Joint ICTP-IAEA Fusion Energy School, 2024
International Centre for Theoretical Physics, Trieste, Italy

INTERACTING NONLINEAR WAVES

MAY 8TH, 2024

Felipe Asenjo
Universidad Adolfo Ibáñez
felipe.asenjo@uai.cl

SUMMARY

Monday 6th

- New nonlinear solutions of the old wave equation
- Accelerating self-modulated nonlinear waves in magnetized plasmas

Tuesday 7th

- Can we surf a gravitational wave?
- Amplification of electromagnetic plasma waves due to gravitational wave
- Amplification of electromagnetic plasma waves due to cosmological gravitational waves

Wednesday 8th

- Interacting quantum and classical waves: Resonant and non-resonant energy transfer to electrons immersed in an intense electromagnetic wave.
- Statistical model for relativistic quantum fluids interacting with an intense electromagnetic wave.

RESONANCE OF QUANTUM AND RELATIVISTIC QUANTUM ELECTRONS WITH INTENSE EM WAVES

Asenjo & Mahajan, Phys. Plasmas 29, 022107 (2022)

We develop a systematic theory of energy exchange between a quantum and a classical wave, demonstrating when the energy transfer may be efficient.

We concentrate on investigating the dynamics of the quantum and relativistic electron wave (EW) in the field of a classical circularly polarized electromagnetic (CPEM) wave (with and without an ambient magnetic field).

These calculations are stimulated by the recent experimental work in photon-induced near-field electron microscopy (PINEM), demonstrating the resonant phase-matching exchange between photons and electrons.

CPEM wave

For analytic simplicity, the EM wave is assumed to be circularly polarized (CPEM). The contravariant components of electromagnetic four potential A^{μ} of a CPEM wave propagating in the z-direction are (the Minkowski signature tensor $\eta^{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$)

$$A^0 = 0 = A^z, \quad A^x = A\cos(\omega t - kz), \quad A^y = -A\sin(\omega t - kz), \tag{1}$$

where A is its constant amplitude and the four-wave vector in the lab frame is $k^{\mu} = [\omega, 0, 0, k]$. Notice that $k^{\mu}A_{\mu} = 0 = \mathbf{k}\cdot\mathbf{A}$, and the Lorentz invariant $A^{\mu}A_{\mu} = A^2$ has no space time dependence;

Non-relativistic quantum electron: Schrödinger equation

$$i\frac{\partial\Psi}{\partial t}=\frac{1}{2m}(\mathbf{p}+q\mathbf{A})^2\Psi,\quad \mathbf{p}=-i\nabla,$$

$$-2im\frac{\partial\Psi}{\partial t}-(\nabla_\perp^2+\partial_z^2)\psi-2iqA[\cos(\omega t-kz)\partial_x\Psi$$
$$-\sin(\omega t-kz)\partial_y\Psi]+q^2A^2\Psi=0,$$

$$\Psi(x,y,z,t)=e^{iK_\perp(x\cos\varphi+y\sin\varphi)}\psi(t,z)$$

$$-2im\frac{\partial\psi}{\partial t}-\partial_z^2\psi+2qAK_\perp\cos(\omega t-kz)\psi+(K_\perp^2+q^2A^2)\psi=0,$$

$$\psi(t,z)=\psi(\zeta)$$

The key is the perpendicular momentum

$$-2im\omega\frac{d\psi}{d\zeta}-k^2\frac{d^2\psi}{d^2\zeta}+2qAK_\perp\cos\zeta\psi+(K_\perp^2+q^2A^2)\psi=0.$$

$$-2im\omega \frac{d\psi}{d\zeta} - k^2 \frac{d^2\psi}{d^2\zeta} + 2qAK_\perp \cos\zeta\psi + (K_\perp^2 + q^2A^2)\psi = 0.$$

