Assessing the second-order correlation function of a quantum state from its Wigner function

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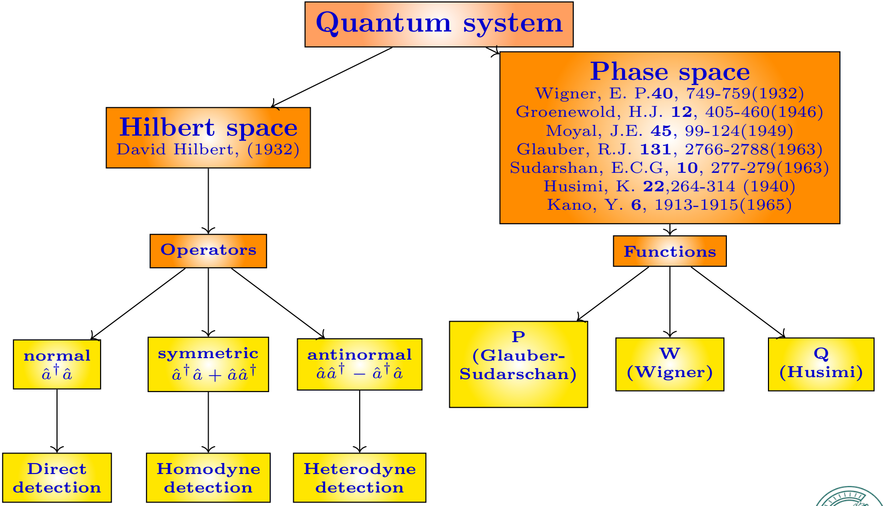

Quantum system
Hilbert space
David Hilbert, (1932)
Phase space
Wigner, E. P.**40**, 749-759(1932)
Groenewold, H.J. **12**, 405-460(1946)
Moyal, J.E. **45**, 99-124(1949)
Glauber, R.J. **131**, 2766-2788(1963)
Sudarshan, E.C.G, **10**, 277-279(1963)
Husimi, K. **22**,264-314 (1940)
Kano, Y. **6**, 1913-1915(1965)
Operators
Functions
normal
$\hat{a}^\dagger \hat{a}$
symmetric
$\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger$
antinormal
$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}$
P
(Glauber-Sudarschan)
W
(Wigner)
Q
(Husimi)
Direct detection
Homodyne detection
Heterodyne detection

Quantum-Quadrature operator

$$E(t) = E_0 cos(\omega t + \theta), = E_0 cos(\theta) cos(\omega t) - E_0 sin(\theta) sin(\omega t)$$

$$= X_1 cos(\omega t) + X_2 sin(\omega t) == X_1 cos(\omega t) + X_2 sin(\omega t)$$

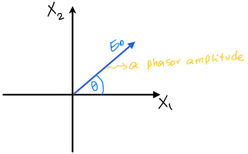
Phasor representation of field

$$a(t) = E_0 e^{-i\theta} e^{-i\omega t} = a e^{-i\omega t}$$

$$a = X_1 + iX_2$$

$$X_1 = \mathrm{Re(a)} = \frac{1}{2}(a + a^*)$$

$$X_2 = \mathrm{Im(a)} = \frac{1}{2i}(a - a^*)$$



Quadrature operator

In the Heisenberg picture, the field operator evolves as

$$E(x,t) = E_1\varepsilon(\underline{\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}})sin(kz) + i(\underline{\hat{a}e^{i\omega t} - \hat{a}^\dagger e^{-i\omega t}})cos(kz),$$

The combination of $\hat{a}^\dagger$ and $\hat{a}$ operators, is called **quadrature**.

The quadrature variables:

$$\hat{X}_1 = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}, \quad \hat{X}_2 = \frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i} \text{ and obey } [\hat{X}_{1i}, \hat{X}_{2j}] = 2i\delta_{i,j},$$

The Hamiltonian of the radiation field of a single mode

$$\hat{H} = \frac{\hbar\omega}{4}(\hat{X}_1^{\,2} + \hat{X}_2^{\,2}) = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}),$$
$$\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2}), \text{ where } \hat{n} = \hat{a}^\dagger\hat{a}.$$

Quantum states

Number states: $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$,

$|vac\rangle = |0\rangle \rightarrow$ vacuum state.

