Joint ICTP-IAEA Fusion Energy School, 2024

International Centre for Theoretical Physics, Trieste, Italy

INTERACTING NONLINEAR WAVES

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THE MAIN PURPOSE IS TO PRESENT TO YOU SOME NEW AND DIFFERENT FORMS OF NONLINEAR INTERACTIONS OF PLASMA WAVES WITH THEIR SURROUNDING MEDIA

SUMMARY

Monday 6th

- New nonlinear solutions of the old wave equation
- Accelerating self-modulated nonlinear waves in magnetized plasmas

Tuesday 7th

- Can we surf a gravitational wave?
- Amplification of electromagnetic plasma waves due to gravitational wave
- Amplification of electromagnetic plasma waves due to cosmological gravitational waves

Wednesday 8th

- Interacting quantum and classical waves: Resonant and non-resonant energy transfer to electrons immersed in an intense electromagnetic wave.
- Statistical model for relativistic quantum fluids interacting with an intense electromagnetic wave.

THE OLD AND RELIABLE WAVE EQUATION

Nonlinear solutions have interesting new effects

RELATIVISTIC PLASMA EQUATIONS

When the velocity of the plasma components are close to the speed of light, the plasma is considered to be relativistic. Also, the temperature can be very high. We can describe a relativistic plasma using a fluid formalism, when the main body of the plasma behave as a whole. Thus, each plasma fluid component j is characterized by its rest-frame density n_j, the energy density ϵ_j, pressure p_j, enthalpy density $h_j = \epsilon_j + p_j$ and temperature T_j.

The equations for the relativistic plasma fluids are the continuity equation, with the relativistic vectorial velocities $\boldsymbol{v}_j$ and the Lorentz factor $\gamma = {}^1/_{\sqrt{1-v^2}}$

$$\frac{\partial(\gamma_j n_j)}{\partial t} + \nabla \cdot (\gamma_j n_j \mathbf{v}_j) = 0$$

The momentum equation for the plasma fluid, where m_j is the mass of the plasma constituents, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, considering an equation of state to link the pressure with the plasma density.

$$m_j \gamma_j \left(\frac{\partial}{\partial t} + \mathbf{v}_j \cdot \nabla \right) (f_j \gamma_j \mathbf{v}_j) = q_j \gamma_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \frac{1}{n_j} \nabla p_j$$

$$f \equiv \frac{h}{mn} = f(T)$$

Lastly, we need to couple the plasma dynamics to Maxwel equations for electromagetism

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\
\frac{\partial \mathbf{E}}{\partial t} + \sum_i q_i n_i \gamma_i \mathbf{v}_i &= \nabla \times \mathbf{B} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \cdot \mathbf{E} &= \sum_i q_i n_i \gamma_i
\end{aligned}$$

From here we can obtain a wave equation for plasma waves. But let me show another more elegant way

MAGNETOFLUID - COVARIANT FORM

Instead of using the vectorial form for relativistic plasmas, it is better to work with a covariant formalism, because, in that case, equations are manifestly invariant under Lorentz transformations.

For flat–spacetimes with $\eta_{\mu\nu} = (-1,1,1,1)$, the plasma fluid 4-velocity $U^\mu = (\gamma, \gamma\mathbf{v})$ satisfies $U_\mu U^\mu = \eta_{\mu\nu} U^\mu U^\nu = -1$.

In this way, the momentum equation for the plasma fluid can be written as

$$U^\nu \partial_\nu \left(m f U^\mu \right) = q F^{\mu\nu} U_\nu - \frac{1}{n} \partial^\mu p$$

