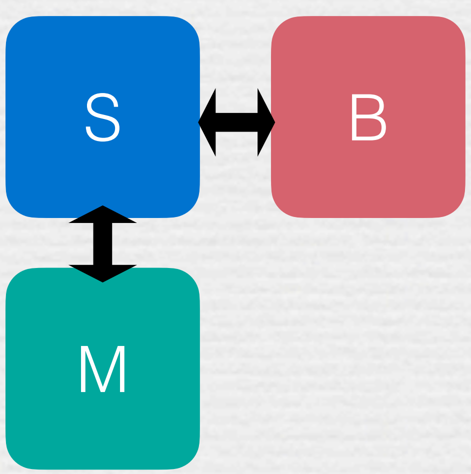


Information is considered as a additional quantity to be incorporated in a **new version of the second law**

$$W \geq \Delta F - kTI$$

$$\Delta F^S_{\text{tot}} = \Delta F^S + \Delta F^S_{meas} = \Delta F^S + kTI$$

Or

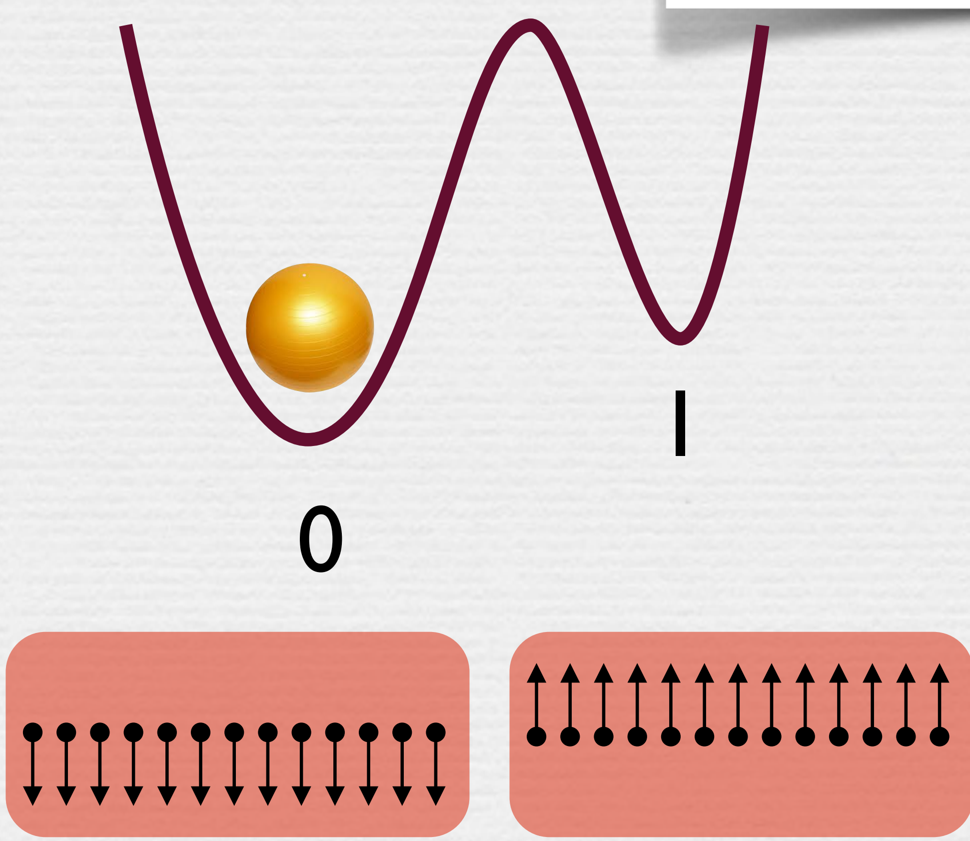


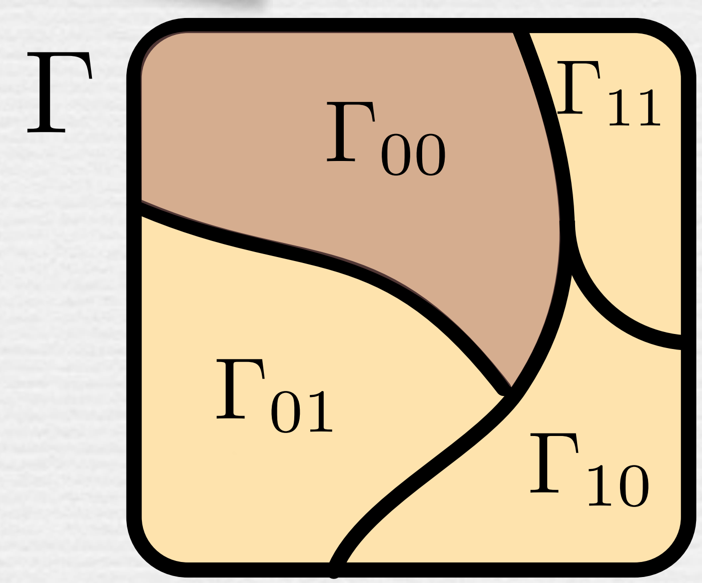
The demon/memory is included as a new physical system in the description and the **original second law applies to the full system.** The thermodynamic cost of the measurement and erasure must be taken into account.

