SHORT INTRODUCTION TO GYROKINETICS

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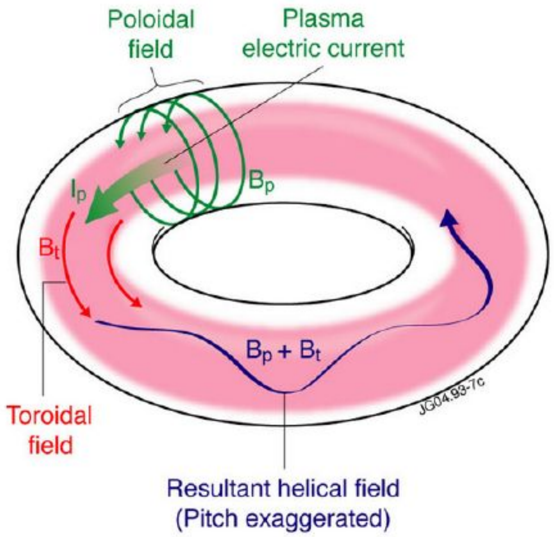
Gyrokinetics is a major success of theoretical and computational plasma physics. It has a long story, from the ‘70s and has now reached the point where we can use it to predict plasma profiles in fusion experiments.

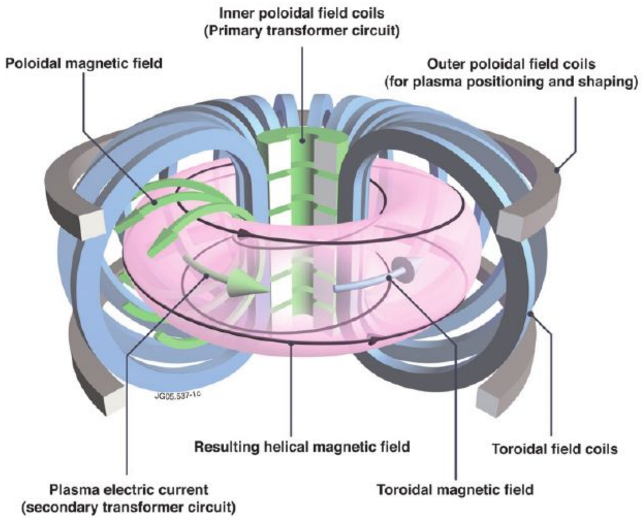
This is a short introduction to it, not complete and not fully rigorous, but there’s a lot of available material.

The following slides profited of inspiring lectures and notes of A. Brizard, G. Hammet, F. Jenko, T.S. Hahm, J.Citrin and many others

TOKAMAK

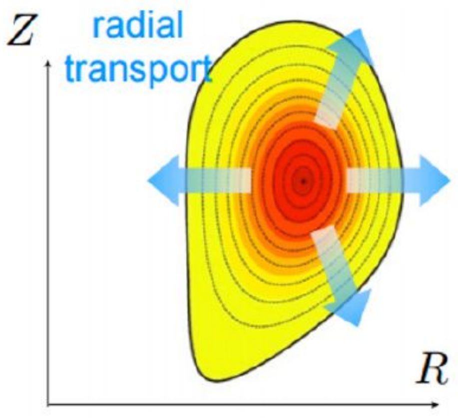
The Tokamak is the leading configuration for magnetic confinement fusion





Plasma confined with helical field and heated with NBI and RF waves.
Energy is lost because of both large scale (MHD) instabilities and small scale fluctuations, turbulence

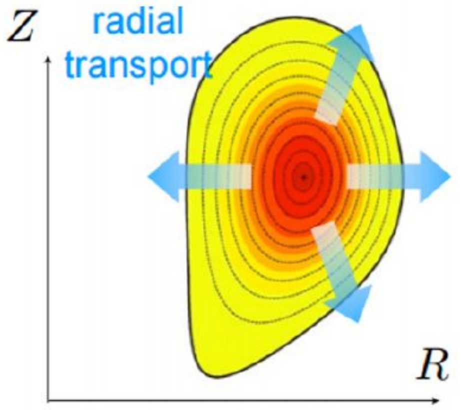
TOKAMAK TRANSPORT



Higher pressure in core would give more fusion power, but effectively this is limited.

We are interested in the radial transport of heat particles and momentum.

TOKAMAK TRANSPORT



$$\chi_{i,e_{neo}} \approx \frac{q^2}{\epsilon^{3/2}} \frac{T_{i,e}}{m_{i,e}\Omega_{i,e}^2 \tau_{i,e}}$$

$$\chi_{i_{neo}} \approx 0.3\, m^2/s$$
$$\chi_{e_{neo}} \approx 0.005\, m^2/s$$

Higher pressure in core would give more fusion power, but effectively this is limited.

We are interested in the radial transport of heat particles and momentum.

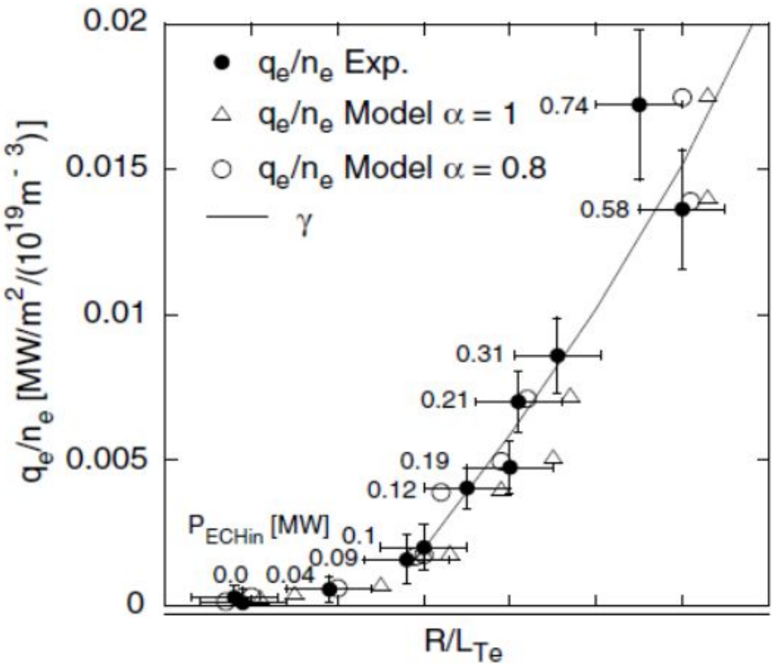
The toroidal geometry gives rise to *neoclassical transport* measured transport is however much higher.

Caused by small-scale collective instabilities driven by gradients in temperature, density, …

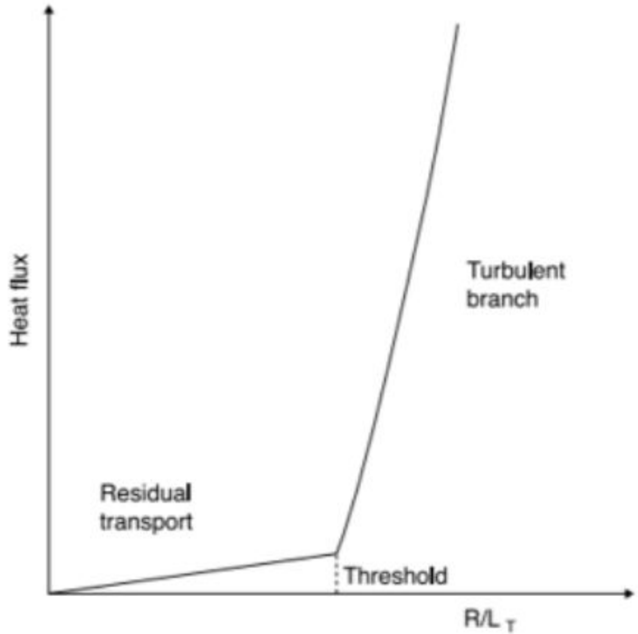
Instabilities saturate at low amplitude due to nonlinear mechanisms

Particles E x B drift radially due to fluctuating electric field

EVIDENCE OF ANOMALOUS STIFF TRANSPORT



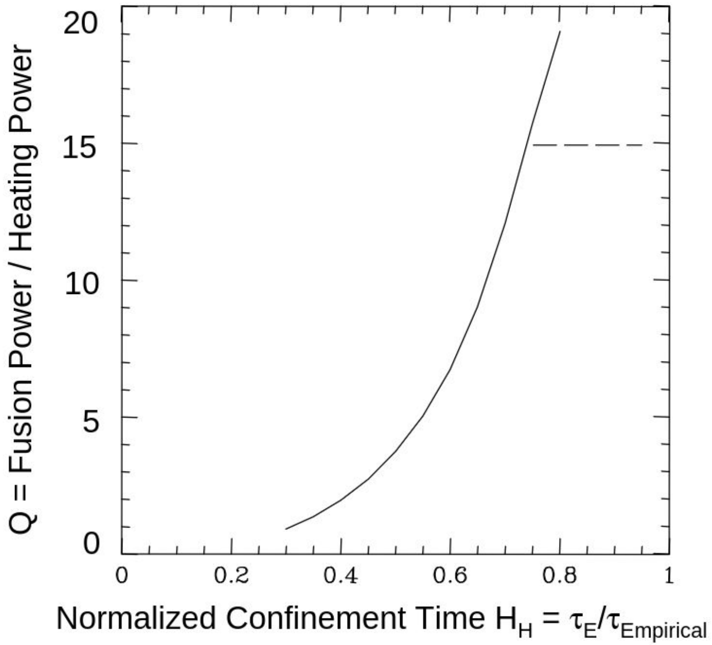
F. Ryter et al, Phys. Rev. Lett. **95** 085001 (2005).