Also, the continuity equation becomes

$$\partial_\mu (n U^\mu) = 0$$

and finally, Maxwell equations acquire the form

$$\partial_\nu F^{\mu\nu} = q n U^\mu$$

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$$\partial_\nu F^{\mu\nu} = q n U^\mu$$

$$U_\nu M^{\mu\nu} = -\frac{T}{q} \partial^\mu \sigma$$

$$M^{\mu\nu} = F^{\mu\nu} + \frac{m}{q} S^{\mu\nu}$$

$$S^{\mu\nu} = \partial^\mu (f U^\nu) - \partial^\nu (f U^\mu)$$

$$\partial^\mu \sigma = -\frac{1}{nT} (\partial^\mu p - mn\partial^\mu f)$$

$$U_\mu \partial^\mu \sigma = 0$$

MAGNETOFLUID – WAVE EQUATION

Let us assume a isentropic fluid with

$$\partial^\mu \sigma = 0$$

The simplest solution is then

$$U_\nu M^{\mu\nu} = 0 \Rightarrow M^{\mu\nu} = 0 \Rightarrow qF^{\mu\nu} = -mS^{\mu\nu}$$

$$qA^\mu = -mfU^\mu$$

Using this in Maxwell equation

$$\partial_\nu F^{\mu\nu} = qnU^\mu \Rightarrow \partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu = -\frac{q^2 n}{mf} A^\mu$$

From where we obtain the covariant form of the wave equation

$$\partial_\nu \partial^\nu A^\mu - \frac{\omega_p^{\ 2}}{f} A^\mu = 0$$

With the Gauge $\partial_\nu A^\nu = 0$

MAGNETOFLUID – WAVE EQUATION

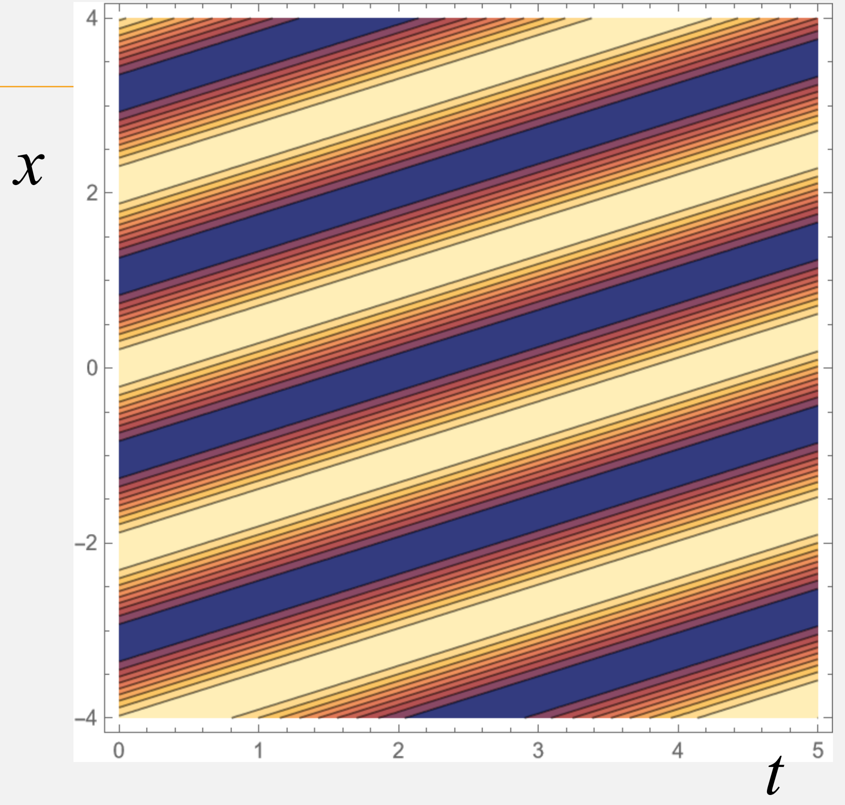
We can write the wave equation as (remembering that there is a Gauge involved)

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\vec{A}-\frac{\omega_p{}^2}{f}\vec{A}=0$$

Linear waves are obtained when $$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)\vec{A}-\frac{\omega_p{}^2}{f}\vec{A}=0$$

$$\vec{A}=\hat{e}\;A\exp(ikx-i\omega t)$$

$$\omega^2=k^2+\frac{\omega_p{}^2}{f}$$



But there are other nonlinear interesting solutions!