5.2 Memories and information reservoirs

An **information device** is a physical system that can adopt one among several meso- or macroscopic states and stays in this state for a relatively long period of time.

Ergodicity breaking



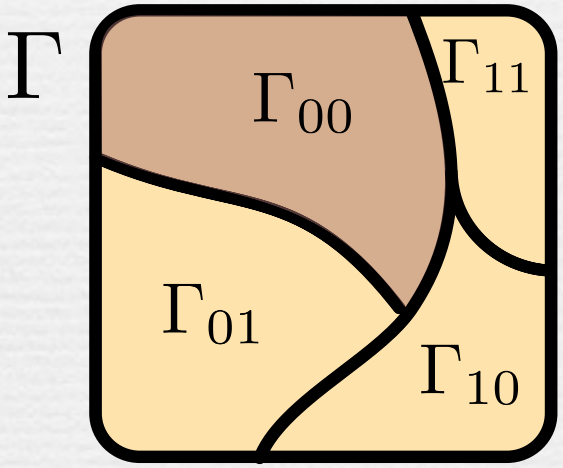


Information-bearing degrees of freedom

(Bennett, Deffner and Jarzynski)

5.2 Memories and information reservoirs

Informational states
- Non-equilibrium states locally in equilibrium
- Depend on history and/or our knowledge



$m \in \{00,01,10,11\}$

Probability of state m: p_m

Partition function of state m:

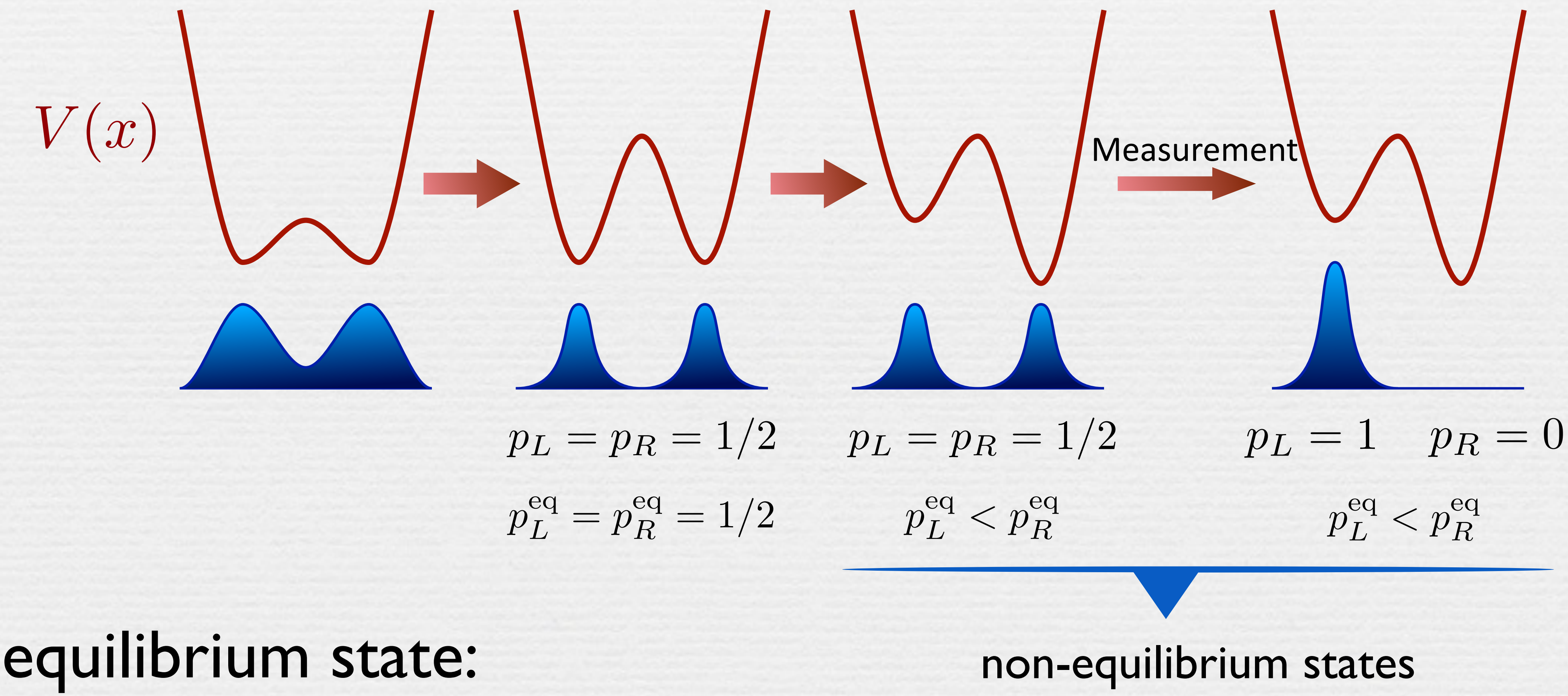
$$Z_m = \int_{\Gamma_m} dq dp \, e^{-\beta H(q,p)}$$

