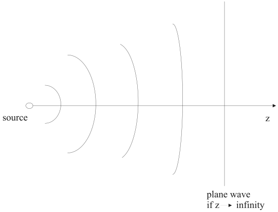
Lecture

Image Formation

Silvania Pereira

Contents

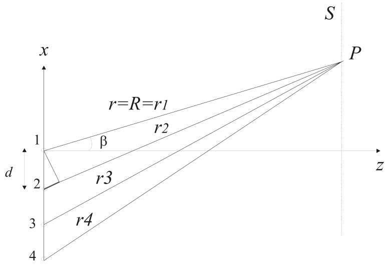
Point source



$$\epsilon(z) = E_0 e^{i(\phi - kz)} \tag{1}$$

$k = \omega/c = 2\pi/\lambda$ and $\phi =$ arbitrary phase

Field due to several point sources



Field at point P:

$$\epsilon = Ae^{-ikr_1} + Ae^{-ikr_2} + Ae^{-ikr_3} + Ae^{-ikr_4} \ldots . Ae^{-ikr_n} \qquad (2)$$

where

$r_1 = r$

$r_2 = r + d \sin \beta$ $\qquad$ $r_3 = r + 2d \sin \beta$

$r_4 = r + 3d \sin \beta$ $\qquad$ $r_N = r + (N-1)d \sin \beta$

Field due to several point sources

Substituting $r_1 ... r_N$ into Eq.(2):

$$\epsilon = Ae^{-ikr}(1 + e^{-ikd\sin\beta} + e^{-i2kd\sin\beta} + e^{-i3kd\sin\beta} + ... e^{-ikd(N-1)\sin\beta}) \tag{3}$$

Geometric series: $\Sigma_{k=0}^{N-1} y^k = \frac{1-y^N}{1-y}$

Field due to several point sources

Substituting $r_1 ... r_N$ into Eq.(2):

$$\epsilon = Ae^{-ikr}(1 + e^{-ikd\sin\beta} + e^{-i2kd\sin\beta} + e^{-i3kd\sin\beta} + ... e^{-ikd(N-1)\sin\beta}) \quad (3)$$

Geometric series: $\Sigma_{k=0}^{N-1} y^k = \frac{1-y^N}{1-y}$

Field due to several point sources

The field can be written as:

$$\epsilon = Ae^{-ikr}\left(\frac{1 - e^{-ikNd\sin\beta}}{1 - e^{-ikd\sin\beta}}\right) \tag{4}$$

$$= Ae^{-ikr}\frac{e^{-ikN\frac{d}{2}\sin\beta}}{e^{-ik\frac{d}{2}\sin\beta}}\left(\frac{\sin(kN\frac{d}{2}\sin\beta)}{\sin(k\frac{d}{2}\sin\beta)}\right) \tag{5}$$

The intensity is given by

$$I = |\epsilon|^2 = |A|^2 \frac{\sin^2(kN\frac{d}{2}\sin\beta)}{\sin^2(k\frac{d}{2}\sin\beta)} \tag{6}$$

For $d \to 0$ and $N \to \infty$, $Nd = a =$constant, and $A_0 = NA$
Small angles β, the sinus term can be approximated as

$$\sin(n\theta) \approx n\theta \tag{7}$$

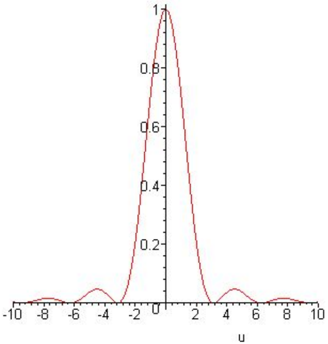
$$\sin(\theta) \approx \theta \tag{8}$$

Diffraction due to several point sources

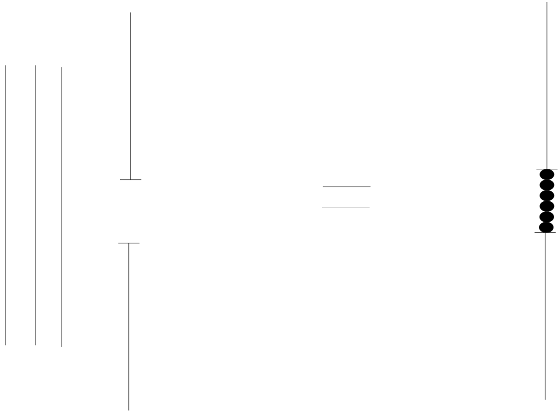
Finally, the intensity can be expressed as:

$$I = |A_0|^2 \frac{\sin^2 u}{u^2} \tag{9}$$

with $u = \frac{ka \sin \beta}{2}$

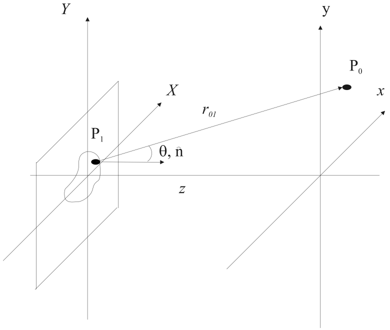


Diffraction pattern due to a single slit



aperture a $=$ Huygens

Diffraction of an arbitrary aperture



z = distance between two planes

θ = angle between normal $\hat{n}$ and the vector r_{01}

$\cos\theta = \frac{z}{r_{01}}$

$$U(P_0) = \frac{1}{i\lambda}\int_{aperture} U(P_1)\frac{e^{ikr_{01}}}{r_{01}}\cos\theta dS \tag{10}$$

Diffraction of an arbitrary aperture

$$U(x,y) = \frac{z}{i\lambda}\int_{aperture} U(X,Y)\frac{\exp^{ikr_{01}}}{(r_{01})^2}dXdY \tag{11}$$

with $r_{01} = \sqrt{z^2 + (x-X)^2 + (y-Y)^2}$

$r_{01} = z\sqrt{1 + (\frac{x-X}{z})^2 + (\frac{y-Y}{z})^2}$

Expanding the square root
$\sqrt{1+b} = 1 + b/2 - b^2/8 + ...$

$r_{01} = z(1 + (\frac{1}{2}\frac{x-X}{z})^2 + \frac{1}{2}(\frac{y-Y}{z})^2)$

$r_{01} \rightarrow$ denominator $r_{01} \approx z$

$r_{01} \rightarrow$ exponent $\rightarrow$ retain since it is related to the phase

Diffraction of an arbitrary aperture

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Fresnel approximation

$$U(x,y) = \frac{e^{ikz}}{i\lambda z}\int_{aperture} U(X,Y)e^{\frac{ik}{2z}((x-X)^2+(y-Y)^2)}dXdY \quad (12)$$

Factoring out the exponential term:

$$U(x,y) = \frac{e^{ikz}}{i\lambda z}e^{\frac{ik}{2z}(x^2+y^2)}\int_{aperture} U(X,Y)e^{\frac{ik}{2z}(X^2+Y^2)}e^{-\frac{ik}{z}(xX+yY)}dXdY \quad (13)$$

This is the Fourier transform of the product of the field with a quadratic phase potential

$$U(x,y) = \frac{e^{ikz}}{i\lambda z}e^{\frac{ik}{2z}(x^2+y^2)}FT((U(X,Y)e^{\frac{ik}{z}(X^2+Y^2)}) \quad (14)$$

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Fraunhofer approximation

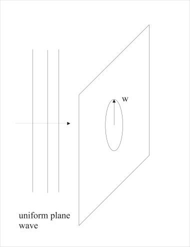
Consider $z >>> \frac{k(X^2+Y^2)_{max}}{2}$

quadratic phase factor becomes unity and the field can be written as:

$$U(x,y) = \frac{e^{i\lambda z}}{ikz} e^{\frac{ik(x^2+y^2)}{2z}} \int_{aperture} U(X,Y) e^{\frac{ik}{z}(xX+yY)} dXdY \quad (15)$$