SOLUTION I

Consider a 2+1 wavepacket satisfying the gauge

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\vec{A}-\frac{\omega_p{}^2}{f}\vec{A}=0$$

$$\vec{A}(x,y,t)=\hat{z}\,A(x,y,t)$$

$$A(x,y,t)=\zeta(\eta,y)\exp\left(i\alpha\xi-i\frac{\omega_p{}^2}{4\alpha f}\eta\right)$$

Solution with accelerating properties

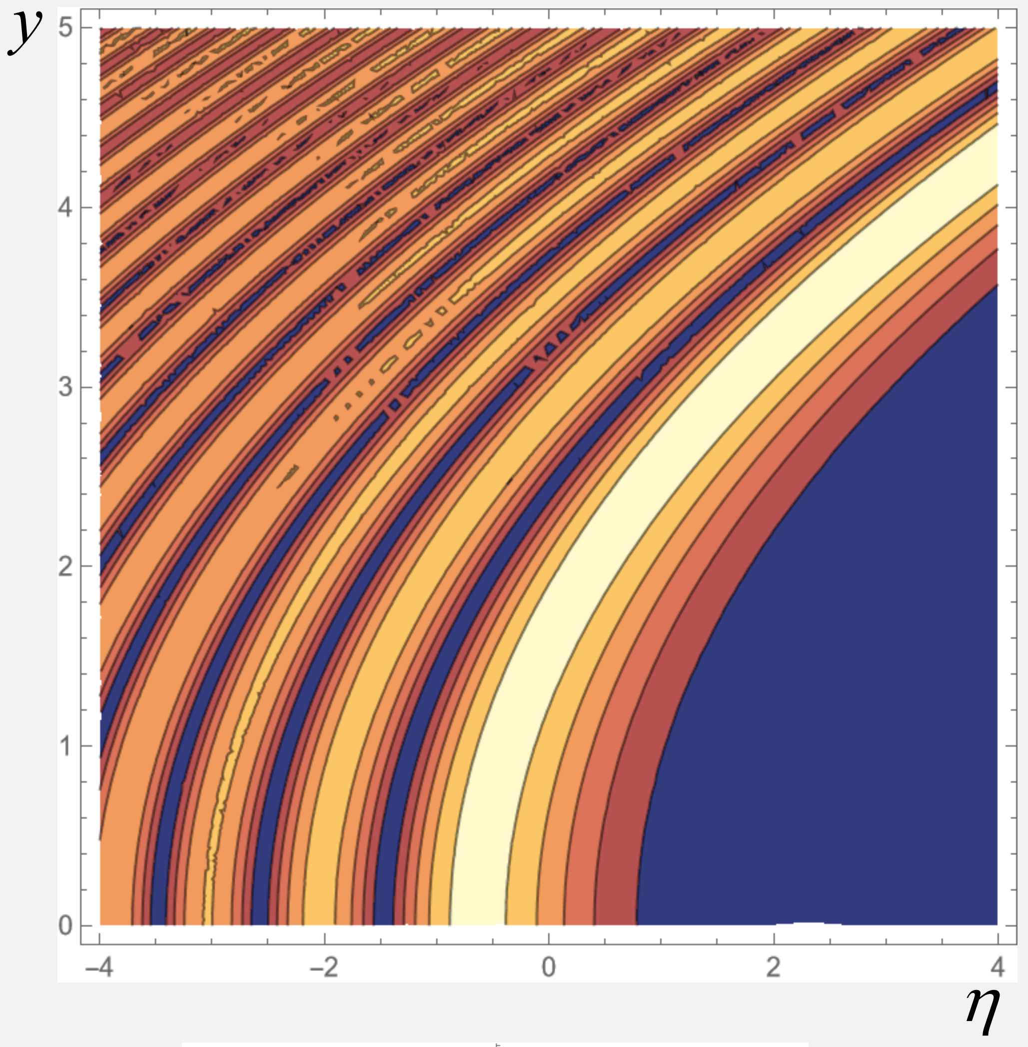
Winkler, Vásquez-Wilson & Asenjo,
Eur. Phys. J. D (2023).