P. Mantica *et al,* Fus. Sci. Tech. **53** 2008

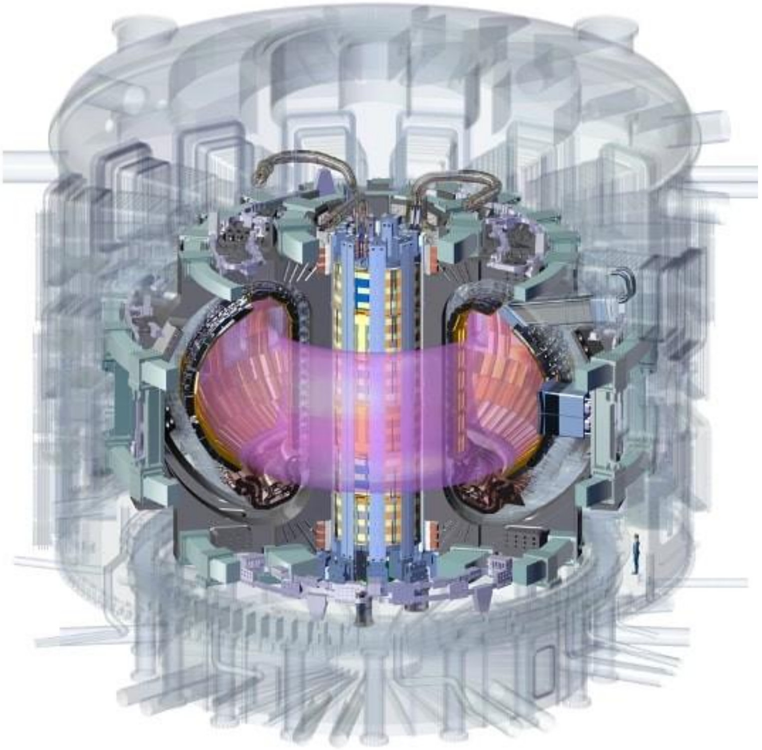
- Sharp discontinuity in heat conductivity above a critical gradient -> destabilization of underlying linear modes
- Heating less effective above critical threshold -> *profile stiffness*

EFFECT ON CONFINEMENT



- Fusion performance depends crucially on energy confinement
- Sensitive dependence of confinement on turbulence causes some uncertainties, but also gives opportunities for significant improvements.
- We need to model turbulence effectively and efficiently to be able to predict, control and optimize it.

WHY GYROKINETICS?



[https://www.iter.org]

- The core plasma is extremely hot, hence almost collisionless. Kinetic effects (e.g. FLR, Landau damping, trapping,...) are crucial in determining the plasma behavior.

- A fluid description is clearly not applicable, we need to resort to a kinetic description **Vlasov-Maxwell (6D):**

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

- <u>Problem</u>: we need to simulate turbulence in a very complex system, which features multiple spatial (R, a, ρ_i, ρ_e) and temporal scales (ω,Ω, t_E)

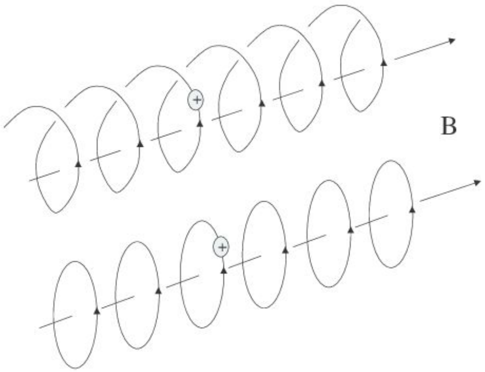
  A fully kinetic description is simply too demanding, although for certain things may be necessary

WHAT IS GYROKINETICS?

Gyrokinetics is in fact a major success of theoretical and computational plasma physics.

Basic idea behind it is that if $\omega << \Omega$ then we can average the fast particle motion and remove it from our system. This eliminates the fastest time scale and gets rid of one phase space dimensions, 6D -> 5D tremendous savings!

Effectively this means that we describe the plasma particles as ring that move in fluctuating fields.



[1] E. A. Frieman and Liu Chen Physics of Fluids, 1982.
[2] T. S. Hahm. Physics of Fluids, 31(9):2670–2673, 1988.
[3] T. S. Hahm, W. W. Lee, and A. Brizard. Physics of Fluids, 1988.
[4] A. Brizard. Physics of Fluids B, 1:1381–1384, 1989.
[5] A. J. Brizard and T. S. Hahm. Reviews of Modern Physics, 79(2):421, 2007.

GYROKINETICS ORDERING

Gyroaveraging is not a deus ex machina, it is motivated by actual experimental properties of plasma turbulence that allow us to introduce an ordering

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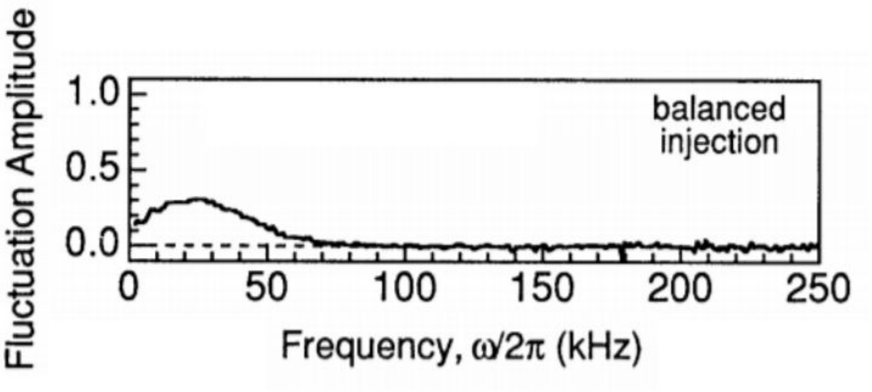
1. Slow temporal variation with respect to the gyromotion

$$\omega/\Omega \sim \epsilon \ll 1$$

Experimental TFTR values:

$$\omega_{peak} \approx \Delta\omega \approx 100\ kHz$$

$$\Omega_i = \frac{eB}{m_i} \approx 100\ MHz$$



[R Fonck PRL 1993]

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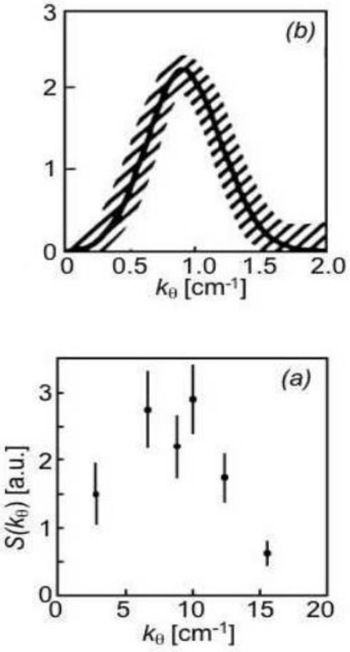
$$\omega/\Omega \sim \epsilon \ll 1$$

2. Large separation between macroscopic scales and Larmor radius

$$\rho/L \equiv \epsilon \ll 1$$



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$$k_{||}/k_{\perp} \sim \epsilon \ll 1$$

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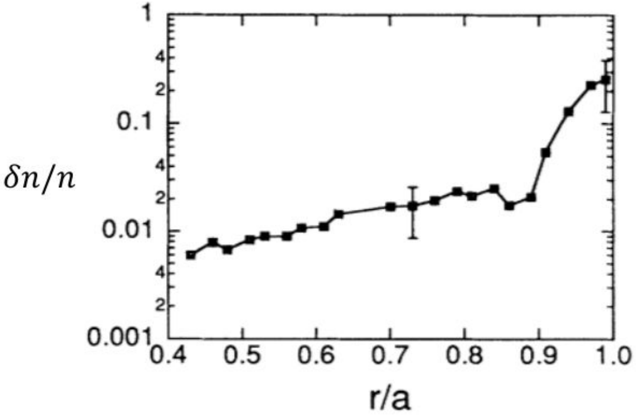
3. Strong anisotropy (only perpendicular gradients can be large)

$$k_{||}/k_{\perp} \sim \epsilon \ll 1$$

4. Small fluctuation amplitude (energy in perturbation smaller than thermal)

$$e\phi/T_e \sim \epsilon \ll 1$$



[R Fonck PRL 1993]

GYROKINETICS HISTORY

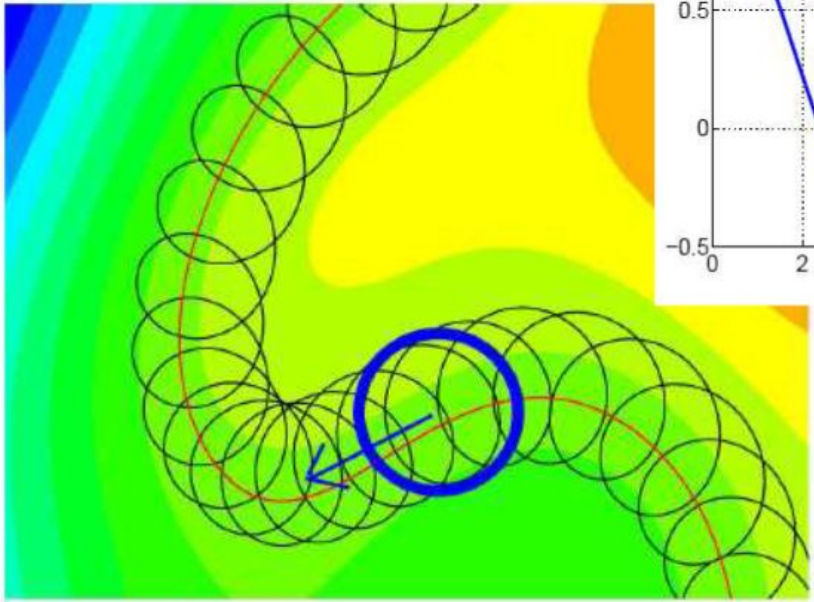
First derivation of nonlinear GK equation is from Frieman and Chen [1982]

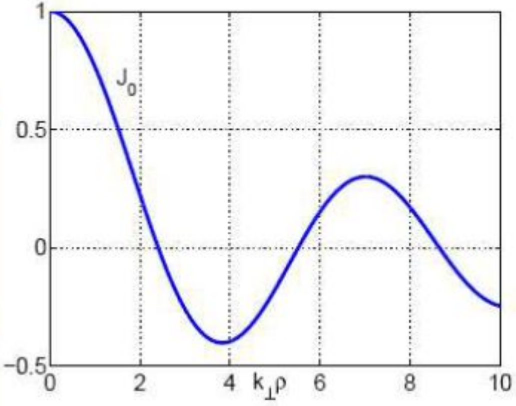
Derivation is based on a separation between background and perturbation, followed by small scale averaging. Complicated and has difficulties, mostly due to challenging in going to higher order and prove conservation properties.