Free energy of state m: $F_m = -kT \ln Z_m$

5.2 Memories and information reservoirs

Informational state of the memory: p_m

Phase-space state of the memory: $p(x) = \frac{p_m}{Z_m} e^{-\beta H(x)}$ if $x \in \Gamma_m$



Global equilibrium state:

$$p_m^{\rm eq} = \frac{e^{-\beta F_m}}{Z} = \frac{Z_m}{Z}$$

5.2 Memories and information reservoirs

Non-equilibrium free energy of a memory:

$$\mathcal{F}(M) = \sum_m p_m F_m - kTS(M)$$

"Bare" free energy

Informational entropy

$$= F_{\text{eq}} + D(p_m || p_M^{\text{eq}})$$

Minimal work for changing the non-equilibrium state of a memory isothermally:

$$p_m \to p'_m$$

$$M \to M'$$

Standard second law applied to the memory:

$$W_{\min} = \mathcal{F}(M') - \mathcal{F}(M)$$

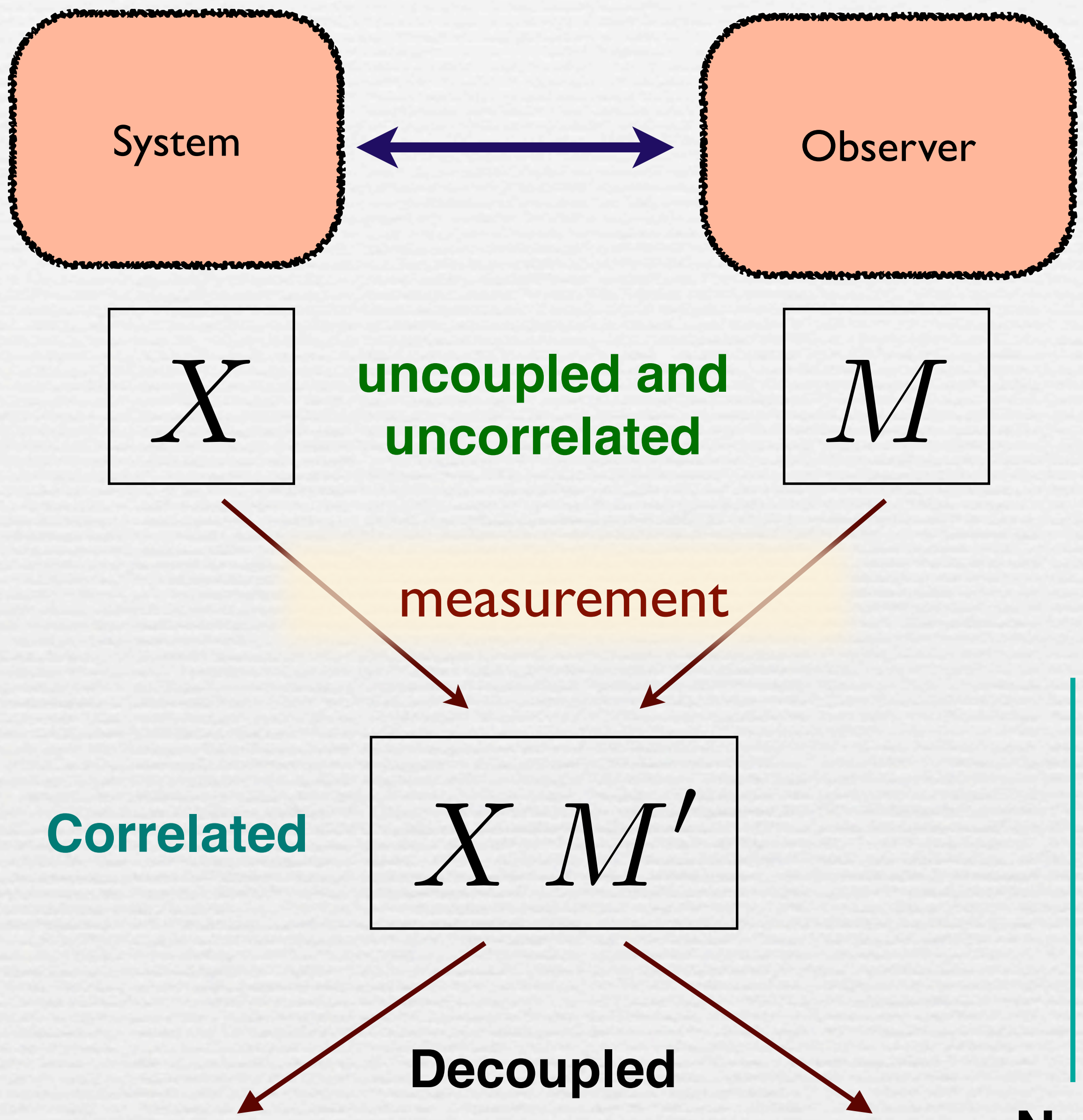
For a symmetric memory: $F_m = F$

$$= kT(S(M) - S(M'))$$

≥ 0 if we order the memory (Landauer's principle)

≤ 0 can act as a "fuel" (information reservoir)

5.3 **Measurement** and erasure



$$p_{m,x} = p_m p_x$$
$$\langle H^{XM} \rangle = \langle H^X \rangle + \langle H^M \rangle$$
$$S(X,M) = S(X) + S(M)$$
$$\mathcal{F}(M,X) = \mathcal{F}(M) + \mathcal{F}(X)$$

$$p'_{m,x} = p'(m|x) p_x$$
$$\langle H^{XM'} \rangle = \langle H^X \rangle + \langle H^{M'} \rangle$$
$$S(X,M') = S(X) + S(M') - I(X,M')$$
$$F(X,M') = \mathcal{F}(M') + \mathcal{F}(X) + kT\,I(M';X)$$

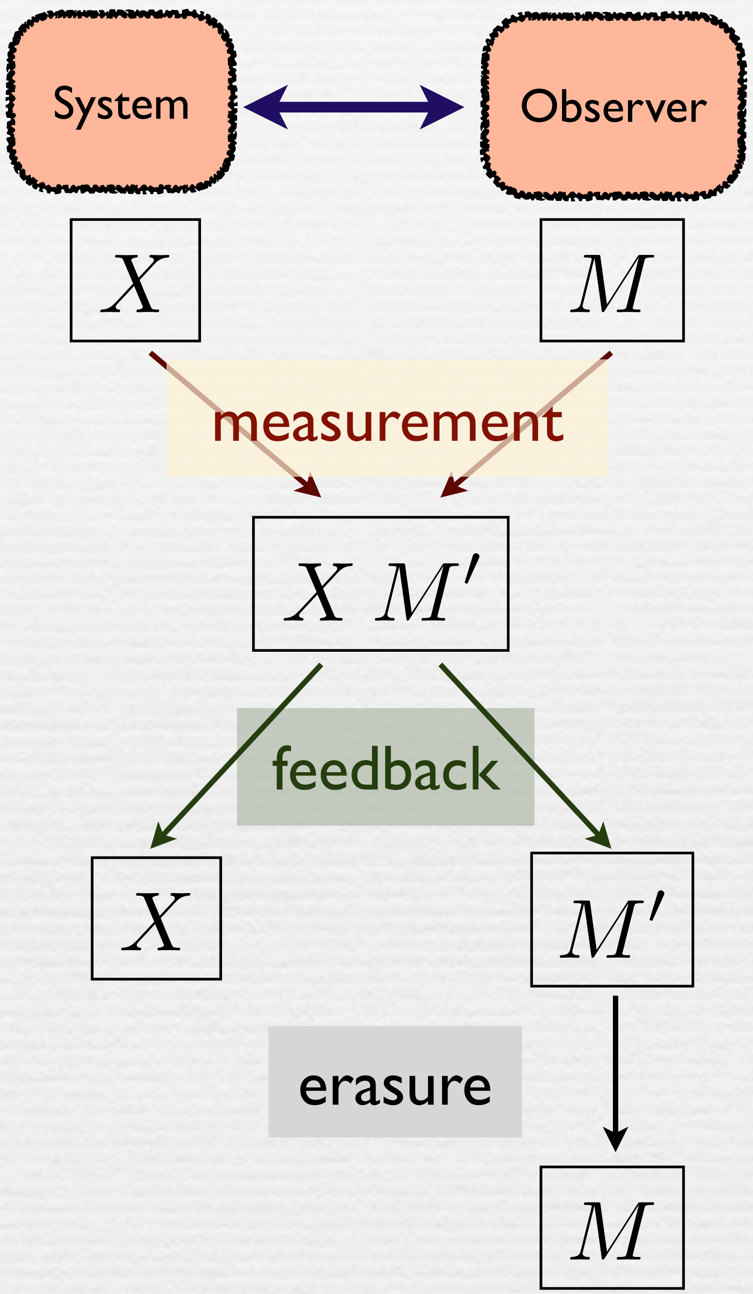
Non-equilibrium free energy of two correlated systems