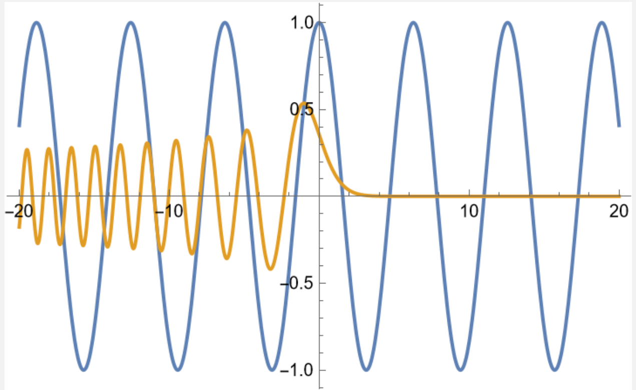
$$\eta=x+t;\quad \xi=x-t$$

$$4i\alpha\frac{\partial\zeta}{\partial\eta}+\frac{\partial^2\zeta}{\partial y^2}=0$$

A Schrödinger-like equation!

$$\zeta(\eta,y)=\mathrm{Ai}\left(2(a\alpha)^{1/3}\left(y-iv\eta-\frac{a}{2}\eta^2\right)\right)\times$$
$$\exp\left(2i\alpha at\left(y-\frac{a}{3}\eta^2\right)+2v\left(-y+a\eta^2+\frac{i}{2}\eta v\right)\right)$$





Cosine (blue)
Airy (yellow)

SOLUTION II

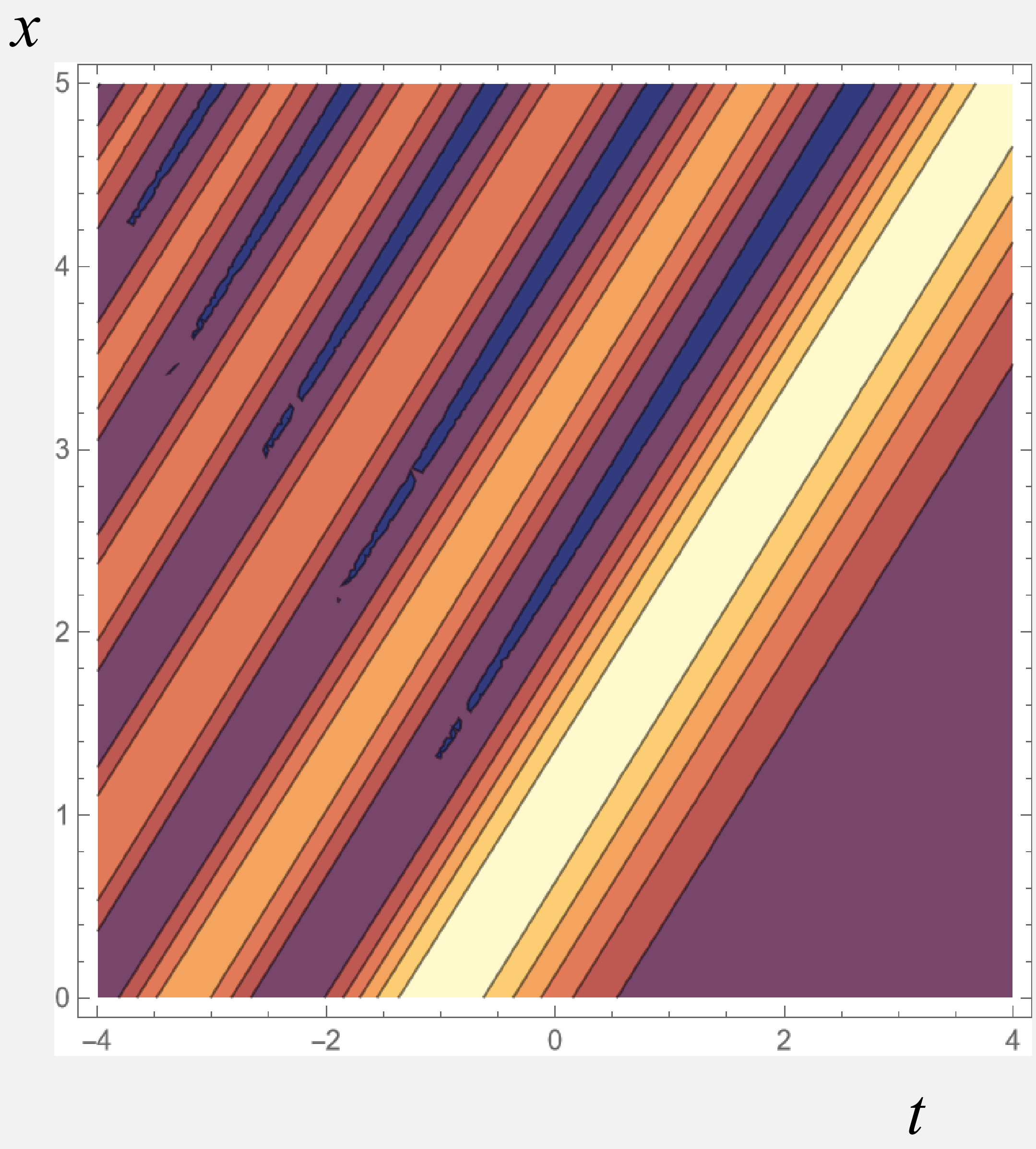
Consider a 1+1 wavepacket satisfying the gauge