Coherent state: $|\alpha\rangle = D(\alpha)|0\rangle$,

$$D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}), D^\dagger(\alpha)D(\alpha) = I,$$

$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \rightarrow$ Eigenstate of annihilation operator,

$\langle\hat{a}^\dagger\hat{a}\rangle = \alpha^*\alpha$ $\rightarrow$Photon number,

A Coherent state is superposition of the number states:

$$|\alpha\rangle = \sum_{n=0}^{\infty}|n\rangle\langle n|\alpha\rangle, |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle,$$

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\exp(-\tfrac{1}{2}|\alpha|^2),$$

$$|\alpha\rangle = \exp(-\tfrac{|\alpha|^2}{2})\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Quantum states

Mixed states: represented by density operator $\hat{\rho}$

$$\hat{\rho} = w_1|\sigma_1\rangle\langle\sigma_1\rangle + w_2|\sigma_2\rangle\langle\sigma_2| + \dots$$

Attention

Superposition state is **Not** mixture of state

Consider the equal superposition state, $|\phi\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$

The density operator is: $\rho_p = |\phi\rangle\langle\phi| = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|a\rangle\langle b| + \frac{1}{2}|b\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|,$

In contrast, the mixture of the states: $\hat{\rho}_m = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|.$

Thermal state: represented as a mixture of number states,

$$\hat{\rho}_{th} = \frac{1}{G}(|0\rangle\langle 0| + \frac{G-1}{G}|1\rangle\langle 1| + (\frac{G-1}{G})^2|2\rangle\langle 2| + \dots)$$

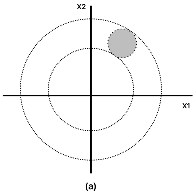
where $G = \frac{1}{1-e^{-\hbar\omega/K_BT}}$

Phasor diagram for a quantized field

Quantum uncertainty principle in the field quadrature:

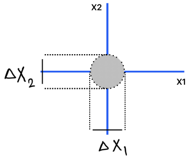
$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

Coherent state



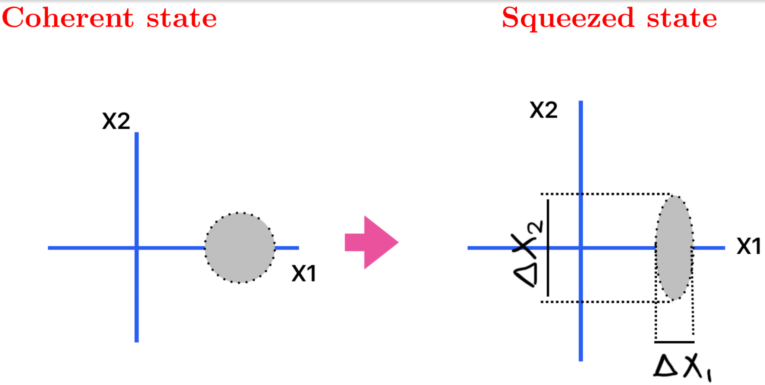
The shaded circle represents the equal uncertainty in the two quadratures. The field phase can lie anywhere withing this uncertainty circle.

Vacuum state



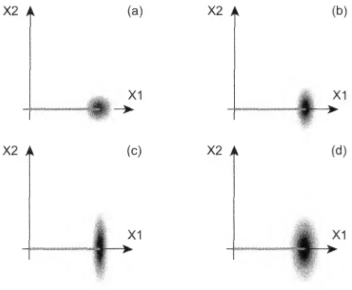
The shaded region of the phasor diagram indicates the random fluctuating field of the vacuum. The uncertainties in the two quadratures are identical and each equal to minimum, $\Delta X_1^{vac} = \Delta X_2^{vac} = \frac{1}{2}$.

Squeezed state



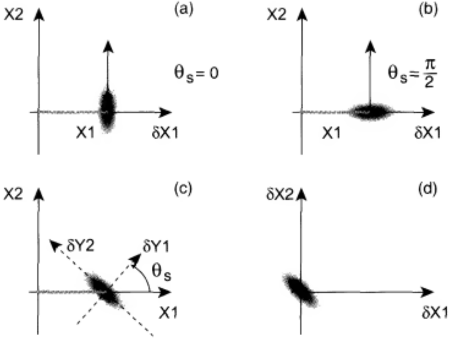
In the squeezed case, the noise fluctuation reduces below the minimum limit in one quadrature only.