The second derivative is negligibly small for the NR electron compared to the first derivative (frequency is larger than mass)

solution

$$\psi = \psi_0 \exp\left(-\frac{i}{2m\omega}(\beta\zeta + \alpha\sin\zeta)\right)$$
$$= \psi_0 \sum_n J_n\left(\frac{\alpha}{2m\omega}\right) \exp\left(-i\left(\frac{\beta}{2m\omega} + n\right)\zeta\right),$$

where $\alpha = 2qAK_\perp$ and $\beta = {K_\perp}^2 + q^2A^2$

Leading order electron energy

$$E = i\left\langle \psi^* \frac{\partial\psi}{\partial t} \right\rangle,$$
$$\langle\ \ \rangle = \frac{1}{2L}\int_{-L}^{L} dz,$$

$$E = \frac{\beta}{2m} + \frac{\alpha}{2m}\langle\cos\zeta\rangle \equiv \frac{K_\perp^2 + q^2A^2}{2m} + \frac{2qAK_\perp}{2m}\langle\cos\zeta\rangle = \frac{K_\perp^2 + q^2A^2}{2m} + \frac{2qAK_\perp}{2m}\frac{\sin(kL)}{kL}\cos(\omega t),$$

Non-relativistic quantum electron gain energy because wave-wave interaction through its perpendicular momentum (perpendicular to the EM propagation).

Non-relativistic quantum electron in external magnetic field

$$\mathbf{A}_0 = B_0 x \hat{e}_y.$$

$$-2im\omega\frac{\partial\psi}{\partial\zeta} - \frac{\partial^2\psi}{\partial x^2} + \left(q^2A^2 + \alpha^2x^2 - 2\alpha qAx\sin\zeta - 2iqA\cos\zeta\frac{\partial}{\partial x}\right)\psi = 0,$$

with $\alpha = qB_0 = m\Omega$, where Ω is the gyrofrequency of the particle.

$$\psi = \psi_0 \exp\left(-i\left((l+1/2)\frac{\Omega}{\omega} + \frac{q^2A^2 + K_x^2}{2m\omega}\right)\zeta - i\frac{2qAK_x}{2m\omega}\sin\zeta\right),$$

Leading order solution;
coupling with perpendicular momentum, which is along the vector potential

$$E = \left(l + \frac{1}{2}\right)\Omega + \frac{K_x^2 + q^2A^2}{2m} + \frac{2qAK_x}{2m}\frac{\sin(kL)}{kL}\cos(\omega t);$$

the latter, now, contains the energy corresponding to the Landau level l. To this order, the interference of the CPEM wave and the ambient field does not appear.

Relativistic quantum electron: Klein-Gordon equation

$$(p_\mu + qA_\mu)(p^\mu + qA^\mu)\Psi = (i\partial_\mu + qA_\mu)(i\partial^\mu + qA^\mu)\Psi = m^2\Psi,$$

$$\partial_t^2\Psi - \nabla^2\Psi - (2iqA[\cos(\omega t - kz)\partial_x\Psi - \sin(\omega t - kz)\partial_y\Psi] + \mathcal{M}^2\Psi = 0,$$

where $\mathcal{M} = \sqrt{m^2 + q^2A^2}$ is the field-renormalized effective mass,

Coupling between the CPEM wave and the perpendicular momentum

$$\Psi(x, y, z, t) = e^{iK_\perp(\cos\varphi x + \sin\varphi y)}\psi(t, z)$$

that obeys

$$\partial_t^2\psi - \partial_z^2\psi + 2qAK_\perp\cos(\omega t - kz)\psi + ({K_\perp}^2 + \mathcal{M}^2)\psi = 0,$$

$$\psi(t, z) = \psi(\zeta)$$

$$(\omega^2 - k^2)\frac{d^2\psi}{d\zeta^2} + [\mu + \lambda\cos\zeta]\psi = 0,$$

where $\mu = K_\perp^2 + m^2 + q^2A^2$ and $\lambda = 2qAK_\perp$ $(\lambda/\mu < 1)$.

Mathieu equation!