However crucial since it proved possible to gyro-average nonlinear terms and keep full FLR-effects for arbitrary $k_\perp \rho$, with rigorous solutions w/o closure problem (not obvious)

Modern derivation is preferable. One uses Lagrangian variational methods to perform coordinate transformations via Lie transforms that remove the fast gyromotion. Easier to preserve invariance property of Hamiltonian, extend to full-f and / or higher order

GYROAVERAGED POTENTIALS





[F. Jenko]

Rings experience a reduced response and react to the fields

- Full Lorentz dynamics
- Gyrokinetic approx.: $\phi^{\text{eff}}(\vec{x}, \rho) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \Phi(\vec{x} + \vec{\rho})$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}\vec{x}} \phi(\vec{k}) J_0(|\vec{k}|\rho)$$

GYROKINETIC DERIVATION

Begin considering the equation of motion for a charged particle in external EM fields, using non-canonical coordinates:

$$\mathcal{L}(\mathbf{x}, \mathbf{v}) = (m\mathbf{v} + q\mathbf{A}(\mathbf{x})) \cdot \mathbf{v} - \mathcal{H}(\mathbf{x}, \mathbf{v}),$$

$$\mathcal{H}(\mathbf{x}, \mathbf{v}) = \frac{1}{2}mv^2 + q\phi(\mathbf{x}).$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

Formally, it is convenient to use one-forms to describe the Hamiltonian system and obtain the eqs. of motion from the generalized Euler-Lagrange equations:

$$\gamma = \gamma_\mu dz^\mu = \mathcal{L}dt, \qquad z^\mu = \{z^i, t\}.$$

$$\omega_{\mu\nu}\frac{dz^\nu}{dt} = 0 \qquad\qquad \omega_{\mu\nu} = \frac{\partial \gamma_\nu}{\partial z^\mu} - \frac{\partial \gamma_\mu}{\partial z^\nu}$$

GYROKINETIC DERIVATION - I

1. Guiding center motion in static fields (only vector potential)

$$\gamma_0(\mathbf{x}, \mathbf{v}) = (m\mathbf{v} + q\mathbf{A}_0) \cdot \mathrm{d}\mathbf{x} - \frac{1}{2}mv^2\mathrm{d}t.$$

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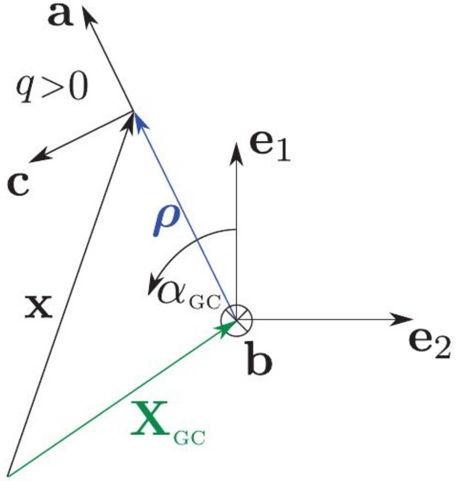
Introduce guiding center coordinates and remove gyration angle with a gauge transform

$$(\mathbf{X}_{\mathrm{GC}}, v_{\|,\mathrm{GC}}, \mu_{\mathrm{GC}}, \alpha_{\mathrm{GC}})$$



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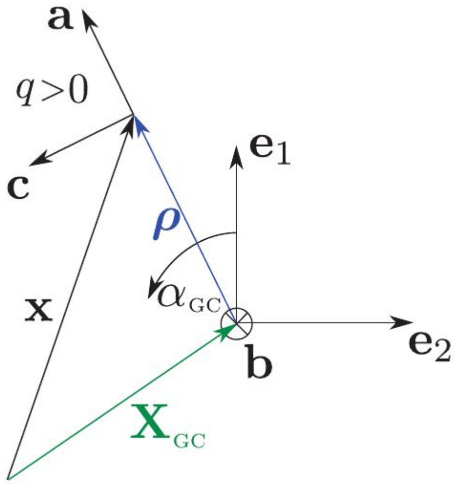
$$(\mathbf{X}_{\mathrm{GC}}, v_{\|,\mathrm{GC}}, \mu_{\mathrm{GC}}, \alpha_{\mathrm{GC}})$$

$$\mathbf{x} = \mathbf{X}_{\mathrm{GC}} + \rho(\mathbf{X}_{\mathrm{GC}},\mu_{\mathrm{GC}})\mathbf{a}(\alpha_{\mathrm{GC}}),$$

$$\mathbf{v} = v_{\|\mathrm{GC}}\mathbf{b} + v_{\perp\mathrm{GC}}(\mathbf{X}_{\mathrm{GC}},\mu_{\mathrm{GC}})\mathbf{c}(\alpha_{\mathrm{GC}}).$$

$$\mathbf{a}(\alpha_{\mathrm{GC}}) = \mathbf{e_1}\cos(\alpha_{\mathrm{GC}}) - \mathbf{e_2}\sin(\alpha_{\mathrm{GC}}),$$

$$\mathbf{c}(\alpha_{\mathrm{GC}}) = -\mathbf{e_1}\sin(\alpha_{\mathrm{GC}}) - \mathbf{e_2}\cos(\alpha_{\mathrm{GC}}),$$



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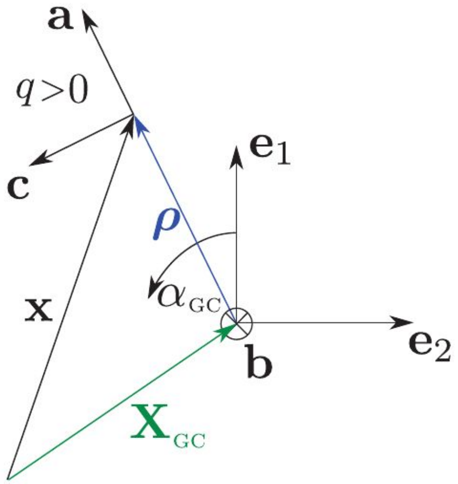

And expand fields

$$\mathbf{A}_0(\mathbf{x}) = \mathbf{A}_0(\mathbf{X}_{\mathrm{GC}}) + \rho\mathbf{a}(\alpha_{\mathrm{GC}}) \cdot \nabla\mathbf{A}_0(\mathbf{X}_{\mathrm{GC}}) + \mathcal{O}(\epsilon_B^2),$$

GYROKINETIC DERIVATION - I

1. Guiding center motion in static fields (only vector potential)

$$\gamma_0(\mathbf{x},\mathbf{v}) = (m\mathbf{v} + q\mathbf{A}_0)\cdot \mathrm{d}\mathbf{x} - \frac{1}{2}mv^2\mathrm{d}t.$$

obtaining

$$\Gamma_0^{\mathrm{GC}} = \Big(mv_{\|\mathrm{GC}}\mathbf{b} + mv_{\perp\mathrm{GC}}\mathbf{c}(\alpha_{\mathrm{GC}}) + q\mathbf{A}_0(\mathbf{X}_{\mathrm{GC}}) + \rho\mathbf{a}(\alpha_{\mathrm{GC}})\cdot\nabla\mathbf{A}_0(\mathbf{X}_{\mathrm{GC}})\Big)$$
$$\cdot\left(\mathrm{d}\left[\mathbf{X}_{\mathrm{GC}} + \rho\mathbf{a}(\alpha_{\mathrm{GC}})\right]\right) - \left(\frac{1}{2}mv_{\|\mathrm{GC}}^2 + \mu_{\mathrm{GC}}B_0\right)\mathrm{d}t.$$

Apply gauge (or just gyro-average)

$$\bar{\mathcal{F}}(\vec{X},\mu) = \frac{1}{2\pi}\oint \mathcal{F}(\vec{X} + \vec{\rho}(\vec{X},\mu,\alpha))d\alpha$$

$$\Gamma_0^{\mathrm{GC}} = \Big(mv_{\|\mathrm{GC}}\mathbf{b} + q\mathbf{A}_0\Big)\cdot\mathrm{d}\mathbf{X}_{\mathrm{GC}} + \frac{\mu_{\mathrm{GC}}B_0}{\Omega}\mathrm{d}\alpha_{\mathrm{GC}} - \left(\frac{1}{2}mv_{\|\mathrm{GC}}^2 + \mu_{\mathrm{GC}}B_0\right)\mathrm{d}t.$$

GYROKINETIC DERIVATION - II

2. Introduce perturbations

$$\phi(\mathbf{x},t) = \phi_1(\mathbf{x},t),$$

$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}_0(\mathbf{x}) + \mathbf{A}_1(\mathbf{x},t),$$

$$\gamma = \gamma_0 + \gamma_1,$$

$$\gamma_1 = q\mathbf{A}_1(\mathbf{x},t)\cdot \mathrm{d}\mathbf{x} - q\phi_1(\mathbf{x},t)\mathrm{d}t.$$

Repeat the procedure to find the perturbed guiding center one form

$$\Gamma_1^{\mathrm{GC}} = q\mathbf{A}_1(\mathbf{X}_{\mathrm{GC}}+\rho,t)\cdot\left[\mathrm{d}\mathbf{X}_{\mathrm{GC}} + \frac{1}{qv_{\perp\mathrm{GC}}}\mathbf{a}(\alpha_{\mathrm{GC}})\mathrm{d}\mu_{\mathrm{GC}} + \frac{mv_{\perp\mathrm{GC}}}{qB_0}\mathbf{c}(\alpha_{\mathrm{GC}})\mathrm{d}\alpha_{\mathrm{GC}}\right] - q\phi_1(\mathbf{X}_{\mathrm{GC}}+\rho,t)\mathrm{d}t,$$

Here we cannot remove the gyro-angle easily (intuitively fluctuation depend on it, so an average will not work). We need a new coordinate transformation to gyrocenter. This involves near identity Lie transform and computation of generators (see refs.)