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)\vec{A}-\frac{\omega_p{}^2}{f}\vec{A}=0$$

$$\vec{A}(x,t)=\hat{z}\,A(x,t)$$

$$A(x,t)=\mathrm{Ai}\left(\frac{\omega_p{}^2}{4\alpha^2 f}\xi+\alpha\sqrt{\eta}\right)\mathrm{Ai}\left(\frac{\omega_p{}^2}{4\alpha^2 f}\xi-\alpha\sqrt{\eta}\right)$$

$$\eta=x+t;\quad \xi=x-t$$



Solution with propagation different to a plane wave

Asenjo & Mahajan, in preparation (2024).

SOLUTION III: NONLINEAR WAVE EQUATION

Nonlinear self-modulation of a circularly polarized electromagnetic or Alfvén wave in a weakly or strongly magnetized relativistic electron–positron plasma with finite temperature

$$i\frac{\partial a}{\partial t}+P\frac{\partial^2 a}{\partial z^2}+Q|a|^2 a=0$$

Nonlinear Schrödinger equation

$$P=c^2/(2\omega);$$

$$Q=\frac{3\lambda\omega_p{}^2\Omega_C{}^2}{\omega^3 f^5};$$

Weakly magnetized; Asenjo, Borotto, Chian, Muñoz & Valdivia, *Phys. Rev.* E (2012).

$$Q=\frac{f\lambda\omega_p{}^2\omega^3}{4\Omega_C{}^4}$$

Strongly magnetized; López, Asenjo, Muñoz, Chian & Valdivia, *Phys. Rev.* E (2012).