Solution $K_\perp = 0$

$$\psi = \psi(0) e^{-i\sqrt{\frac{m^2+q^2A^2}{\omega^2-k^2}}(\omega t - kz)}, \qquad \langle E \rangle = \frac{i}{\psi}\frac{\partial \psi}{\partial t} = (m^2 + q^2A^2)^{1/2} \frac{\omega}{\sqrt{\omega^2 - k^2}},$$

Leading Solution

$$\psi = \psi_s(\zeta) e^{-i\frac{\sqrt{\mu}}{\sqrt{\omega^2-k^2}}\zeta} \equiv \psi_s(\zeta) e^{-iS\zeta},$$

where $\psi_s(\zeta)$ represents the slow variation [compared to the eikonal, i.e., $(d\psi_s/d\zeta)/(\psi_s) \ll S = \sqrt{\mu/(\omega^2 - k^2)}$] of the wave function.

$$\frac{1}{\psi_s}\frac{d\psi_s}{d\zeta} = \frac{i\lambda}{2S(\omega^2 - k^2)} \cos \zeta$$

that integrates to

$$\psi_s = e^{i\frac{qAK_\perp}{[\mu(\omega^2-k^2)]^{1/2}}\sin \zeta} = e^{i\alpha \sin \zeta}$$

Energy

$$\begin{aligned}
\langle E \rangle = i\left\langle \Psi^* \frac{\partial \Psi}{\partial t} \right\rangle &= \frac{\omega}{2L}\int_{-L}^{L} (S + \alpha \cos \zeta) dz \\
&= -\frac{\omega}{2kL}\int_{\omega t + kL}^{\omega t - kL} (S + \alpha \cos \zeta) d\zeta \\
&= \omega S + \alpha \omega \cos \omega t \frac{\sin kL}{kL}.
\end{aligned}$$

Rate of change of particle energy; acceleration

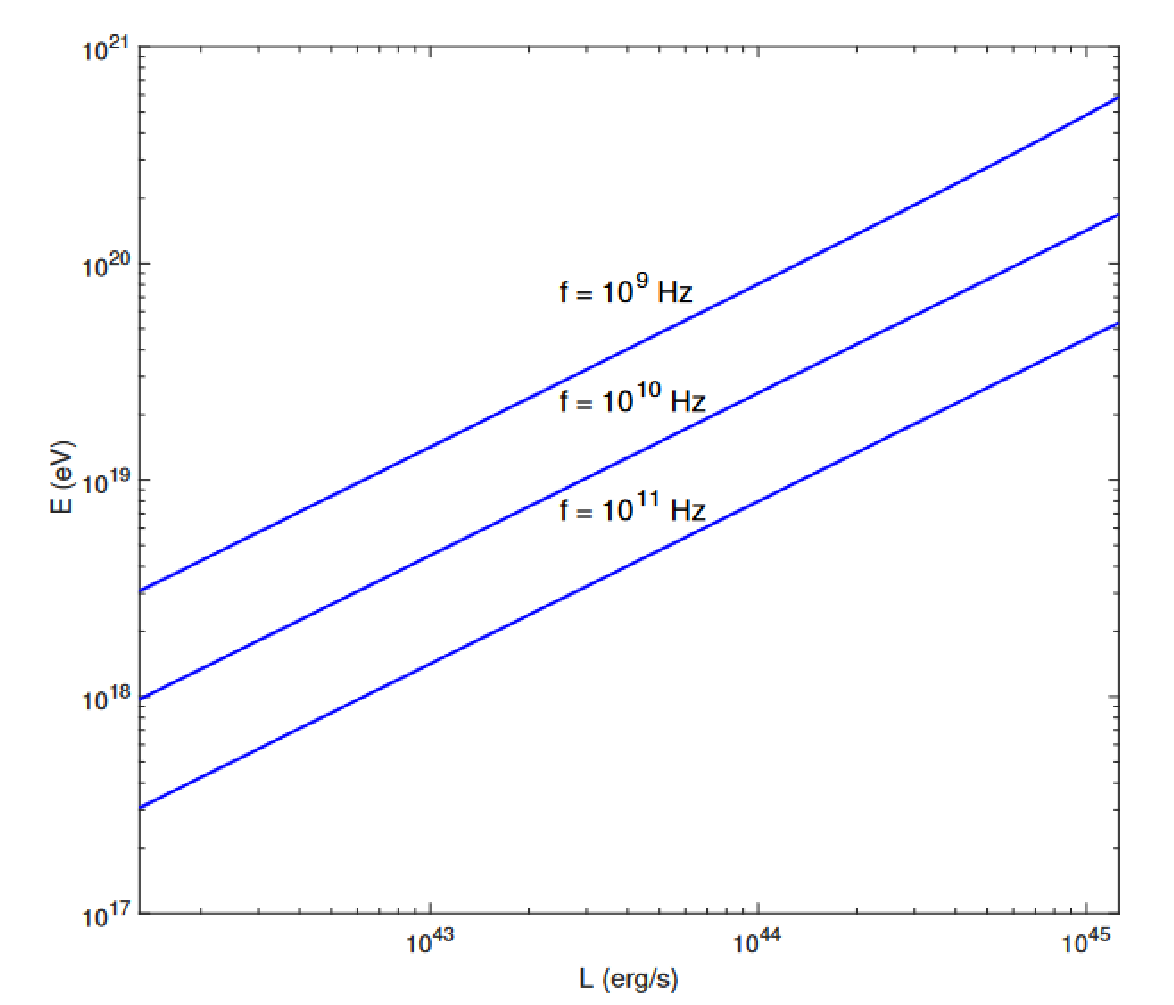
$$\begin{aligned}
\left| \frac{d\langle E \rangle}{dt} \right| &= \left\langle \frac{1}{2\pi}\int_{-\pi}^{\pi} d(\omega t) \left(\frac{d\langle E \rangle}{dt} \right)^2 \right\rangle^{1/2} \\
&= \frac{\omega}{4(\omega^2 - k^2)^{1/2}} \frac{2qAk_\perp}{(m^2 + k_\perp^2 + q^2A^2)^{1/2}} \frac{\sin (kL)}{kL}
\end{aligned}$$