$$W^{\mathrm{meas}}_{\mathrm{min}} = \Delta\mathcal{F}(M,X) = \Delta\mathcal{F}(M) + kT\,I(M';X)$$

$$W^{\mathrm{meas}} \geq \Delta F(M) + kTI(M';X)$$

reduction of entropy
or increase of free energy
due to measurement

5.3 Measurement and **erasure**



Non-equilibrium free energy

$$\mathcal{F}(X) + \mathcal{F}(M)$$

Minimal work

$$W^{\mathrm{meas}} = \Delta\mathcal{F}(M) + kT\, I(X;M')$$

$$\mathcal{F}(X) + \mathcal{F}(M') + kT\, I(X;M')$$

$$W^{\mathrm{feedback}} = -kT\, I(M';X)$$

$$\mathcal{F}(X) + \mathcal{F}(M')$$

$$W^{\mathrm{eras}} = -\Delta\mathcal{F}(M)$$

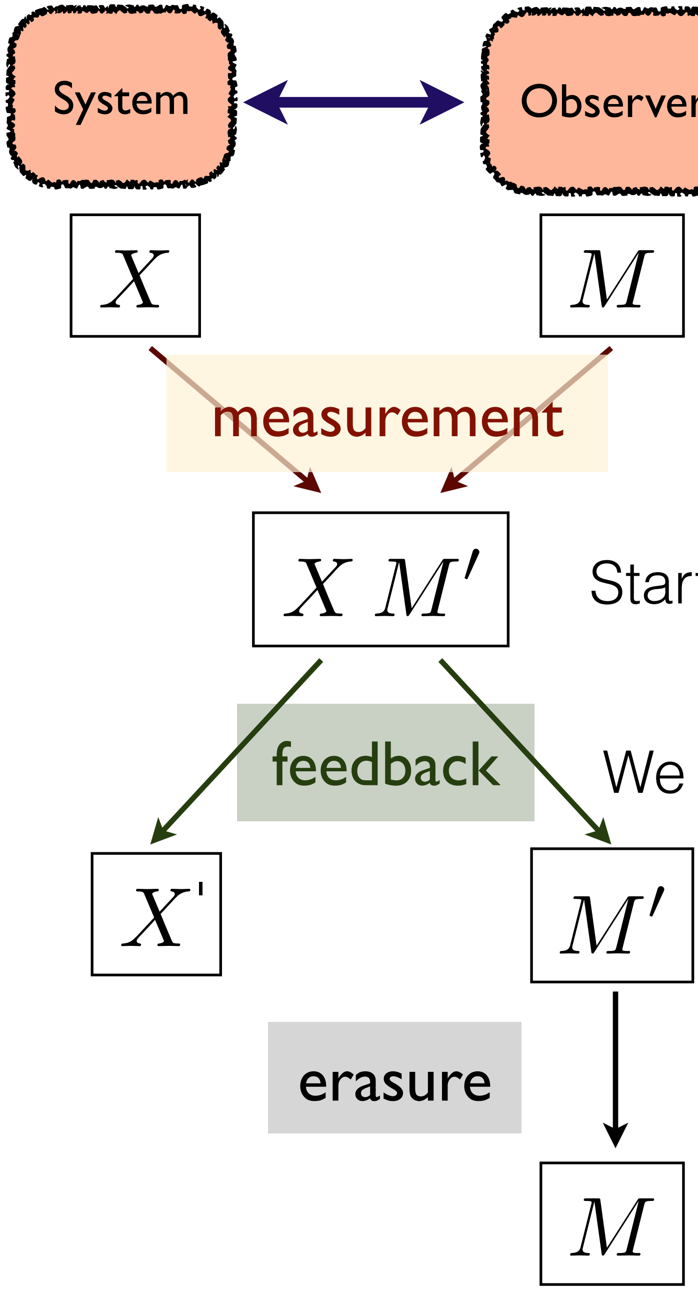
$$\mathcal{F}(X) + \mathcal{F}(M)$$

Cost of feedback

$$W^{\mathrm{meas}}_{\mathrm{min}} + W^{\mathrm{eras}}_{\mathrm{min}} = kT\, I(M';X)$$

Sagawa, Ueda. PRL (2009).

5.3 **Feedback**