This is the Fourier transform of the aperture distribution evaluated at frequencies:
$f_x = x/\lambda z, f_y = y/\lambda z$ with $\lambda = 2\pi/k$

The circular aperture



Consider a screen of transmission function:

$$t_A(q) = left\{ \begin{array}{l} circ(q/w) = 1, \quad q \leq w \\ 0, \quad q \geq w \end{array}$$

$q =$ radius coordinate, and w = radius of the aperture

circular symmetry since $g(r, \theta) = g(r) \rightarrow$ Fourier Bessel Function

$q = \sqrt{X^2 + Y^2}, X = q \cos\theta, Y = q \sin\theta$ and $\theta = arctan(\frac{Y}{X})$

$f_x = \rho \cos\phi, f_y = \rho \sin\phi, \rho = \sqrt{f_x^2 + f_y^2}$ and $\phi = arctan(\frac{f_x}{f_y})$

Circular aperture

The Fourier transform of an aperture $g(X, Y)$ is given by:

$$G_0(f_x, f_y) = \int g(X, Y) e^{-i2\pi(f_x X + f_y Y) dXdY} \tag{16}$$

and in cylindrical coordinates:

$$G_0(\rho, \phi) = \int_0^{2\pi} \int_0^{\infty} g(q) e^{-i2\pi qp(\cos\theta\cos\phi + \sin\theta\sin\phi)} qdqd\theta \tag{17}$$

Integral in θ is equal to the Bessel function of first kind, order 0 defined as:

$$J_0(2\pi q\rho) = \int_0^{2\pi} e^{-i2\pi qp(\cos(\theta-\phi))} d\theta \tag{18}$$

Thus

$$G_0(\rho, \phi) = 2\pi \int_0^{\infty} g(q) J_0(2\pi q\rho) qdq \tag{19}$$

FT of a circularly symmetric function is also circularly symmetric.

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The Fourier transform of an aperture $g(X, Y)$ is given by:

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FT of a circularly symmetric function is also circularly symmetric.

The circular aperture

Substituting $g(q)$ by the transmission function $g(q) = circ(q/w)$ we have that:

$$G_0(\rho) = 2\pi \int_0^w J_0(2\pi q\rho) q dq \tag{20}$$

Make the substitutions: $2\pi q\rho = q'$ we get:

$$G_0(\rho) = 2\pi \int_0^{2\pi\rho w} \frac{1}{(2\pi\rho)^2} J_0(q') q' dq' = A \frac{J_1(2\pi\rho w)}{\rho w \pi} \tag{21}$$

where: $A = \pi w^2 =$ area of the aperture and $J_1 =$ Bessel function of the first kind, order 1.

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The circular aperture

Finally we have that:

$$U(r) = \frac{e^{ikz}}{i\lambda z} e^{\frac{ikr^2}{2z}} BT(U(q)_{\rho=r/\lambda z}) \tag{22}$$

where BT= Bessel transform

$$U(r) = \frac{e^{ikz}}{i\lambda z} e^{\frac{ikr^2}{2z}} \frac{A}{i\lambda z}(2\frac{J_1(kwr/z)}{kwr/z}) \tag{23}$$

And the intensity

$$I(r) = (\frac{A}{i\lambda z})^2(2\frac{J_1(kwr/z)}{kwr/z})^2 \tag{24}$$

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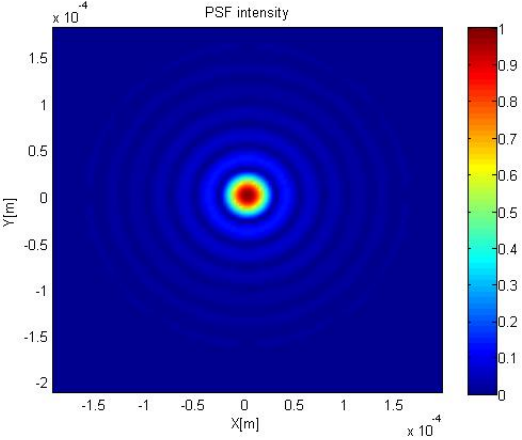
The circular aperture: Airy pattern