APPLICATION: RESONANT ENERGIZATION OF PARTICLES BY RADIO AGN, MAHAJAN & OSMANOV (A&A, 2022)

They apply the essentials of this theory to the particular case of a plasma in the magnetospheres of a radio AGN that emits copious EM energy in the radio frequency range

They found that the resonant mechanism is dominant under the two most important impeding (cooling) processes: (1) the inverse Compton (IC) scattering of the charged particles with the ambient photon field, and (2) synchrotron radiation when relatively strong magnetic fields are present.

The energy increase with luminosity, up to several hundreds to thousand ExaeV.



NOW WE CAN CONSTRUCT A STATISTICAL MODEL FOR RELATIVISTIC QUANTUM FLUIDS INTERACTING WITH AN INTENSE ELECTROMAGNETIC WAVE

Mahajan & Asenjo, Physics of Plasmas 23, 056301 (2016).

At leading order for relativistic quantum electrons interacting with intense EM wave

$$E = i\int\left(\psi^*\frac{\partial\psi}{\partial t}\right)d\zeta,$$

$$K_z = -i\int\left(\psi^*\frac{\partial\psi}{\partial z}\right)d\zeta.$$

$$E^2 = K_z^2 + \mathcal{M}^2 + (1-\alpha)K_\perp^2 \equiv m^2 + q^2A^2 + K_z^2 + a^2K_\perp^2,$$

$$a = 1-\alpha; \quad \alpha \approx \frac{3q^2A^2}{2(\mathcal{M}^2 + K_\perp^2)}$$

Anisotropic in momentum

Statistical average

In the fluid-rest frame $U^\mu = (1,0,0,0)$

The covariant relativistic M-B distribution is

$$f = Ne^{-\frac{K_\mu U^\mu}{T}},$$

where U^μ is the bulk four velocity of the K-G fluid, and $K^\mu = (E, aK_x, aK_y, K_z)$ is the microscopic particle energy-momentum four vector obeying $K^\mu K_\mu = \mathcal{M}^2$; the new invariant mass of the particle is the field renormalized mass $\mathcal{M}$.

$$f_R = Ne^{-\frac{E}{T}} = N\exp\left[-\frac{\sqrt{K_z^2 + \mathcal{M}^2 + a^2K_\perp^2}}{T}\right].$$

Distribution in the rest frame

Flux tensor

$$\Gamma^{\mu} = \int \frac{d^3K}{E} K^{\mu} f_R ,$$

Number density in rest-frame

$$n_R \equiv \Gamma^0 = \frac{N\mathcal{M}^3}{a^2} \int d^3p \exp\left(-\zeta\sqrt{1+p^2}\right), \qquad (41)$$

where $\zeta = \mathcal{M}/T$, and we have defined the adjusted momentum variables $p_z = K_z/\mathcal{M}$, $p_x = aK_x/\mathcal{M}$, $p_y = aK_y/\mathcal{M}$. In the p variables ($p_0 = E/\mathcal{M}$), $p^{\mu}p_{\mu} = 1$.