Start from a system of state X on which we have the information I (via M')

We can apply a process which can uses (or not) the obtained information
i.e., the most general possible (given that it only acts on the system)

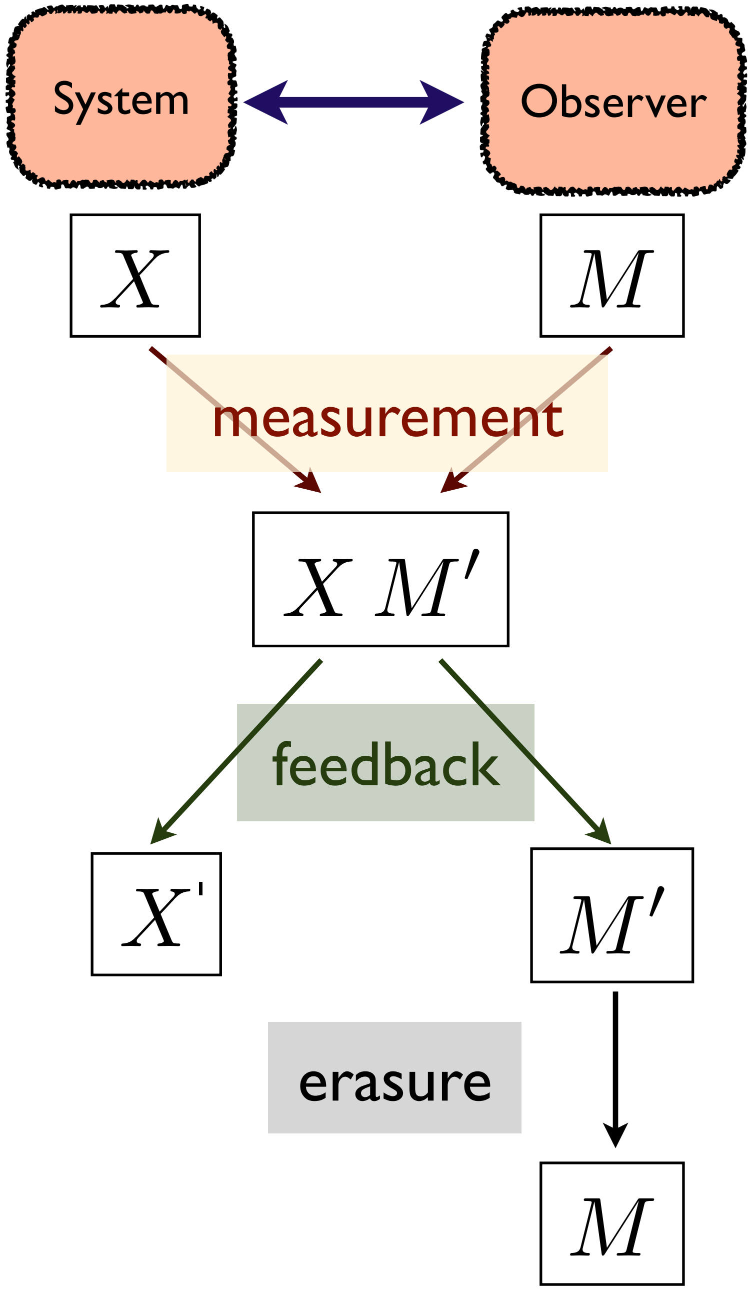
The second law on the free energy hence reads:

Remaining mutual information

$$W_{\rm Feedback} \geq \Delta F^{XM}_{\rm Feedback} = \Delta F^{X}_{\rm Fb} + \underbrace{\cancel{\Delta F^{M}_{\rm Fb}}}_{=0} + kT\big(\underbrace{I(S';M')}_{\geq 0} - I(S,M')\big)$$

$$\geq \Delta F^{X}_{\rm Fb} - kTI(S,M')$$

5.3 **Second Law**



The second law on a system on which we have information:

$$W \geq \Delta F^X - kTI(S, M')$$

The second law on the full system SM:

$$W_{\mathrm{tot}} = W_{\mathrm{Meas}} + W_{\mathrm{Fb}} + W_{\mathrm{Erasure}}$$