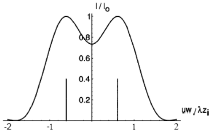


The width of the central lobe is given by: $d = 1.22\frac{\lambda z}{w}$
where z is the distance between the aperture plane and obeservation plane

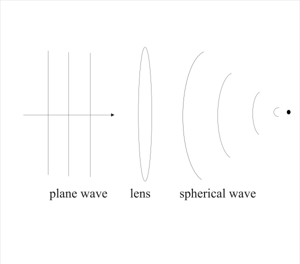
Raileigh criterion

Rayleigh criterion: two incoherent point sources are barely resolved by a circular pupil system when the center of the Airy pattern generated by one point source falls on the first zero of the Airy pattern of the second source. The minimum resolvable separation will be ;

$$\delta = 0.61\lambda z_i / w$$



Lenses



transmission function:

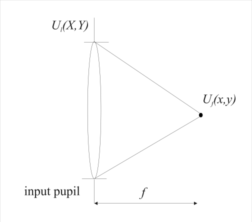
$$t_l(x, y) = e^{-\frac{ik}{2f}(x^2+y^2)} \tag{25}$$

where

$$\frac{1}{f} = (n-1)(\frac{1}{R_1} - \frac{1}{R_2}) \tag{26}$$

with R_1, R_2 the radius of curvature of the two surfaces of the lens

Lenses



Field at pupil: $P(X,Y) = \begin{cases} 1, & \text{inside lens aperture} \\ 0, & \text{outside lens aperture} \end{cases}$

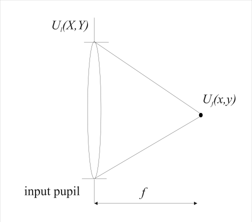
Field just after the lens:

$$U_l'(X,Y) = P(X,Y)e^{-\frac{ik}{2f}(x^2+y^2)} \tag{27}$$

Propagate this field a distance f

$$U_f(x,y) = \frac{e^{\frac{ik}{2f}(x^2+y^2)}}{i\lambda f}\int_{aperture} U_l'(X,Y)e^{\frac{ik}{2f}(X^2+Y^2)}e^{-\frac{ik}{2f}(xX+yY)}dXdY \tag{28}$$

Lenses



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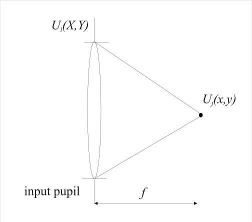

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Lenses

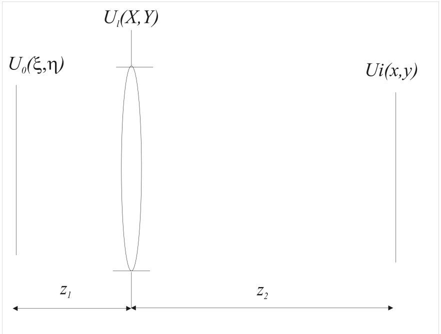
Substituting $U_l'(X, Y)$, the phase factor cancels and we obtain

$$U_f(x, y) = \frac{e^{\frac{ik}{2f}(x^2+y^2)}}{i\lambda f} \int_{aperture} P(X, Y) e^{-\frac{ik}{2f}(xX+yY)} dXdY \quad (29)$$

This is the Fraunhofer pattern of the field at the input pupil of the lens at the points:

$f_x = x/\lambda f$ and $f_y = y/\lambda f$

Image formation



Consider $h(x, y, \xi, \eta)$= field produced at (x, y) by an unit amplitude point source at (ξ, η)

$$U_i(x, y) = \int h(x, y, \xi, \eta) U_0(\xi, \eta) d\xi d\eta \tag{30}$$

Image formation

Field at the lens is a diverging spherical wave. In the paraxial approximation we have:

$$U_l(X,Y) = \frac{1}{i\lambda z} e^{i\frac{k}{2z_1}((X-\xi)^2+(Y-\eta)^2)} \tag{31}$$

