7. Thermodynamic cost of measurement and erasure.

Consider now the observer as a physical memory with energy $\mathcal{H}_{obs}(m)$ and non-equilibrium free energy:

$$\mathcal{F}(M) = \mathcal{F}(p_m ; \mathcal{H}_{obs}(m)) = \langle \mathcal{H}_{obs}(m) \rangle - kT\, H(M)$$

An ideal measurement on a system whose state is a random variable X with distribution $p_X(x)$ consists of:

1) Initially X and M are independent and non-interacting:

$$\mathcal{H}(x,m) = \mathcal{H}_s(x) + \mathcal{H}_{obs}(m) \qquad p_{XM}(x,m) = p_X(x)\, p_m$$

2) In the measurement the observer changes: $M \to M'$ and correlates with X, which is not altered. After measurement observer and system decoupled:

$$p_{XM'}(x,m) = p_X(x)\, p(m|x)$$

$\leftarrow$ measurement device

Let us calculate the non-equilibrium free energy:

Before: $\mathcal{F}(X) + \mathcal{F}(M)$

After: $\langle \mathcal{H}_s(x) \rangle + \langle \mathcal{H}_{obs}(m) \rangle - kT\, H(X, M')$

$$= \mathcal{F}(X) + \mathcal{F}(M') + kT\, I(X, M')$$

where we have used the mutual information:

$$I(X,M') = H(X) + H(M') - kT\,I(X,M')$$

The change of free energy due to the measurement is:

$$\Delta \mathcal{F}_{meas} = \Delta \mathcal{F}_{obs} + kT\,I(X,M') \leq W_{meas.}$$

We get a similar expression as in sec. 4, but now with a different interpretation: there, kTI appeared as the result of a change in our information about the system X ($p(x) \to p(x|m)$). Here is the result of correlating two physical systems: X and M.

Restoring the observer to its initial state $M' \to M$ needs also some work, usually referred to as the work or cost of erasure:

$$\text{Erasure:} \quad M' \longrightarrow M$$

$$\Longrightarrow \quad \Delta \mathcal{F}_{eras} = -\Delta \mathcal{F}_{obs} \leq W_{eras.}$$

Combinig the expressions for the minimal work:

$$W_{eras} + W_{meas} \geq kT\,I(X,M') = W_{extract}^{max}$$

That is, the extracted work in the feedback is balanced by the work to measure and erase.

Example:

Szilard / Bennett scenario

1) Error free measurement: $I(X, M') = H(M')$

2) Symmetric memory: $\Delta \mathcal{F}_{obs} = kT[H(M) - H(M')]$

$W_{meas} \geq kTH(M)$ ← (= 0 Bennet/Landauer erasure = restore-to-zero)

$W_{eras} \geq kT(H(M') - H(M))$

The whole picture of feedback

$\mathcal{F}(X) + \mathcal{F}(M)$

$W_{meas} \geq \Delta \mathcal{F}_{obs} + kTI$ measurement

$\mathcal{F}(X) + \mathcal{F}(M') + kTI(X, M')$

$W_{fb} \geq -kTI$ feedback

$\mathcal{F}(X) + \mathcal{F}(M')$

$W_e \geq -\Delta \mathcal{F}_{obs}$ erasure

$\mathcal{F}(X) + \mathcal{F}(M)$

$$\begin{tikzcd}
X \arrow[dr] & & M \arrow[dl] \\
& X M' \arrow[dl] \arrow[dr] & \\
X' = X \arrow[d] & & M' \arrow[d] \\
X & & M
\end{tikzcd}$$