$\geq \Delta F^M + kTI(S, M')$

$\geq -\Delta F^M$

$\geq \Delta F^X_{\mathrm{Fb}} - kTI(S, M')$

$$\geq \Delta F^X_{\mathrm{Fb}} = \Delta F^X$$

Because the system' state only changes during the feedback step

We retrieve the standard second law of the system S

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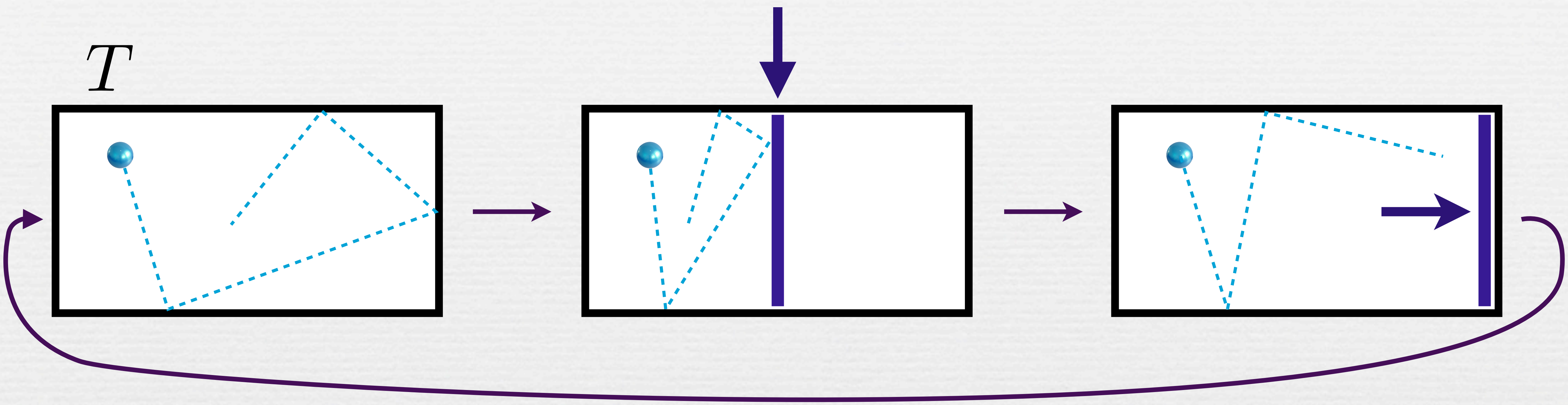
Additional content

The Szilard engine

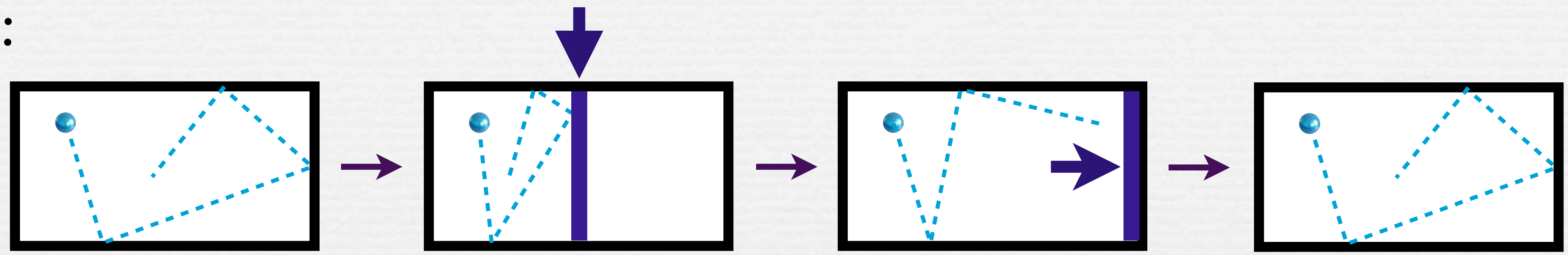


$$W_{\text{extract}} = \int_{V_{\text{init}}}^{V_{\text{fin}}} P dV = \int_{V_{\text{init}}}^{V_{\text{fin}}} \frac{kT}{V} dV = kT \log \frac{V_{\text{fin}}}{V_{\text{init}}}$$