and after the lens

$$U_l'(X,Y) = U_l(X,Y) e^{-i\frac{k}{2f}(X^2+Y^2)} P(X,Y) \tag{32}$$

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Image formation

Field at the lens is a diverging spherical wave. In the paraxial approximation we have:

$$U_l(X, Y) = \frac{1}{i\lambda z} e^{i\frac{k}{2z_1}((X-\xi)^2+(Y-\eta)^2)} \tag{31}$$

and after the lens

$$U_l'(X, Y) = U_l(X, Y) e^{-i\frac{k}{2f}(X^2+Y^2)} P(X, Y) \tag{32}$$

Image formation

Propagate field $U_l'(X, Y)$ a distance z_2

$$h(x, y, \xi, \eta) = \frac{1}{i\lambda z_2} \int U_l'(X, Y) e^{-\frac{ik}{2z_2}((x-X)^2+(y-Y)^2)} dXdY \quad (33)$$

Substituting $U_l'(X, Y)$ we have

$$h(x, y, \xi, \eta) =$$

$$\frac{1}{\lambda^2 z_1 z_2} \int P(X, Y) e^{\frac{ik}{2}(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f})(x^2+y^2)} e^{-ik(\frac{\xi}{z_1}+\frac{x}{z_2})X+(\frac{\eta}{z_1}+\frac{y}{z_2})Y} dXdY \quad (34)$$

where constant phase phase have been omitted

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Image formation

Lens law : $\frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f} = 0$, so that $h(x, y, \xi, \eta) =$

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and defining $M = -\frac{z_2}{z_1}$

$h(x, y, \xi, \eta) =$

$$\frac{1}{\lambda^2 z_1 z_2} \int P(X, Y) e^{\frac{-ik}{z_2}((x - M\xi)X + (y - M\eta)Y)} dXdY \tag{36}$$

Impulse response is the Fraunhofer pattern of the lens aperture centered in the coordinates $x = M\xi, y = M\eta$

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Relation between object and image

$$U_i(x, y) = \int h(x, y, \xi, \eta) U_0(\xi, \eta) d\xi d\eta \tag{37}$$

where $U_0(\xi, \eta)$ can be seen as the geometrical optics prediction of the image, as if the impulse function would be a delta function

$U_0(\xi, \eta) =$ amplitude transmitted by the object at point (ξ, η) and $h(x, y, \xi, \eta) =$ amplitude response to a point source at point (ξ, η) given by:

$$h(x, y, \xi, \eta) = \frac{1}{\lambda^2 z_1 z_2} \int P(X, Y) e^{\frac{-ik}{z_2}((x - M\xi)X + (y - M\eta)Y)} dXdY \tag{38}$$

Frequency response for diffraction limited coherent image

Amplitude transfer function (ATF or MTF)

$$H(f_x, f_y) = \int h(x, y) e^{-i2\pi(f_x x + f_y y)} dxdy \tag{39}$$

Fourier transform of the amplitude response function
But $h(x, y)$ is the Fourier transform of the pupil function, thus

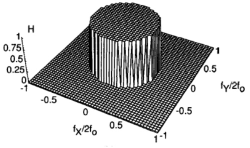
$$H(f_x, f_y) = P(\lambda z_i f_x, \lambda z_i f_y) \tag{40}$$

Example: circular pupil

$$P(X,Y) = circ(\frac{\sqrt{X^2+Y^2}}{w}) \tag{41}$$

The MTF is straighforward:

$$H(f_x,f_y) = circ(\frac{\sqrt{f_x^2+f_y^2}}{w/\lambda z_i}) \tag{42}$$



where $f_0 = \frac{w}{\lambda z_i}$

Incoherent imaging

Obey the intensity convolution integral and the Optical transfer function (OTF) is defined as:

$$\mathscr{H}(f_x, f_y) = \frac{\int |h(x,y)|^2 e^{-i2\pi(f_x x + f_y y)} dxdy}{\int |h(x,y)|^2 dxdy} \qquad (43)$$

where the denominator is a normalisation factor

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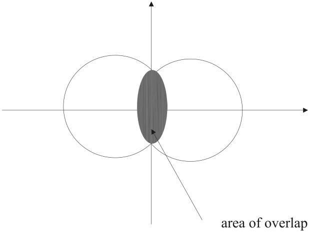
Relationship between MTF and OTF