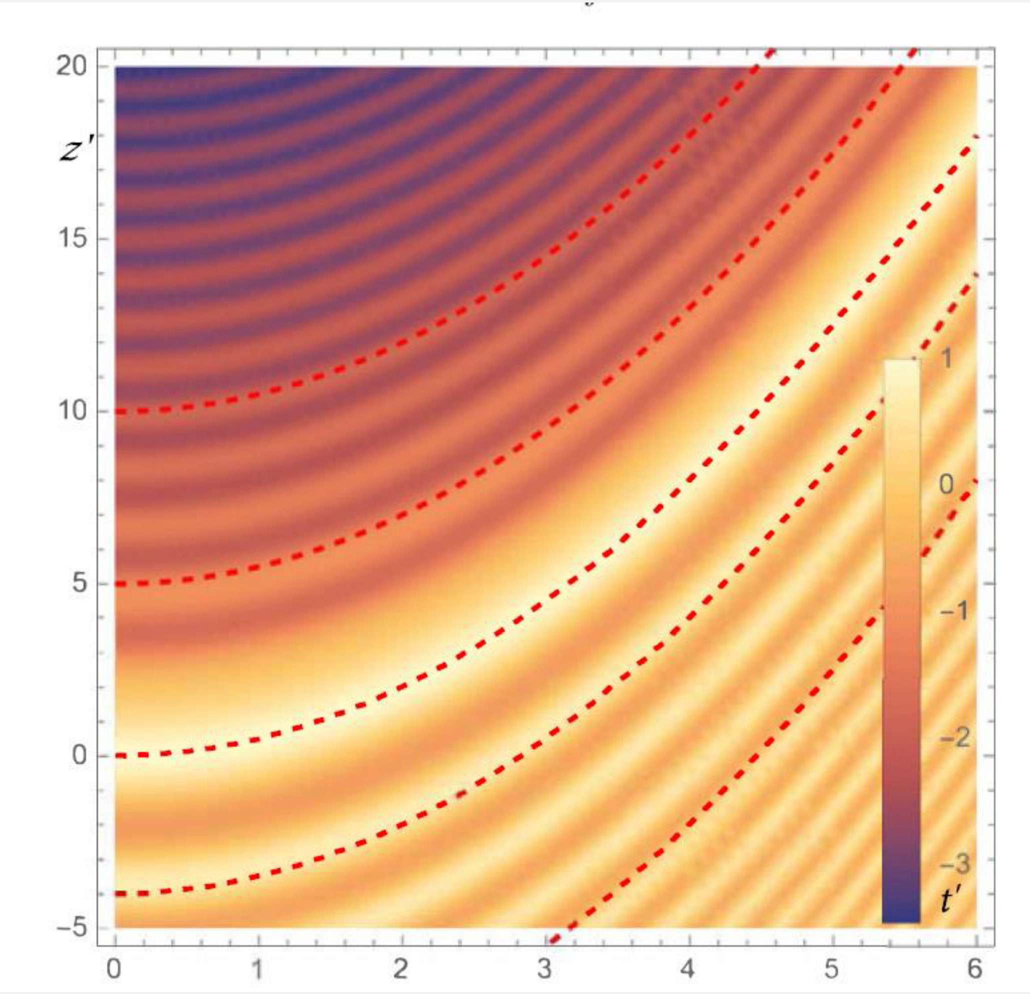
Non-accelerating soliton (well-known) solution

$$a(t,z)=\operatorname{sech}\left(\sqrt{\frac{Q}{2P}}(z-vt)\right)\exp\left(\mathrm{i}\frac{v}{2P}z-\mathrm{i}\left(\frac{v^2}{4P}-\frac{Q}{2}\right)t\right).$$

SOLUTION III: NONLINEAR WAVE EQUATION

$$i\frac{\partial a}{\partial t}+P\frac{\partial^2 a}{\partial z^2}+Q|a|^2 a=0$$

Nonlinear Schrödinger equation



New Accelerating nonlinear wavepacket solution;

Asenjo, *J. Plasma Phys.* (2024),

$$a(t,z)=f(\xi)\exp(i\eta(t,z))$$

$$\xi=\sqrt{\frac{Q}{2P}}\left(z-vt-\sqrt{\frac{Q^3P}{8}}t^2\right)$$

$$\eta(t,z)=\frac{Q}{2}\sqrt{\frac{Q}{2P}}\left(tz+\frac{v}{Q}\sqrt{\frac{2}{QP}}z-\frac{v^2}{Q^2}\sqrt{\frac{Q}{2P}}t-vt^2-\frac{Q}{3}\sqrt{\frac{QP}{2}}t^3\right),$$

Painlevé equation

$$\frac{d^2 f}{d\xi^2}-\xi f+2f^3=0$$

CONCLUSIONS

- Linear and nonlinear plasma wave equations have interesting nonlinear effects in the propagation of wavepackets.
- These are accelerations (in space, time, or spacetime), implying that the máximum intensity of plasma wavepacket can, for example, follow curved trajectories.
- Tomorrow, we will see how other kind of nonlinear effects appears due to nonlinear interacctions of plasma and the surrounding media.