Energy-momentum tensor in the rest-frame

$$T_R^{\mu\nu} = \int \frac{d^3K}{E} K^{\mu} K^{\nu} f_R .$$

Energy density

$$\begin{aligned} \mathcal{E} = T_R^{00} &= \int d^3K E f \\ &= \frac{n_R \zeta \mathcal{M}}{K_2(\zeta)} \int d^3p \sqrt{1+p^2} \exp\left(-\zeta\sqrt{1+p^2}\right) \\ &= \frac{n_R \zeta \mathcal{M}}{K_2(\zeta)} \frac{d^2}{d\zeta^2}\left(\frac{K_1(\zeta)}{\zeta}\right), \end{aligned}$$

parallel pressure ($P_{\parallel} = P_z = T_R^{zz}$)

$$\begin{aligned} P_{\parallel} = \int \frac{d^3K}{E} K^z K^z f = \frac{n_R \zeta \mathcal{M}}{K_2(\zeta)} \int_0^{\infty} dp p^2 \\ \times \frac{p_z^2}{\sqrt{1+p^2}} \exp\left(-\zeta\sqrt{1+p^2}\right) = nT, \end{aligned}$$

the perpendicular pressure ($P_{\perp} = T_R^{xx} = T_R^{yy}$)

$$P_{\perp} = \frac{n_R T}{a^2} = \frac{P_{\parallel}}{a^2} .$$

Now we are ready to construct the general energy-momentum tensor for the Klein-Gordon fluid in the CPEM

$$T^{\mu\nu} = -\eta^{\mu\nu} P_{\|} + \beta^{\mu\nu}(P_{\|} - P_{\perp}) + (\epsilon + P_{\|}) U^{\mu} U^{\nu} ,$$

$$\beta^{\mu\nu} = \frac{F^{\mu\kappa} F_{\kappa}{}^{\nu}}{F^{\alpha\beta} F_{\alpha\beta}} + \frac{k^{\mu} k^{\nu}}{k^{\alpha} k_{\alpha}} - \frac{A^{\mu} A^{\nu}}{A^{\alpha} A_{\alpha}}$$

In the rest-frame, it is reduced to the previously calculated

$$\beta^{00} = 0 = \beta^{33} \text{ and } \beta^{11} = -1 = \beta^{22}.$$

Lastly, Klein-Gordon current and closure with Maxwell's equations

We can take the microscopic particle current derived from Klein-Gordon equation, and averaging over distribution function

$$\langle J_0 \rangle = \frac{n_R}{\mathcal{M}},$$

$$\langle \mathbf{J}_\perp \rangle = \frac{q n_R K_1(\zeta)}{\mathcal{M} K_2(\zeta)} \mathbf{A}_\perp .$$

$$\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - \nabla^2 \mathbf{A}_\perp = -\frac{4\pi q^2 n_R K_1(\zeta)}{\mathcal{M} K_2(\zeta)} \mathbf{A}_\perp ,$$

Yields the dispersion relation with new transparency effects

$$\omega^2 - k^2 = \frac{{\omega_p}^2}{\Gamma_f \Gamma_{th}},$$

$$\Gamma_f = \mathcal{M} / m = \sqrt{1 + q^2 A^2 / m^2}$$

$$\Gamma_{th} = K_2(\zeta) / K_1(\zeta)$$

$$\zeta = \mathcal{M} / T$$

The most simple effects of this relativistic quantum plasma in intense EM waves is propagation of waves at lower frequencies in plasma

CONCLUSIONS

- Relativistic quantum particles can be energized by intense EM waves. A perpendicular momentum is needed.

- A plasma fluid theory, formed by relativistic quantum particles interacting with intense EM waves, can be constructed.

- The fluid plasma presents the nonlinear effects of its constituents, and it can be seen through modifications to the dispersión relation of EM waves.

- We can do something similar to Dirac electrons. In that case, the energy positive and negative solutions, and spin up and down, couple to the EM field.