Phasor diagram comparing squeezed and other states



(a) coherent state (b) minimum uncertainty squeezed state which is narrower than the coherent state in one direction, (c)Squeezed state with excess noise (d)An symmetric noisy but not squeezed state. It is described by an ellipse, but no projection is narrower than the coherent state.

Different types of squeezed states



(a) amplitude, (b) phase squeezed state, (c) quadrature squeezed state, (d) vacuum quadrature squeezed state

Generation of Squeezed Light

Squeezing

- **Four wave mixing, Kerr effect** $\chi^{(3)}$
  Najafabadi, et al **545** 129717
  Kalinin, et al, **6** 202200143
- **Parametric down conversion** $\chi^{(2)}$ **(SPDC)**
- **Self induced transparency (SIT)**
  Najafabadi, et al, accepted PRR.

Photon number distribution for different states:

Coherent state:

$$P_{coh} = \exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}$$

$$(\Delta n)^2_{coh} = \langle n^2 \rangle - \langle n \rangle^2$$
$$= |\alpha|^2 = \langle n \rangle,$$
$$\Delta n = \sqrt{n}.$$

Poissonian distribution

Number(Fock) state:

$$P_{fock} = \delta$$

$$\langle \Delta n \rangle^2_{fock} = 0,$$
$$\Delta n < \sqrt{n}.$$

Sub-Poissonian distribution

Thermal state:

$$P_{th} = \frac{\langle n \rangle^n}{(1+\langle n \rangle)^{n+1}}$$

$$\langle \Delta n \rangle^2_{th} = \langle n \rangle^2 + \langle n \rangle,$$
$$\Delta n > \sqrt{n}.$$

Super-Poissonian distribution

Squeezed state:

Sub-Poissonian for the squeezed quadrature
Super-Poissonian for the not squeezed quadrature

L.Mandel, PRL, (1982)

Measuring light statistics

Classical second order intensity correlation function

$$g^2_{Class}(0) = \frac{\langle I(t)I(t+\tau)\rangle}{(\langle I(t)\rangle)^2} = \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle^2}$$

Using the second quantization

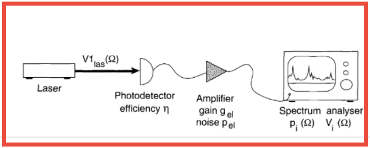
$$\hat{E}_k(t) = E_k^+(t) + \hat{E}_k^-(t) \text{ with } \hat{E}_k^+ \propto \hat{a}_k.\exp(-i(\omega_k t - \vec{k}\cdot\vec{r})),\ \hat{E}_k^- \propto \hat{a}_k^\dagger.\exp(i(\omega_k t - \vec{k}\cdot\vec{r}))$$

Quantum second order correlation function

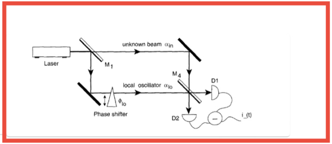
$$g^2_{Qm} = \frac{\langle \hat{E}_k^{(-)}(t)\hat{E}_k^{(-)}(t+\tau)\hat{E}_k^{(+)}(t+\tau)\hat{E}_k^{(+)}(t)\rangle}{\langle \hat{E}_k^{(-)}(t)\hat{E}_k^{(+)}(t)\rangle^2} = \frac{\langle \hat{a}_k^\dagger\hat{a}_k^\dagger\hat{a}_k\hat{a}_k\rangle}{\langle \hat{a}_k^\dagger\hat{a}_k\rangle^2} = \frac{\langle \hat{n}(\hat{n}-1)\rangle}{\langle \hat{n}\rangle^2},$$

$$g^2_{Qm} = \frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2} = \frac{(\Delta n)^2 + \langle n\rangle^2 - \langle n\rangle}{\langle n\rangle^2} = 1 + \frac{(\Delta n)^2 - \langle n\rangle}{\langle n\rangle^2}$$

$$g^{(2)}_{coh}(0) = 1,\quad g^{(2)}_{th}(0) = 2,\quad g^{(2)}(0)_{sq} = 3 + \frac{1}{\langle \hat{n}\rangle}$$



