4. Information and the second law

Non-equilibrium free energy makes easy the incorporation of information to the second law

4.1. Memories and information reservoirs.

Consider a memory with informational states $m=1,2,\dots$ These are meso- or macroscopic states defined by regions Γ_m covering the whole phase space $\Gamma = \bigcup_m \Gamma_m$.

They have a local partition function and local free energy

$$Z_m = \int_{\Gamma_m} dx\, e^{-\beta H(x)} \qquad F_m = -kT \ln Z_m$$

Consequently, to change the state of a memory for M (p_m) to M' (p'_m) we need to perform a minimal work:

$$W_{min} = \Delta \mathcal{F} = \mathcal{F}(M') - \mathcal{F}(M)$$

In the case of a symmetric memory:

$$W_{min} = kT\left[H(M) - H(M')\right]$$

A positive work is necessary if we want to order ($H(M') < H(M)$) a memory. Example: restore-to-zero (Landauer principle):

$$M = 0,1 \text{ with prob. } 1/2 \longrightarrow M' = 0 \text{ with prob. } 1 \text{ (erasure)}$$

$$\Rightarrow W^{eras}_{min} = kT \ln 2.$$

Conversely, we can use a highly ordered memory (low $H(M)$) to extract energy:

$$H(M') > H(M) \Longrightarrow W_{min} < 0 \quad (\Rightarrow \text{extractable work})$$

Information reservoirs (Mandal, Jarzinsky, 2012) (Barato, Seifert, 2014)

The state of the memory is given by the probability p_m to observe the informational state m. This state is a random variable M.

The state of the memory $\rho(x)$ in the phase space reads:

$$\rho(x) = \frac{p_m}{Z_m} e^{-\beta H(x)} \quad \text{if } x \in \Gamma_m$$

and it is equal to the global equilibrium state if and only if $p_m^{eq} = Z_m / Z$.

Let us calculate the non-equilibrium free energy of the memory:

$$\mathcal{F}(M) = \int_{\Gamma} dx \left[H(x) + kT \ln \rho(x) \right] \rho(x)$$

$$= \sum_m \int_{\Gamma_m} \left[\cancel{H(x)} + kT \left(\ln p_m - \beta \cancel{H(x)} - \ln Z_m \right) \rho(x) \right.$$

$$= \sum_m p_m F_m + kT \sum_m p_m \ln p_m = \langle F_m \rangle - kT H(M)$$

which can be also written as:

$$\mathcal{F}(M) = F_{eq} + D(p_m \| p_m^{eq})$$

For a symmetric memory $F_m = F_1$ and $p_m = 1/N_{is}$ where N_{is} is the total number of informational states.

4.2. Measurement and feedback.

After measurement of an observable with outcome m:

$$p(x) \longrightarrow p(x|m) = \frac{p(x,m)}{p(m)} = \frac{p(x)}{p(m)}\, p(m|x)$$

The free energy change is (we assume that the Hamiltonian H does not change):

$$\Delta\mathcal{F}_{meas} = \mathcal{F}(p(x);H) - \sum_m p(m)\mathcal{F}(p(x|m);H) = kT\left[H(X)-H(X|M)\right]$$

$$\Rightarrow \Delta\mathcal{F}_{meas} = kT\,I(X,M) \geq 0$$ Measurement increases the free energy.

We can extract energy in a cycle with measurement (feedback process = the protocol depends on the measurement outcome).

1) Start with the system at equilibrium with $\mathcal{F}_0 = F_{eq}(\lambda(0))$

2) Make a measurement $\mathcal{F}_1 = \mathcal{F}_0 + kT\,I(X,M)$

3) Drive the system back to the initial state. The work satisfies:

$$W \geq \Delta\mathcal{F} = \mathcal{F}_0 - \mathcal{F}_1 = -kT\,I(X,M)$$

If the cycle is reversible we can extract a work:

$$W_{extract} = -W = kT\,I(X,M)$$

If the measurement is error-free, $I(X,M) = H(M)$ and

$$W_{extract} = kT\,H(M)$$

Example: In Szilard engine $p_m = \frac{1}{2}$ for $m =$ Left, Right.

$$H(M) = 1 \text{ bit} = \ln 2 \text{ nats} \Rightarrow W_{extract} = -kT \ln 2$$

Szilard engine is reversible (hence optimal)

Szilard engine with error measurement → Exercise 2.

In general, the second law for a process, starting and ending in equilibrium and with a measerement in between, reads

$$\boxed{W \geq \Delta F - kT I}$$ where I is the information gain in the measurement