GYROKINETIC DERIVATION - II

2. Introduce perturbations

After some algebra we obtain the gyrokinetic equations of motion

$$\dot{\mathbf{X}} = \frac{\mathbf{b}}{qB^*_{0\|}} \times \nabla H + mv_\| \frac{\mathbf{B}^*}{mB^*_{0\|}},$$

$$\dot{v}_\| = -\frac{\mathbf{B}^*}{mB^*_{0\|}} \cdot \left(\nabla H + q\mathbf{b}\dot{\bar{A}}_{1\|}\right),$$

$$H = \frac{1}{2}mv_\|^2 + \mu B_0 + q\bar{\phi}_1 + \mu \bar{B}_{1,\|}.$$

$$\mathbf{B}^* = \mathbf{B}_0 + \frac{B_0}{\Omega} v_\| \nabla \times \mathbf{b} + \nabla \times \left(\mathbf{b}\bar{A}_{1\|}\right),$$

GYROKINETIC DERIVATION - II

2. Introduce perturbations

We obtain the gyrokinetic equations of motions

$$\dot{\mathbf{X}} = \frac{\mathbf{b}}{qB^*_{0\parallel}} \times \nabla H + m v_\parallel \frac{\mathbf{B}^*}{mB^*_{0\parallel}},$$

$$\dot{v}_\parallel = -\frac{\mathbf{B}^*}{mB^*_{0\parallel}} \cdot \left(\nabla H + q\mathbf{b}\dot{\bar{A}}_{1\parallel} \right),$$

$$H = \frac{1}{2} m v_\parallel^2 + \mu B_0 + q\bar{\phi}_1 + \mu \bar{B}_{1,\parallel}.$$

$$\mathbf{B}^* = \mathbf{B}_0 + \frac{B_0}{\Omega} v_\parallel \nabla \times \mathbf{b} + \nabla \times \left(\mathbf{b}\bar{A}_{1\parallel} \right),$$

$$\dot{\mathbf{X}} = v_\parallel \mathbf{b} + \frac{B_0}{B^*_{0\parallel}} \left(\mathbf{v}_\xi + \mathbf{v}_c + \mathbf{v}_{\nabla B} \right)$$

$$\mathbf{v}_\xi = \frac{1}{B_0} \mathbf{b} \times \nabla \bar{\xi}_1$$

$$\mathbf{v}_c = \frac{v_\parallel^2}{\Omega} \left(\nabla \times \mathbf{b} \right)_\perp,$$

$$\mathbf{v}_{\nabla B} = \frac{\mu}{m\Omega} \mathbf{b} \times \nabla B$$

FINALLY, THE GYROKINETIC EQUATION READS

$$\frac{df_j}{dt} = \frac{\partial f_j}{\partial t} + \dot{\vec{X}} \cdot \frac{\partial f_j}{\partial \vec{X}} + \dot{v}_\parallel \frac{\partial f_j}{\partial v_\parallel} + \dot{\mu} \frac{\partial f_j}{\partial \mu} + \dot{\alpha} \frac{\partial f_j}{\partial \alpha} = 0 \, .$$

with

$$\begin{aligned}
\dot{\vec{X}} &= \vec{v}_G = v_\parallel \vec{b}_0 + \frac{B_0}{B^*_{0\parallel}} (\vec{v}_E + \vec{v}_{\nabla B} + \vec{v}_c) \, , \\
\dot{v}_\parallel &= -\frac{1}{m_j \, v_\parallel} \vec{v}_G \cdot (q_j \vec{\nabla} \bar{\phi}_1 + q_j \, \vec{b}_0 \, \dot{\bar{A}}_{1\parallel} + \mu \vec{\nabla} B_0) \, , \\
\dot{\mu} &= 0 \quad , \\
\dot{\alpha} &= \Omega_j + \frac{q_j^2}{m_j} \left(\frac{\partial \bar{\Phi}_1}{\partial \mu} - v_\parallel \frac{\partial \bar{A}_{1\parallel}}{\partial \mu} \right) \quad ,
\end{aligned}$$

Now substitute, do a gyroaverage to drop α, which is cyclic variable

FINALLY, THE GYROKINETIC EQUATION READS

Equivalently:

$$\frac{\partial f_j}{\partial t} + \left[v_\parallel \vec{b}_0 + \frac{B_0}{B^*_{0\parallel}}(\vec{v}_E + \vec{v}_{\nabla B} + \vec{v}_c)\right] \cdot \left[\vec{\nabla} f_j - \frac{1}{m_j v_\parallel}\left(q_j \vec{\nabla}\bar{\phi}_1 + q_j \vec{b}_0 \ \dot{\bar{A}}_{1\parallel} + \mu \vec{\nabla} B_0\right)\frac{\partial f_j}{\partial v_\parallel}\right] = 0$$

At this point things can vary.

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At this point things can vary.

A common procedure for the core of fusion devices is to explicitly split between a static background and perturbation

$$f_j = f_{0j} + f_{1j}$$

This is the so-called delta-f splitting which has numerical advantages (savings)

$$\left.\frac{df_{0j}}{dt}\right|_{\text{u. t.}} = \left[v_\parallel \vec{b}_0 + \frac{B_0}{B^*_{0\parallel}}(\vec{v}_{\nabla B} + \vec{v}_c)\right] \cdot \left(\vec{\nabla} f_{0j} - \frac{1}{m_j v_\parallel}\mu \vec{\nabla} B_0 \frac{\partial f_{0j}}{\partial v_\parallel}\right) = 0$$

We solve only for the perturbation assuming a specific background (not evolved, although one can)

WHAT ABOUT THE FIELDS?

3. They must be computed consistently

$$
\begin{aligned}
-\nabla^2\phi &= \frac{1}{\epsilon_0}\sum_j q_j \int f_{1j}(\mathbf{x},\mathbf{v},t)\mathrm{d}^3\mathbf{v} = \frac{1}{\epsilon_0}\sum_j q_j n_{1j}(\mathbf{x}), \\
-\nabla^2\mathbf{A} &= \mu_0 \sum_j q_j \int f_{1j}(\mathbf{x},\mathbf{v},t)\mathbf{v}\mathrm{d}^3\mathbf{v} = \mu_0 \sum_j n_j \mathbf{j}_j(\mathbf{x}),
\end{aligned}
$$

Need to account for the fact that Maxwell eqs. are in particle space, not gyrocenters -> “pull back” transformation.

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3. They must be commuted consistently

$$-\nabla^2\phi = \frac{1}{\epsilon_0}\sum_j q_j \int f_{1j}(\mathbf{x},\mathbf{v},t)\mathrm{d}^3\mathbf{v} = \frac{1}{\epsilon_0}\sum_j q_j n_{1j}(\mathbf{x}),$$

$$-\nabla^2\mathbf{A} = \mu_0 \sum_j q_j \int f_{1j}(\mathbf{x},\mathbf{v},t)\mathbf{v}\mathrm{d}^3\mathbf{v} = \mu_0 \sum_j n_j \mathbf{j}_j(\mathbf{x}),$$

Need to account for the fact that Maxwell eqs. are in particle space, not gyrocenters -> “pull back” transformation.

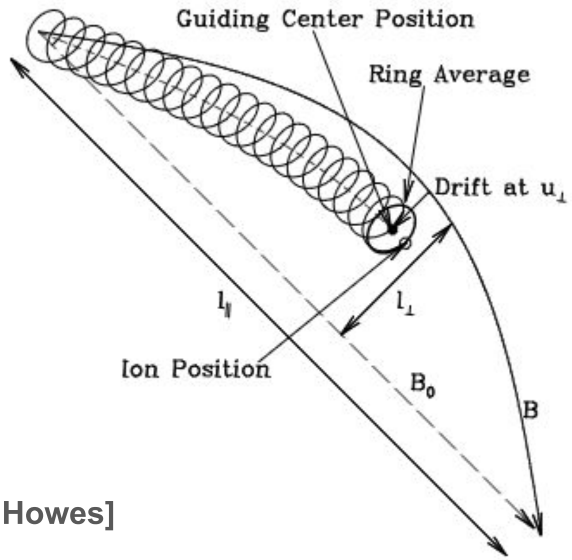
$$\nabla^2\phi_1 = -\frac{1}{\epsilon_0}\sum_j 2\pi\frac{q_j}{m_j}\int \left[\left\langle \{B_0 F_{1,j}\}|_{\mathbf{X}=\mathbf{x}-\boldsymbol{\rho}} \right\rangle - q_j\phi_1 B_0 \frac{F_{0j}}{T_{0,j}} + \left\langle \left\{ \left[q_j\bar{\phi}_{1j} + \mu\bar{B}_{1\|j} \right] B_0 \frac{F_{0,j}}{T_{0,j}} \right\} \Big|_{\mathbf{X}=\mathbf{x}-\boldsymbol{\rho}} \right\rangle \right] \mathrm{d}v_\| \mathrm{d}\mu,$$

$$\nabla_\perp^2 A_{1\|} = -\mu_0 \sum_j 2\pi \frac{q_j}{m_j}\int v_\| \langle \{B_0 f_{1,j}\}|_{\mathbf{X}=\mathbf{x}-\boldsymbol{\rho}} \rangle \mathrm{d}v_\| \mathrm{d}\mu.$$

$$(\nabla\times B_1)_\perp = \mu_0 \sum_j 2\pi q_j \left(\frac{2}{m_j}\right)^{3/2} \int \left\langle \left\{ \mathbf{c} B^{3/2} \left[f_{1j} - (q_j\tilde{\phi}_1 - \mu\bar{B}_{1\|,j})\frac{F_{0,j}}{T_{0,j}} \right] \right\} \Big|_{\mathbf{X}=\mathbf{x}-\boldsymbol{\rho}} \right\rangle \sqrt{\mu}\mathrm{d}v_\|\mathrm{d}\mu,$$

GYROKINETIC DERIVATION - IN SUMMARY

The complex 6D motion has been reduced to charged rings moving in self consistent fluctuating fields



[G. Howes]

Obtained a 5D system of nonlinear PDEs (plus fields)

The derivation so far neglected collisions, which however are playing an important role and they have to accounted for by a proper additional term on the RHS

A DIGRESSION: COORDINATE SYSTEM I

- Recall the GK ordering: turbulence is field aligned, much longer wavelengths parallel than perpendicular to the magnetic field, $k_{//} << k_{\perp}$

- For an axisymmetric system, one can introduce a straight field line coordinate system (ψ, χ, ϕ) where magnetic field lines are straight in a (χ, ϕ) plane at constant ψ.