Direct detection



Homodyne detection

Measure the 2nd order correlation function directly

Reconstruct the Wigner function

Calculate the 2nd order correlation function from the Wigner distribution

Basis for the Wigner distribution

$$\hat{x}, \hat{p} \rightarrow [\hat{x}, \hat{p}] = i,$$
$$\hat{U}(x) = \exp(-ix\hat{p}), \rightarrow \hat{U}\hat{U}^{\dagger} = 1,$$
$$\hat{V}(p) = \exp(-ip\hat{x}), \rightarrow \hat{V}\hat{V}^{\dagger} = 1,$$
$$\hat{U}(x^{'})|x\rangle = |x + x^{'}\rangle,$$
$$\hat{V}(p^{'})|p\rangle = |p + p^{'}\rangle,$$

The commutation relation in the Weyleform

$$\hat{V}(p)\hat{U}(x) = e^{ixp}\hat{U}(x)\hat{V}(p),$$

A general Displacement operator in terms of Weyle form:

$$\hat{D}(x,p) = \hat{U}(p)\hat{V}(x)e^{ixp/2} = exp[i(p\hat{x} - x\hat{p})].$$

Stratonovich-Weyle quantizer $\hat{w}(x,p)$

$$\hat{w}(x,p) = \frac{1}{(2\pi)^2}\int_{R^2} -i(px^{'} - xp^{'})\hat{D}(x^{'},p^{'})dx^{'}dp^{'}.$$

Wigner-weyl map

Consider $\hat{A}$ to be an operator in Hilbert space:

$$a(x,p) = \mathrm{Tr}[\hat{A}\hat{w}(x,p)],$$
$$\hat{A} = \frac{1}{(2\pi)^2}\int_{R^2} a(x,p) w(\hat{x,p}) dxdp.$$

Wigner function:

$$W_\rho(x,p) = \mathrm{Tr}[\hat{\rho}\hat{w}(x,p)],$$
$$\hat{\rho} = \frac{1}{(2\pi)^2}\int \hat{w}(x,p) W_\rho(x,p).$$

Second order Correlation function in Wigner representation

For a single mode field, the $\mathbf{g^2(0)} = \frac{\langle \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}} \hat{\mathbf{a}} \rangle}{\langle \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}} \rangle^2} = \frac{\langle \hat{\mathbf{n}}(\hat{\mathbf{n}} - \mathbf{1}) \rangle}{\langle \hat{\mathbf{n}} \rangle^2}$, where

$$\text{with } [\hat{a}, \hat{a}^\dagger] = 1, \quad \hat{n} = \hat{a}^\dagger \hat{a}$$
$$\hat{n}_W = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) = \hat{n} + \frac{1}{2}, \quad \hat{n}_W^2 = \hat{n}^2 + \hat{n} + \frac{1}{2}$$

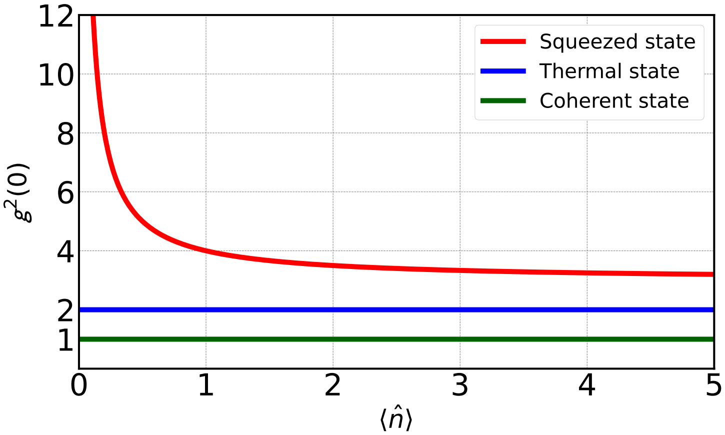
$$\hat{\mathbf{n}}^\mathbf{2} = \hat{\mathbf{n}}_\mathbf{W}^\mathbf{2} - \hat{\mathbf{n}}_\mathbf{W}$$

g^2 **based on the symmertic ordering photon number**:

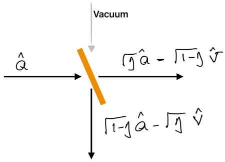
$$\mathbf{g^{(2)}(0)} = \frac{\langle \hat{\mathbf{n}}_\mathbf{W}^\mathbf{2} \rangle - \mathbf{2} \langle \hat{\mathbf{n}}_\mathbf{W} \rangle + \frac{\mathbf{1}}{\mathbf{2}}}{(\hat{\mathbf{n}}_\mathbf{W} - \frac{\mathbf{1}}{\mathbf{2}})^\mathbf{2}} \text{ where:}$$
$$\hat{n} = \hat{x}^2 + \hat{p}^2$$
$$\langle \hat{\mathbf{n}}_\mathbf{W} \rangle = \frac{\mathbf{1}}{\mathbf{2}} \int (\hat{\mathbf{x}}^\mathbf{2} + \hat{\mathbf{p}}^\mathbf{2}) \mathbf{W}_\rho(\mathbf{x}, \mathbf{p}) \mathbf{dxdp},$$

$$\langle \hat{\mathbf{n}}_\mathbf{W}^\mathbf{2} \rangle = \frac{\mathbf{1}}{\mathbf{4}} \int (\hat{\mathbf{x}}^\mathbf{2} + \hat{\mathbf{p}}^\mathbf{2})^\mathbf{2} \mathbf{W}_\rho(\mathbf{x}, \mathbf{p}) \mathbf{dxdp}.$$

Frame Title


12
10
8
6
4
2
1
0
1
2
3
4
5
$g^2(0)$
$\langle \hat{n} \rangle$
Squeezed state
Thermal state
Coherent state

Does $g^(2)(0)$ changes under attenuation?



Field passing through BS with the linear loss.

$$\hat{a}_{out} = \sqrt{\eta}\hat{a}_{in} - \sqrt{1-\eta}\mathrm{v}$$
$$\hat{a}^{\dagger}_{out} = \sqrt{\eta}\hat{a}^{\dagger}_{in} - \sqrt{1-\eta}\mathrm{v}$$

$$\hat{a}^{\dagger}\hat{a} = \eta\hat{a}^{\dagger}\hat{a} - \sqrt{\eta(1-\eta)}\hat{a}^{\dagger}\mathrm{v} - \sqrt{\eta(1-\eta)}\hat{a}^{\dagger}\mathrm{v} + (1-\eta)\mathrm{v}^{\dagger}\mathrm{v}$$

Taking the expectation value $\langle\psi, 0|...|\psi, 0\rangle$

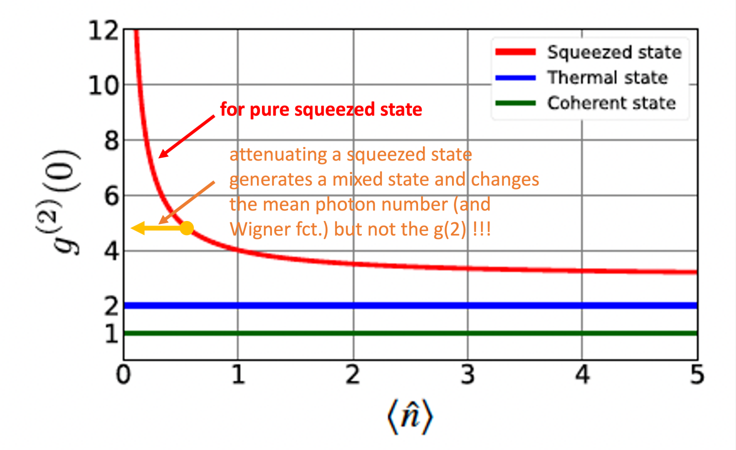
$$\langle\hat{a}^{\dagger}\hat{a}\rangle = \eta\langle\hat{a}^{\dagger}\hat{a}\rangle$$

Similarly

$$\langle\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\rangle = \cdots = \eta^2\hat{a}^{\dagger 2}\hat{a}^2$$

Thus, $g^2(0)$ is independent of losses!