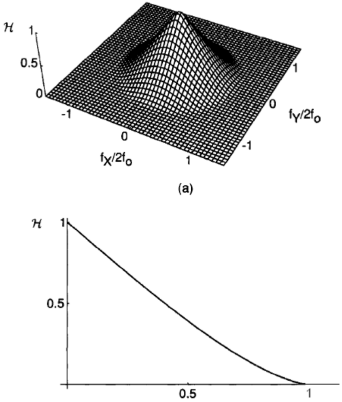
$$\mathscr{H}(f_x, f_y) = \frac{\int H(p + f_x/2, q + f_y/2) H^*(p - f_x/2, q - f_y/2) dpdq}{\int |H(p,q)|^2 dpdq} \tag{44}$$

The OTF is the normalized autocorrelation function of the MTF

Geometrical interpretation:
denominator: area of the pupil
numerator: area of overlap of two displaced pupil functions: one centered at $(\lambda z_i f_x/2, \lambda z_i f_y/2)$ and the other at $(-\lambda z_i f_x/2, -\lambda z_i f_y/2)$



OTF circular aperture



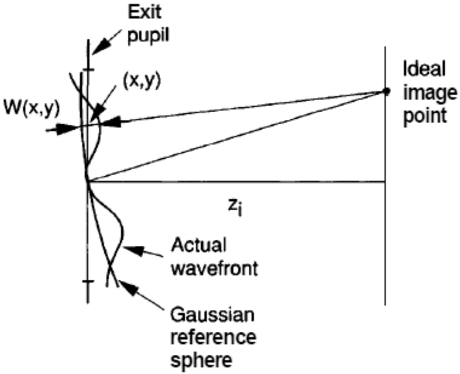
$f_0 = \frac{w}{\lambda z_i}$

Aberrations

Generalized pupil function

$$\mathscr{P}(X, Y) = P(X, Y) \times e^{ikW(X,Y)} \tag{45}$$

$W(X, Y)$= aberration function



Amplitude and Optical Transfer Functions

Amplitude transfer function (ATF or MTF)

$$H(f_x, f_y) = \mathscr{P}(\lambda z_i f_x, \lambda z_i f_y) = P(\lambda z_i f_x, \lambda z_i f_y) e^{ikW(\lambda z_i f_x, \lambda z_i f_y)} \tag{46}$$

Optical transfer function

$$\mathscr{H}(f_x, f_y) = \frac{\int e^{ik(W(x+\lambda z_i f_x/2, y+\lambda z_i f_x/2) - W(x-\lambda z_i f_x/2, y-\lambda z_i f_x/2))} dxdy}{\int_{A(0,0)} dxdy} \tag{47}$$

Zernike polynomials

The aberration function $W(X, Y)$ can be written in polar coordinates $W(\rho, \theta)$, with ρ being the radial coordinate within the unit circle and θ the polar angle

For optical systems that have pupil with circular symmetry, the aberrations can be represented in terms from Zernike polymonials

Zernike polynomials

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Zernike polynomials

Zernike polynomials are orthogonal and normalized within unit circle. The aberration function can then be written as:

$$W(\rho, \theta) = \Sigma_n^k \Sigma_{m=-n}^n W_n^m z_n^m(\rho, \theta) \tag{48}$$

where $z_n^m(\rho, \theta)$ are the Zernike polynomials and W_n^m is the coefficient of the expansion

Zernike polynomials

$$z_n^m(\rho,\theta) = N_n^m R_n^{|m|}(\rho)\cos m\theta \qquad m \geqslant 0, 0 \leqslant \rho \leqslant 1, 0 \leqslant \theta \leqslant 2\pi \tag{49}$$

$$= -N_n^m R_n^{|m|}(\rho)\sin m\theta \qquad m < 0, 0 \leqslant \rho \leqslant 1, 0 \leqslant \theta \leqslant 2\pi \tag{50}$$