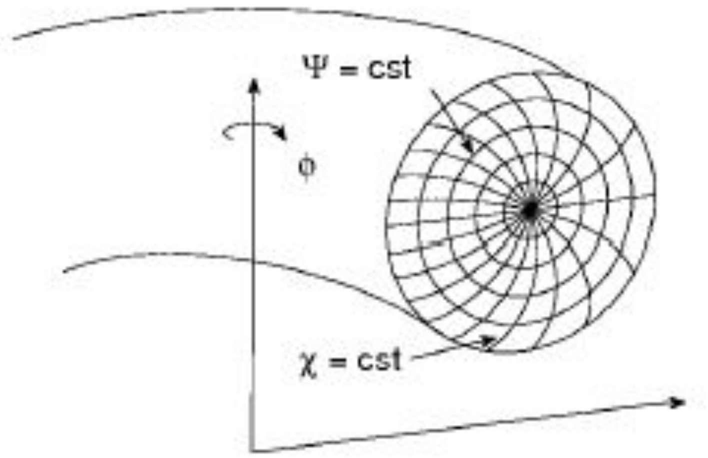
- χ is the straight-field line poloidal angle, defined as

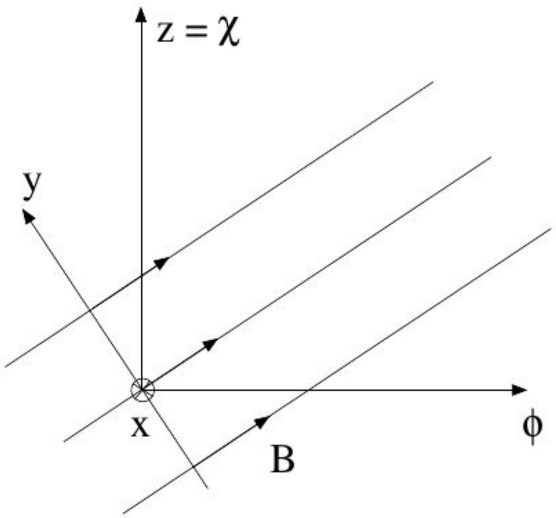
$$\chi = 2\pi \int_0^\theta \frac{\mathbf{B}\cdot\nabla\varphi}{\mathbf{B}\cdot\nabla\theta'} d\theta' \Big/ \oint \frac{\mathbf{B}\cdot\nabla\varphi}{\mathbf{B}\cdot\nabla\theta'} d\theta'$$
$$= 2\pi \int_0^\theta \frac{1}{R^2}\frac{1}{\mathbf{B}\cdot\nabla\theta'} d\theta' \Big/ \oint \frac{1}{R^2}\frac{1}{\mathbf{B}\cdot\nabla\theta'} d\theta'$$
$$= \frac{F(\psi)}{q(\psi)} \int_0^\theta \frac{1}{R^2}\frac{1}{\mathbf{B}\cdot\nabla\theta'} d\theta',$$

$$\frac{d\varphi}{d\chi} = \frac{B\cdot\nabla\varphi}{B\cdot\nabla\chi} = q(\psi)$$

q is the safety factor (# toroidal revolutions for one poloidal)

A DIGRESSION: COORDINATE SYSTEM II





From straight-field-line we construct a field aligned coordinate system by simply

$$x = f(\psi) \qquad y = C_y(q\chi - \varphi) - y_0 \qquad z = \chi$$

$$B = \mathcal{C}(x)\nabla x \times \nabla y \qquad\qquad \mathcal{C}(x) = \left(\frac{d\mathcal{C}(x)}{d\psi} C_y\right)^{-1}$$

Essential for compute time savings

Major theoretical speedups

G.Hammett

relative to original Vlasov/pre-Maxwell system on a naïve grid, for ITER $1/\rho_* = a/\rho \sim 1000$

- Nonlinear gyrokinetic equations
  - eliminate plasma frequency: $\omega_{pe}/\Omega_i \sim m_i/m_e$ — x10^3
  - eliminate Debye length scale: $(\rho_i/\lambda_{De})^3 \sim (m_i/m_e)^{3/2}$ — x10^5
  - average over fast ion gyration: $\Omega_i/\omega \sim 1/\rho_*$ — x10^3
- Field-aligned coordinates
  - adapt to elongated structure of turbulent eddies: $\Delta_{||}/\Delta_\perp \sim 1/\rho_*$ — x10^3
- Reduced simulation volume
  - reduce toroidal mode numbers (i.e., 1/15 of toroidal direction) — x15
  - $L_r \sim a/6 \sim 160\ \rho \sim 10$ correlation lengths — x6
- **Total speedup** — x10^{16}
- For comparison: Massively parallel computers (1984-2009) — x10^7

GYROKINETIC SIMULATIONS AND CODES

Beware, turbulence modeling remains extremely challenging and simulations cannot be performed easily. Some simulations may require months of runtime (i.e. several million core hrs) to be performed.

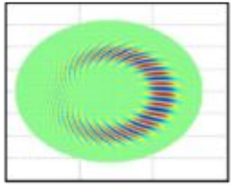
Gyrokinetic codes are highly optimized and still need to use the best numerical schemes to really perform.

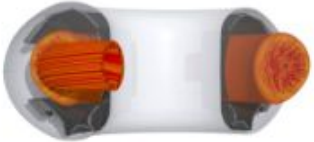
There are several GK codes, which can be classified in various ways

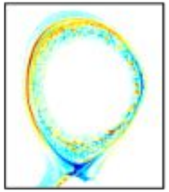
a) Numerical methods: grid based (FD, FV, DG), PIC based, semi-lagrangian or meshless
b) For application: “core” codes vs. edge codes
c) Local vs. global
d) Full-f vs. delta-f vs. total-f

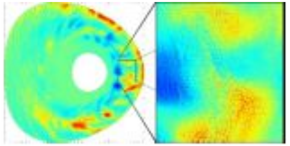
A non complete list: GENE, GS2, ORB5, GT5D, GKW XGC, GKeyll, GYRO, ...

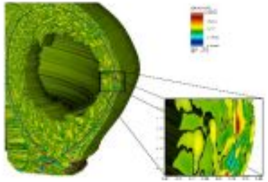
THE POWER OF GK AND COMPUTER SCIENCE

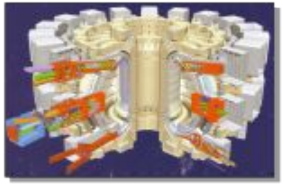


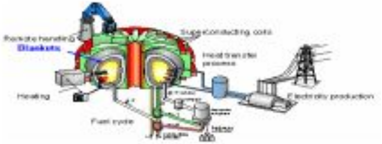












Gigaflops

5-D electrostatic ion physics in simplified circular cylindrical geometry

Teraflops

Core: 5D ion-scale electromagnetic physics in torus

Edge: ion+neutral electrostatic physics in torus

Petaflops

Core: 5D i+e scale electrostatic physics

Edge: ions + electrons + neutrals, electrostatic physics

Exaflops

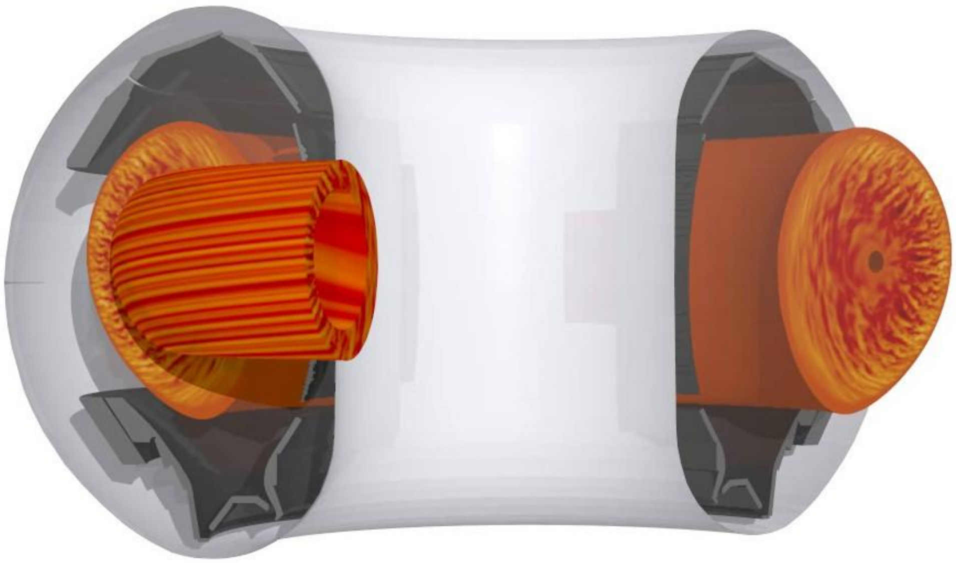
Core-edge coupled 5D electromagnetic study of whole-device ITER, including turbulence, transport, large-scale instability, plasma-material interaction, rf heating, and energetic particles

Beyond

A WDMApp that includes engineering, components burn control, 6D whole device modeling

THE GENE CODE - [Jenko et al., Physics of Plasmas 7, 1904 (2000)]

- Eulerian nonlinear gyrokinetic code for microturbulence, highly tuned for core conditions
- Fortran2008 + MPI/HIP/Sycl/CUDA
- Parallelism: up to 262,144 cores on Titan
- Publicly available, O(100) scientific institutions (amounting to 250-300 world-wide user base)
- Part of Unified European Application Benchmark Suite (PRACE)



Typical "every-day" production runs:

- 8,000 – 32,000 cores
- Up to about 10 million core-hours
- Up to about 10 billion grid points

http://genecode.org

THE GENE CODE

The underlying GK system of equations for each species are solved on a fixed grid in phase space.