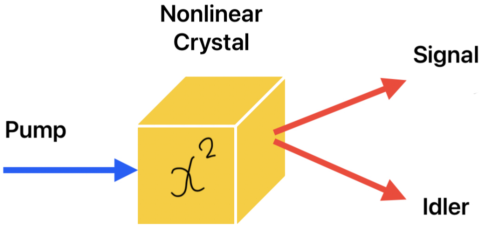
g^2 for Coherent, thermal and Squeezed state



Attenuating the Squeezed field does not change the $g^2(0)$

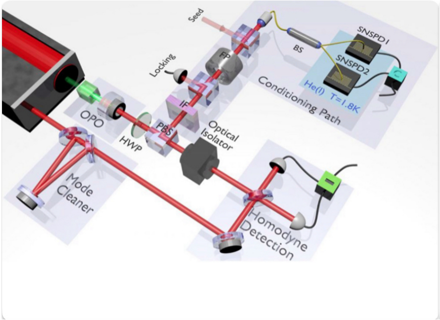
Spontaneous parametric down Conversion (SPSD)

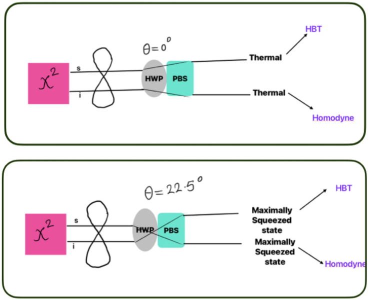
Spontaneous parametric down conversion (SPDC)



Ep = Ei+Es

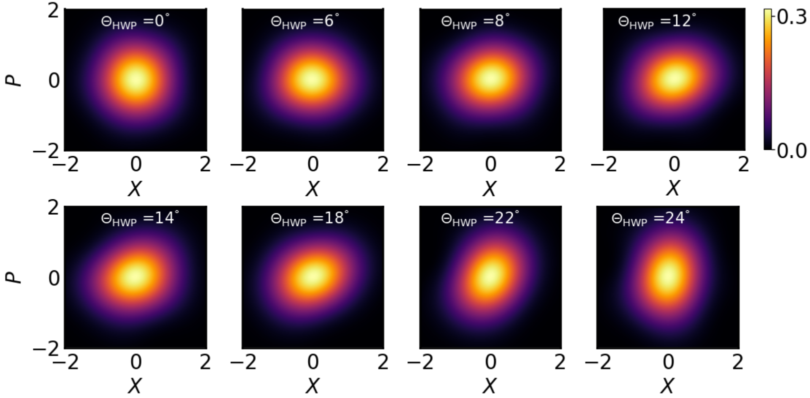
Kp = Ks+Ki





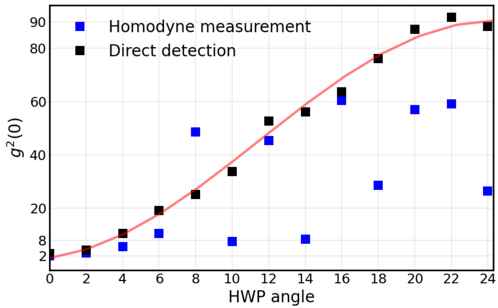
Hanna Le Jeannic, et al , PRL **120**(073603)(2018).

Wigner function gained from Homodyne detection



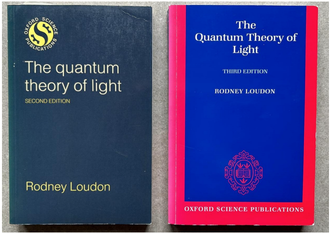
Wigner functions for several HWP angles showing the transition from a thermal state corresponding to the angle 0° to a squeezed vacuum state corresponding to 22.5° for two different laser power.

$g^{(2)}(0)$ gained from both direct and Homodyne detection



Values of $g^{(2)}(0)$ as a function of the angle of the wave plate. The results obtained from both direct photon counting and via the Wigner function reconstructed from Homodyne detection.

Thank you for your attention



It is our pleasure to dedicate this work to Rodney Loudon, who will be remembered as a pioneer of quantum optics.

Submitted to the Philosophical Transactions of the Royal Society A.
Also check out K. Laiho, T. Dirmeier, et al , PLA, 435, 12805, (2022)