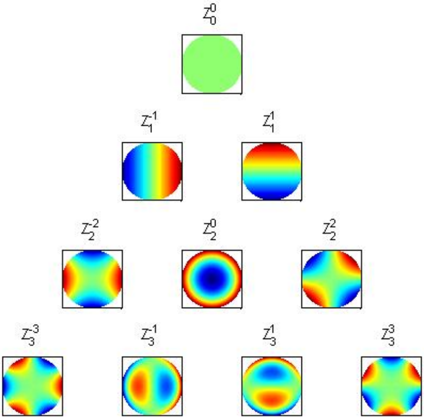
for a given n, m can take the values $-n, -n+2, -n+4 \dots n$, and $N_n^m(\rho)$ is the normalization factor $N_n^m = \sqrt{\frac{2(n+1)}{1+\delta_{m0}}}$ ($\delta_{m0} = 1$ for $m = 0$ and 0 otherwise)

$R_n^{|m|}(\rho)$ is the radial polynomial
$R_n^{|m|}(\rho) = \Sigma_{s=0}^{(n-|m|)/2} \frac{(1-)^s(n-s)!}{s!(0.5(n+|m|)-s)!(0.5(n-|m|)-s)!}\rho^{n-2s}$

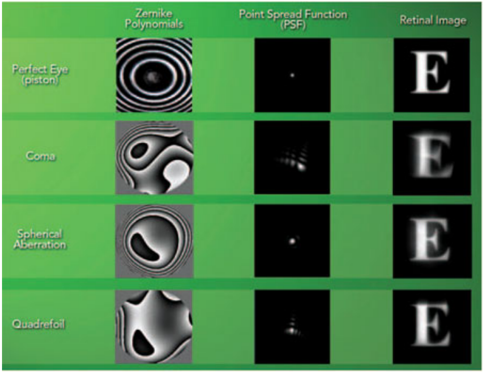
Zernike polynomials

mode j	order n	frequency m	$Z_n^m(\rho,\theta)$	Meaning
0	0	0	1	Constant term, or Piston
1	1	-1	$2\rho\sin(\theta)$	Tilt in y-direction, Distortion
2	1	1	$2\rho\cos(\theta)$	Tilt in x-direction, Distortion
3	2	-2	$\sqrt{6}\rho^2\sin(2\theta)$	Astigmatism with axis at $\pm 45^\circ$
4	2	0	$\sqrt{3}\left(2\rho^2-1\right)$	Field curvature, Defocus
5	2	2	$\sqrt{6}\rho^2\cos(2\theta)$	Astigmatism with axis at 0° or 90°
6	3	-3	$\sqrt{8}\rho^3\sin(3\theta)$	
7	3	-1	$\sqrt{8}\left(3\rho^3-2\rho\right)\sin(\theta)$	Coma along y-axis
8	3	1	$\sqrt{8}\left(3\rho^3-2\rho\right)\cos(\theta)$	Coma along x-axis
9	3	3	$\sqrt{8}\rho^3\cos(3\theta)$	
10	4	-4	$\sqrt{10}\rho^4\sin(4\theta)$	
11	4	-2	$\sqrt{10}\left(4\rho^4-3\rho^2\right)\sin(2\theta)$	Secondary Astigmatism
12	4	0	$\sqrt{5}\left(6\rho^4-6\rho^2+1\right)$	Spherical Aberration, Defocus
13	4	2	$\sqrt{10}\left(4\rho^4-3\rho^2\right)\cos(2\theta)$	Secondary Astigmatism
14	4	4	$\sqrt{10}\rho^4\cos(4\theta)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Zernike polynomials



Effects of aberrations in the PSF and image



Conclusion

- Propagation of Light
- Interference and Diffraction
- Fresnel and Fraunhofer Diffraction
- Image Formation
- Optical and amplitude tranfer functions
- Aberrations

References: Introduction to Fourier Optics, J. Goodman
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