Essential features:

- Combination of finite difference, finite volume and spectral methods
- Exploit turbulence field-alignment
- Regular cartesian block structured grids in velocity space
- Scalable algorithms and 5-D domain decomposition
- Auto-tuning: best MPI layout and optimal subroutines self-determined during initialization phase
- Explicit time-scheme to favor scalability. Time scheme self-chosen to minimize time to solutions
- Time step maximized at initialization and adapted during computation

GENE IS IN FACT A FAMILY OF CODES

Extremely flexible mode of operation:

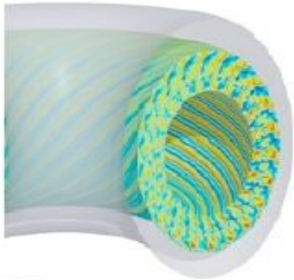
- Flux-tube (local): Fourier in x (radial) and y (binormal) directions; **[Jenko et al. PoP 01]**

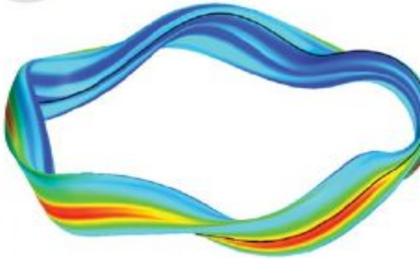
- x-Global (1st evolution) => Full-torus: Real Space in x but Fourier in y; **[Goerler et al. JCP 10]**

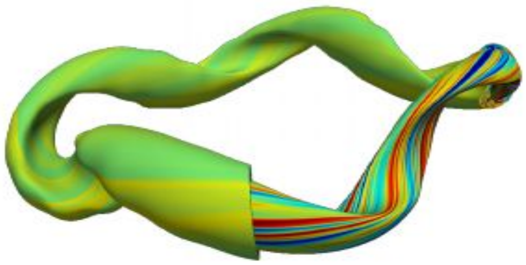
- y-Global (2nd evolution) => Full-surface: Fourier in x (local) but Real space in y; **[Xanthopoulos et al. PRL 14]**

- GENE-3D (3rd evolution) => Full-torus and Full-surface: Real space in x and y; **[Maurer et al. JCP2023]**

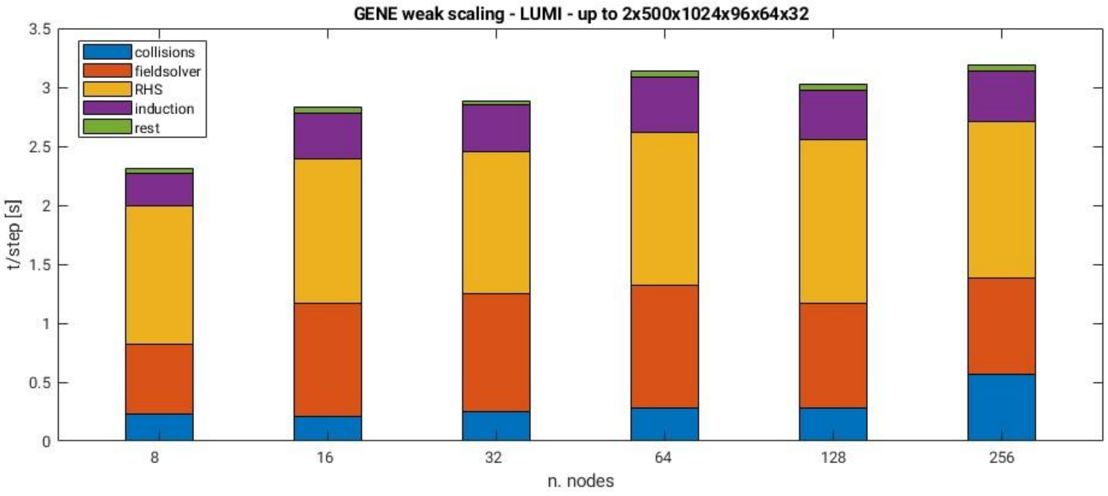
\+ more recently GENE-X for edge applications







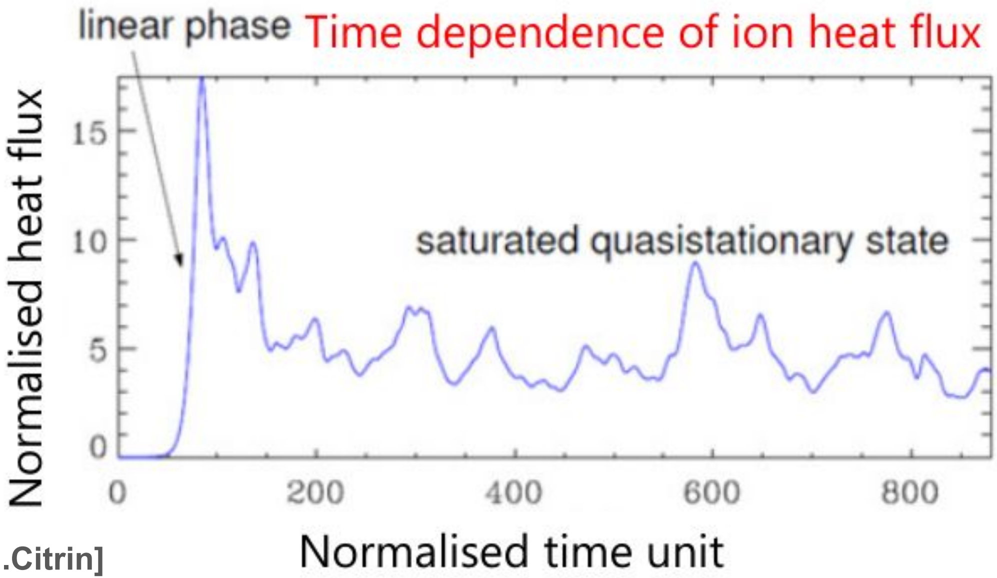
USING HPC EFFICIENTLY IS CRUCIAL



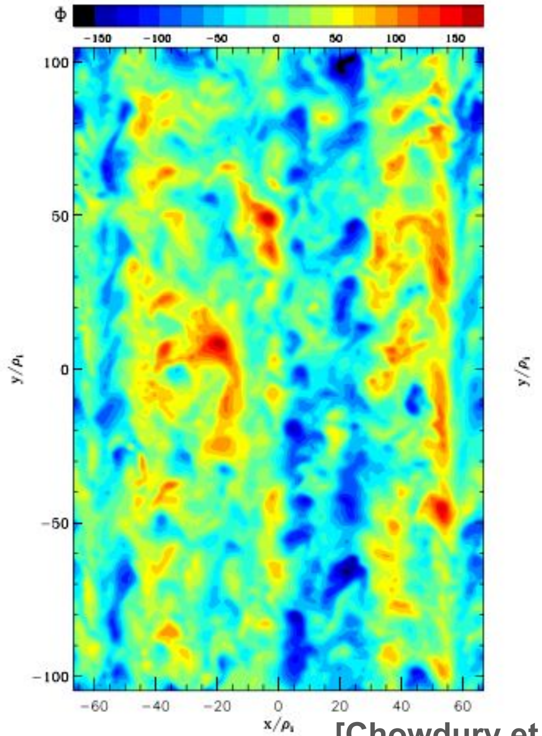
Code massively parallel, runs on all platforms including ExaScale

HOW DOES A SIMULATION LOOK LIKE?

Provide necessary inputs: geometry, plasma species, gradients, …



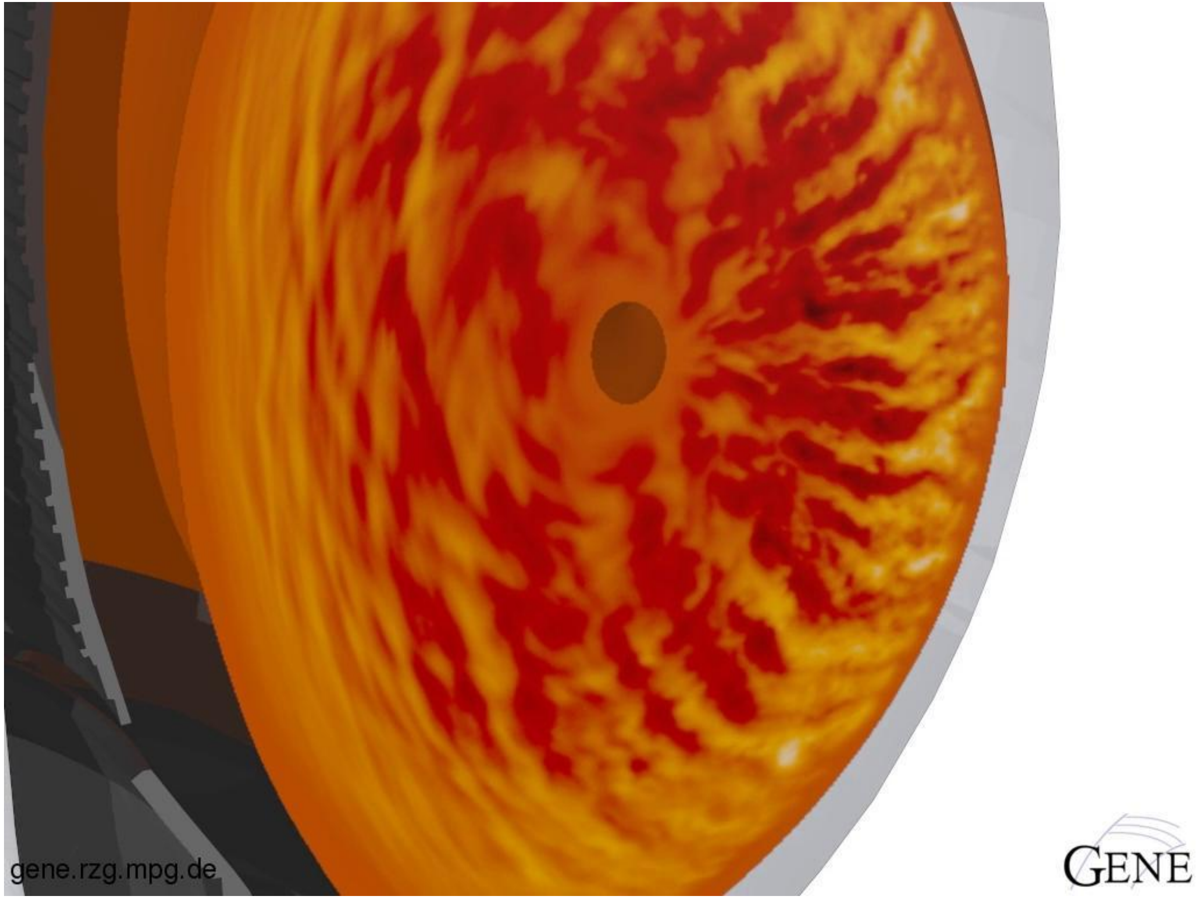
[J.Citrin]