Work done on the system: $\boxed{W = -W_{\text{extract}} = -kT \log 2}$

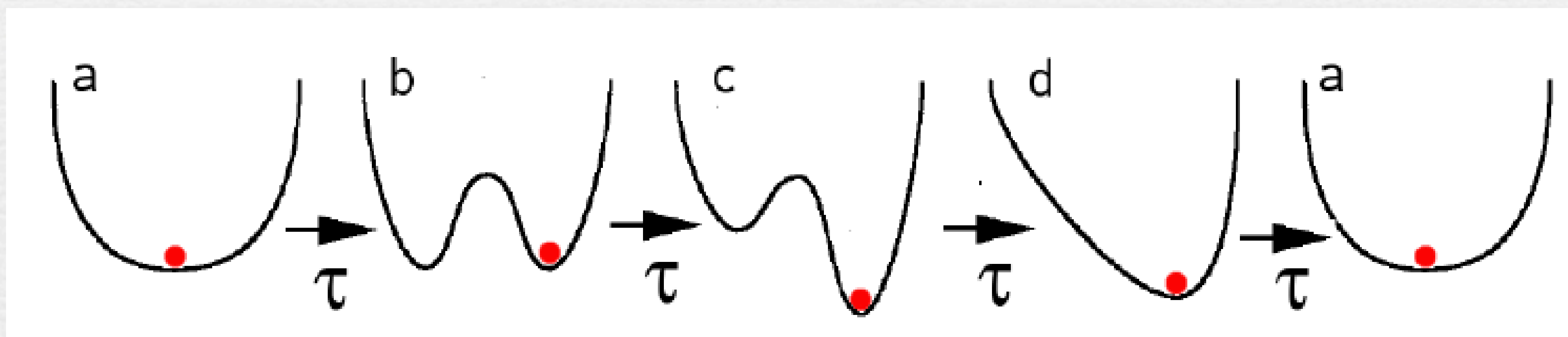
The Szilard engine

- One particle gas:



- Brownian particle:

Kawai, JMRP, van den Broeck. PRL **98**, 080602 (2007).

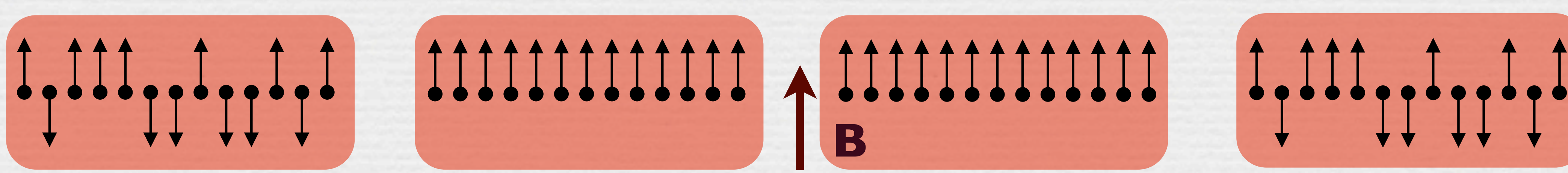


- Ising model:

JMRP. Chaos **11**, 725 (2001)

Coupling Field

(J, B)

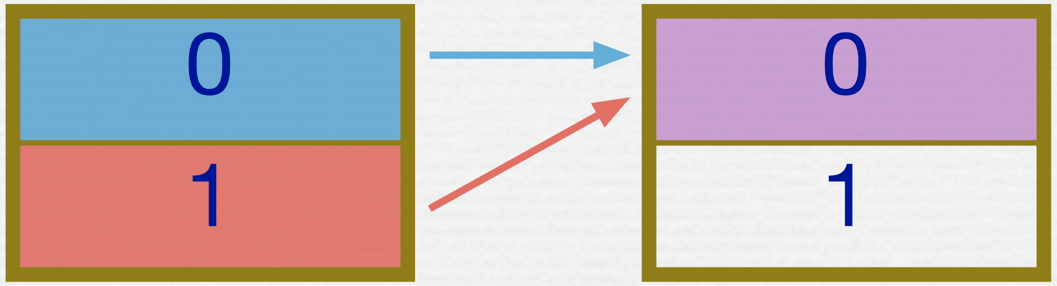


$$(0,0) \rightarrow (J,0) \rightarrow (J,\pm B) \rightarrow (0,\pm B) \rightarrow (0,0)$$

Measurement

Landauer's principle

RESTORE-TO-ZERO process (erasure):



- available phase space volume shrinks by two.
- A heat $kT\log 2$ must be dissipated.