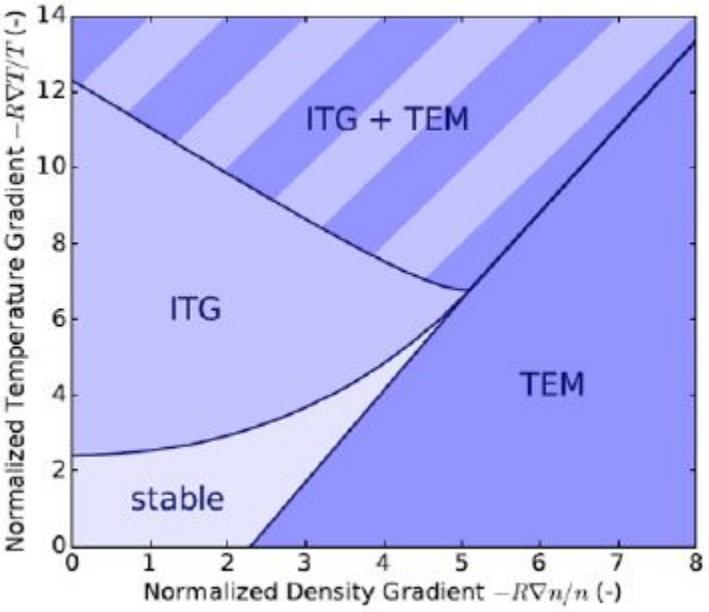
[Chowdury et al PoP 2012]

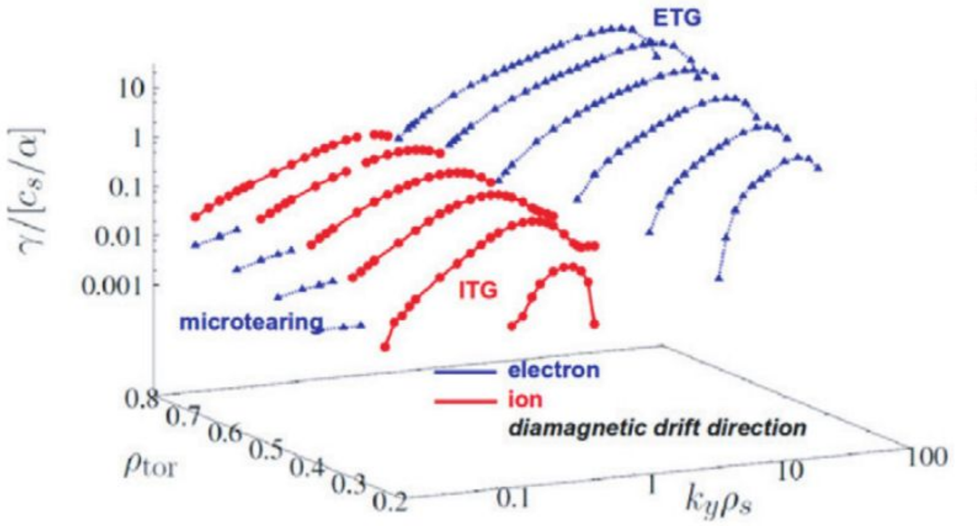
Instabilities grow linearly at different scales, until their amplitude is large enough for them to interact nonlinearly and saturate.

SOME APPLICATIONS - NONLINEAR SATURATION



SOME APPLICATIONS - LINEAR INSTABILITIES

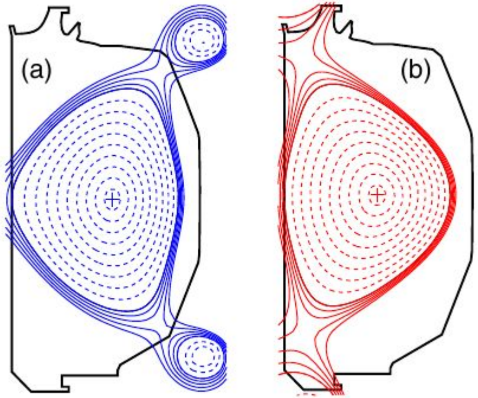




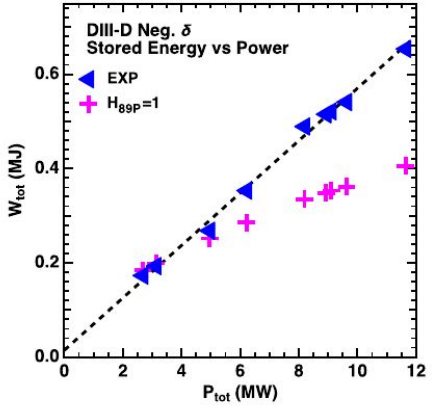
[Jenko et al, NF 2013]

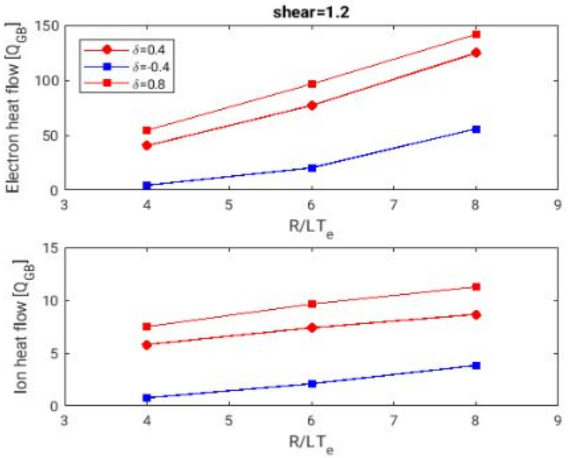
Different kinds of microinstabilities drive different kinds of plasma turbulence with different signatures -> fingerprint concept **[Kotschenreuther et al. NF 2021]**

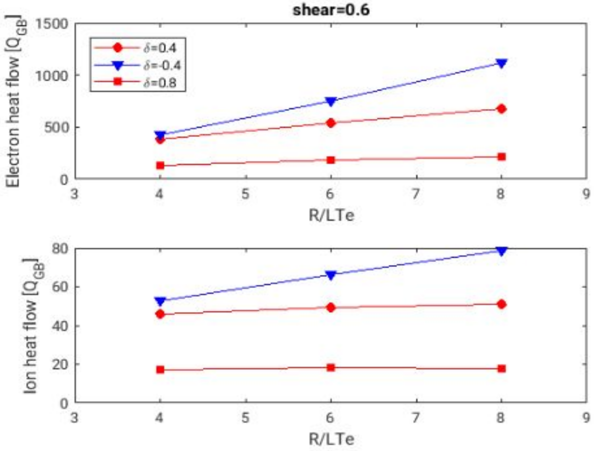
SOME APPLICATIONS - PLASMA SHAPING



[Austin et al. PRL 2019]



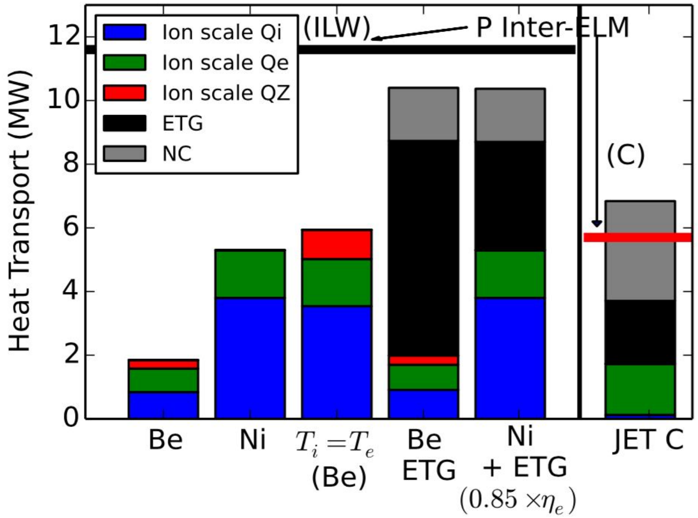




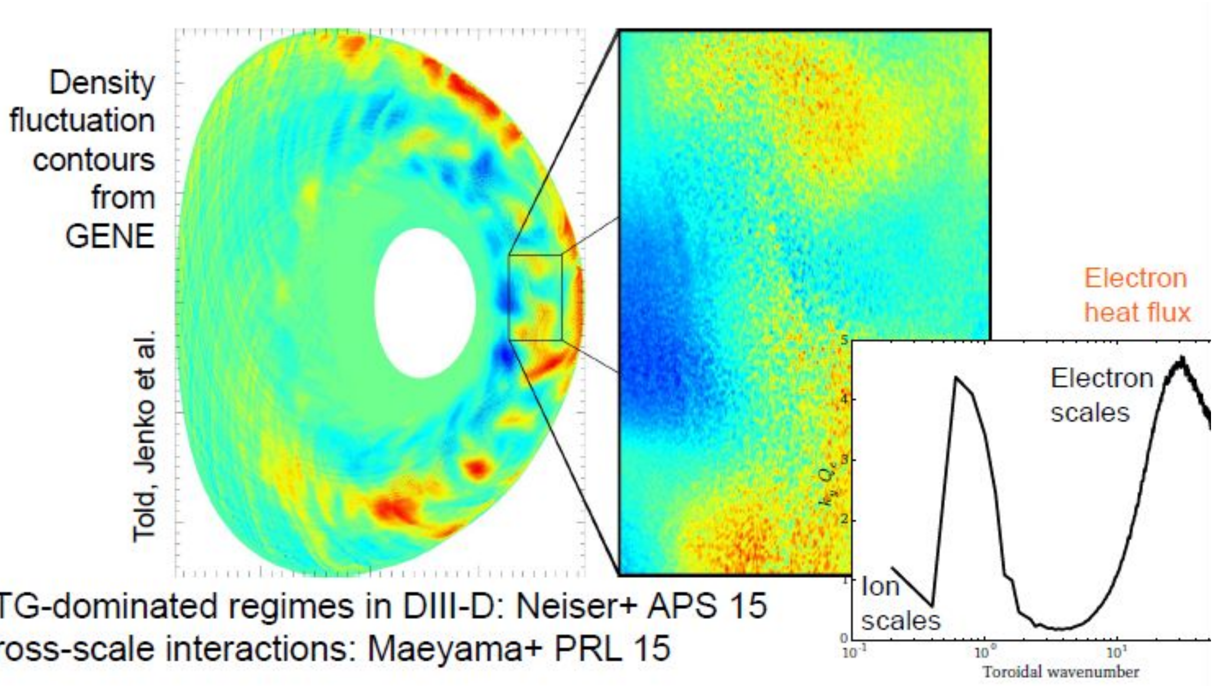
SOME APPLICATIONS - PEDESTAL INSTABILITIES

GENE simulations reproduce and explain JET-ILW trends

- Decrease in confinement on JET since installation of ITER-like wall (ILW)
- High pedestal top temperature no longer accessible
- Improves with impurity seeding (in contrast with carbon wall)
- Direct comparison between JET-C and JET-ILW pedestal transport with GENE **[Hatch et al NF 2019, 2022, 2023]**



SOME APPLICATIONS - MULTISCALE TCV



ETG-dominated regimes in DIII-D: Neiser+ APS 15
Cross-scale interactions: Maeyama+ PRL 15

SUMMARY

- Transport in tokamaks (and other fusion devices) is anomalous. Gradients in background profiles drive collective modes that produce large radial outward fluxes of heat, particles and momentum.

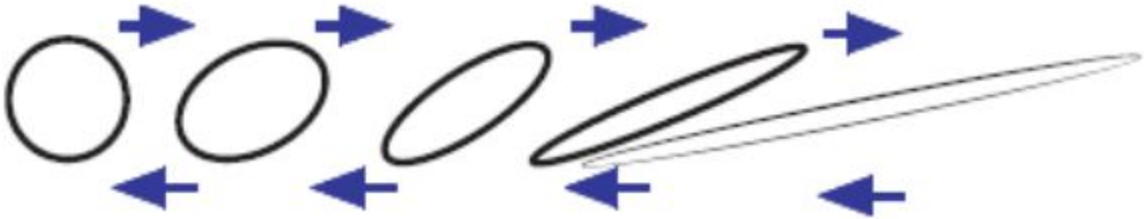
- Nonlinear gyrokinetic theory provides a standard framework for studying turbulence and reproduce experiments.

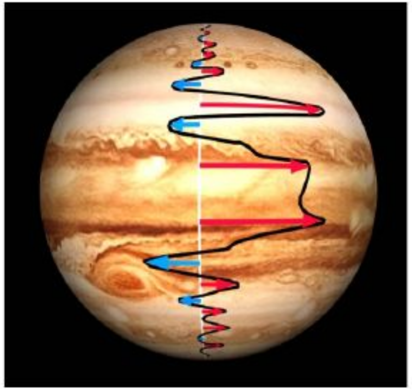
- Reached a maturity in the models and a computing power sufficient to enable predictions of profiles and performance.

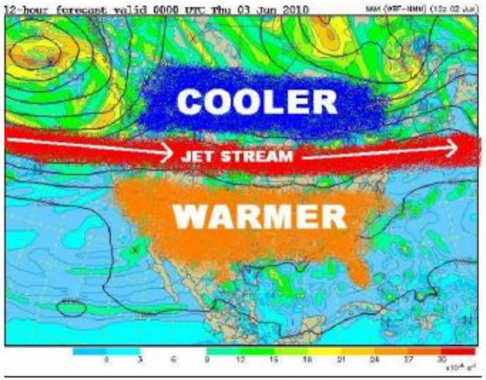
- There are still a lot of open questions: e.g. fast ion and turbulence interaction, LH transition, isotope scaling, more in general reactor design and optimization.

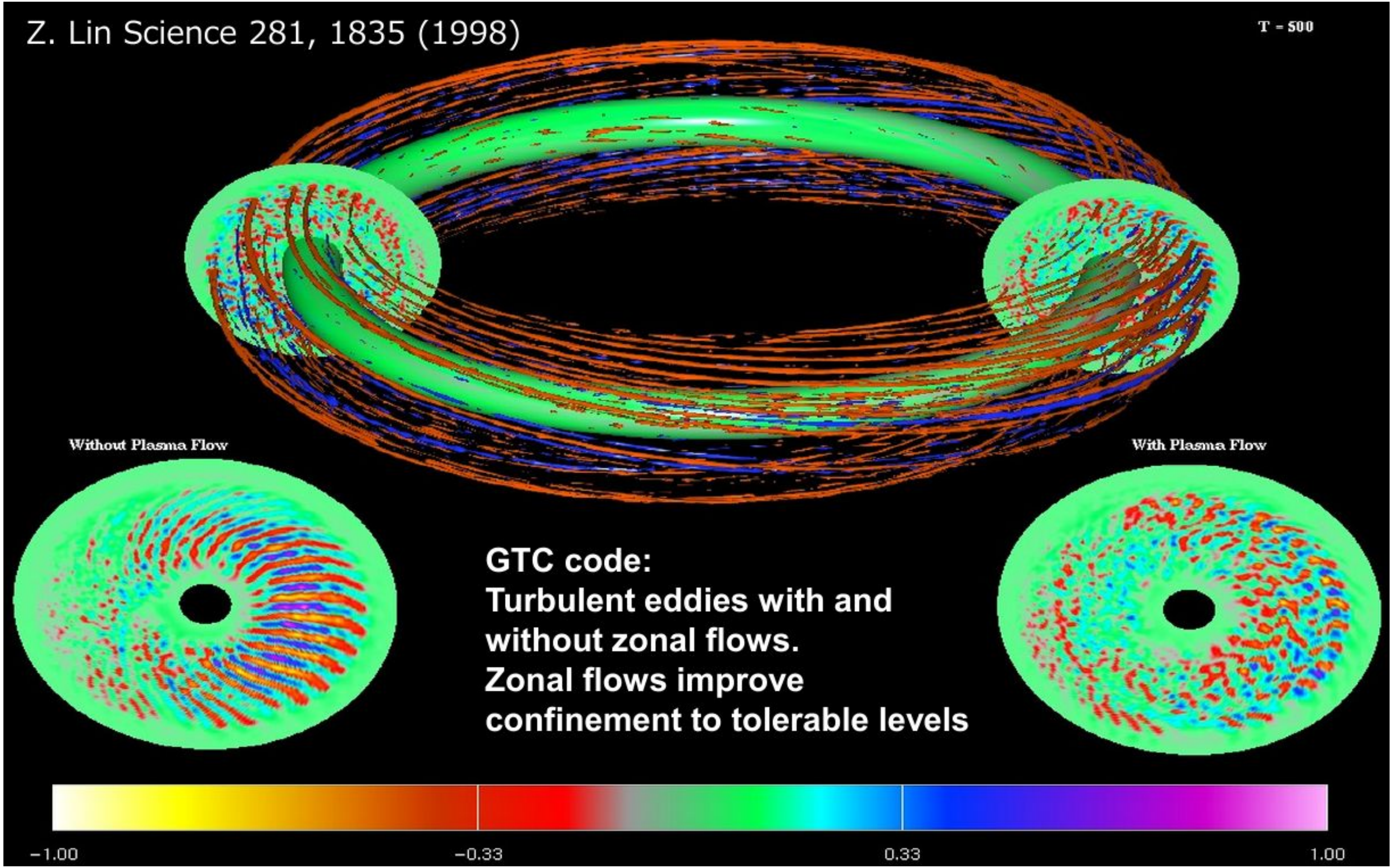
ZONAL FLOWS

Through nonlinear coupling, the linearly stable zonal flows are excited and can provide a very effective mechanism for turbulence saturation (other saturation mechanisms exist)








Z. Lin Science 281, 1835 (1998)
T = 500
Without Plasma Flow
With Plasma Flow
GTC code:
Turbulent eddies with and
without zonal flows.
Zonal flows improve
confinement to tolerable levels
−1.00
−0.33
0.33
1.00

Assume complete scale separation with brackground, periodic BC (flux tube limit)

$$\delta f = \sum_k \delta f_k(z)\, e^{-i(\boldsymbol{k}\cdot\boldsymbol{x})}, \quad \delta\phi = \sum_k \delta\phi_k\,(z) e^{-i(\boldsymbol{k}\cdot\boldsymbol{x})}$$

Landau damping (hence low $k_\parallel$)

Wave-particle resonances

Nonlinear mode coupling! (3 wave coupling here)

$$\frac{\partial}{\partial t}\delta f_k = (v_\parallel \nabla_\parallel + k_\perp v_{\nabla B})\delta f_k - \frac{1}{B}\sum_{k'} \hat{\boldsymbol{b}}\cdot(\boldsymbol{k}\times\boldsymbol{k}')\,\delta f_{k'}\delta\phi_{k-k'}$$
$$-\frac{\mathrm{qF_M}}{\mathrm{T}}(v_\parallel \nabla_\parallel + k_\perp v_{\nabla B} - \omega^*)\,J_0(k_\perp\rho)\delta\phi_k$$

Driving gradients

+ gyrokinetic Poisson equation for $\delta\phi_k$

Trapped electrons — Trapped ions — Passing ions

$$\sum_s \left(1 + Z_{eff}\frac{T_e}{T_i} - f_t\left\langle\frac{\omega_k - \omega_e^*}{\omega_k - \omega_{De}}J_0(k_\perp\delta_e)\right\rangle_t - \frac{n_i Z_i^2}{n_e}\frac{T_e}{T_i}\left(f_t\left\langle\frac{\omega_k - \omega_e^*}{\omega_k - \omega_{Di}}J_0(k_\perp\delta_i)\right\rangle_t + f_p\left\langle\frac{\omega_k - \omega_i^*}{\omega_k - k_\parallel v_{\parallel i} - \omega_{Di}}J_0(k_\perp\rho_i)\right\rangle_p\right)\right)\delta\phi_k = 0$$

$$
\begin{aligned}
\mathcal{L}\,dt = \Gamma \quad &= \quad \Gamma_0 + \Gamma_1 \\
&= \quad (m\,v_\parallel\,\vec{b}_0 + q\,\vec{A}_0 + q\,\bar{A}_{1\parallel}\,\vec{b}_0)\cdot d\vec{X} + \frac{m}{q}\,\mu\,d\alpha \\
&\quad\; - \left[\frac{1}{2}m\,v_\parallel^2 + q\,\bar{\Phi}_1 + \mu\,(B_0 + \bar{B}_{1\parallel})\right]\,dt\ ,
\end{aligned}
$$