Transport Barriers - The GK charge Flux Constraint

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Transport Barriers in Magnetized Plasmas- General Theory with Dynamical Constraints

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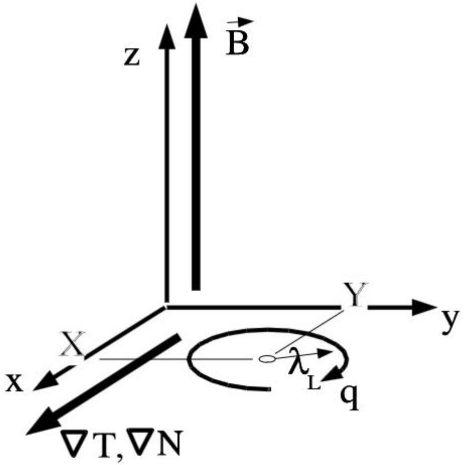
A fundamental dynamical constraint - that fluctuation induced charge-weighted particle flux must vanish- can prevent instabilities from accessing the free energy in the strong gradients characteristic of Transport Barriers (TBs). Density gradients, when larger than a certain threshold, lead to a violation of the constraint and emerge as a stabilizing force. This mechanism, then, broadens the class of configurations (in magnetized plasmas) where these high confinement states can be formed and sustained. The need for velocity shear, the conventional agent for TB formation, is obviated. The most important ramifications of the constraint is to permit a charting out of the domains conducive to TB formation and hence to optimally confined fusion worthy states; the detailed investigation is conducted through new analytic methods and extensive gyrokinetic simulations.

Simplest system possible ITG adiabatic electrons

Particle motion in an B field with an external force

$$m\frac{d\vec{v}}{dt} = q\,\vec{v}\times\vec{B} + \vec{F},$$



Is the superposition of a gyromotion and a drift due to F with velocity

$$\vec{v}_F = \frac{\vec{F}\times\vec{B}}{qB^2},$$

The equilibrium solution is a Maxwellian distribution whose mean velocity is

$$\vec{V}_d(x=0) = \frac{T}{qB}\left[\left.\frac{d\ln\mathcal{N}}{dx}\right|_{x=0} + \left.\frac{d\ln T}{dx}\right|_{x=0}\right]\vec{e}_y = \frac{T}{qB}\left[\left.\frac{d\ln N}{dx}\right|_{x=0} + \left.\frac{d\ln T}{dx}\right|_{x=0} - \frac{F}{T}\right]\vec{e}_y$$
$$= \frac{1}{qB^2}\left(-\frac{\nabla p}{N} + \vec{F}\right)\times\vec{B},$$

i.e. superposition of diamagnetic drifts and F induced drift

Simplest system possible ITG adiabatic electrons

Let's consider a kinetic description of a collisionless plasma, satisfies Vlasov

$$\left[\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + \frac{1}{m}\left(q\vec{v} \times \vec{B} + \vec{F}\right) \cdot \frac{\partial}{\partial \vec{v}}\right] f = 0.$$

And add a perturbation

$$\phi = \hat{\phi}(x) \exp i(k_y y + k_z z - \omega t)$$

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Each distribution function will respond with a similar perturbation

$$f = f_0 + \delta f \qquad \delta f = \hat{\delta f}\exp i(k_y y + k_z z - \omega t).$$

Satisfying the linearized Vlasov equation

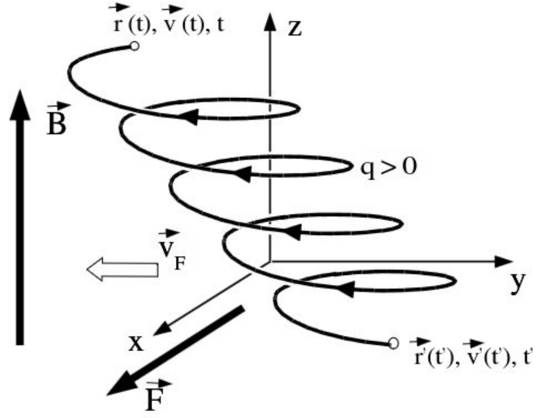
$$\left.\frac{D}{Dt}\right|_{u.t.p.}\delta f = \left[\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{r}} + \frac{1}{m}\left(q\vec{v}\times\vec{B}+\vec{F}\right)\cdot\frac{\partial}{\partial \vec{v}}\right]\delta f = \frac{q}{m}\nabla\phi\cdot\frac{\partial f_0}{\partial \vec{v}}.$$

Simplest system possible ITG adiabatic electrons

We can solve for the perturbation integrating along unperturbed trajectories

$$\delta f(\vec{r},\vec{v},t) = \frac{q}{m}\int_{-\infty}^{t} dt' \; \nabla\phi \cdot \frac{\partial f_0}{\partial \vec{v}}\bigg|_{\vec{r}'(t'),\vec{v}'(t'),t'}$$

$$\frac{d\vec{r}'}{dt'} = \vec{v}',$$
$$\frac{d\vec{v}'}{dt'} = \frac{1}{m}\left(q\vec{v}' \times \vec{B} + \vec{F}\right),$$



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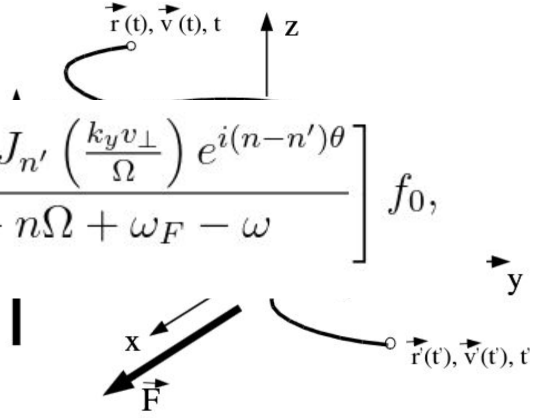
$$\begin{aligned} \frac{d\vec{r}'}{dt'} &= \vec{v}', \\ \frac{d\vec{v}'}{dt'} &= \frac{1}{m}\left(q\vec{v}'\times\vec{B}+\vec{F}\right), \end{aligned}$$

$$\hat{\delta f} = -\frac{q\hat{\phi}}{T}\left[1-(\omega_d'-\omega)\sum_{n,n'=-\infty}^{+\infty}\frac{J_n\left(\frac{k_y v_\perp}{\Omega}\right)J_{n'}\left(\frac{k_y v_\perp}{\Omega}\right)e^{i(n-n')\theta}}{k_z v_z + n\Omega + \omega_F - \omega}\right] f_0,$$



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Which can be solved and used to derive a dispersion relation from the dielectric function

$$\delta\hat{f} = -\frac{q\hat{\phi}}{T}\left[1 - (\omega_d' - \omega)\sum_{n,n'=-\infty}^{+\infty} \frac{J_n\left(\frac{k_y v_\perp}{\Omega}\right) J_{n'}\left(\frac{k_y v_\perp}{\Omega}\right) e^{i(n-n')\theta}}{k_z v_z + n\Omega + \omega_F - \omega}\right] f_0,$$

$$\begin{aligned} -\triangle\phi &= k^2\phi = \frac{1}{\epsilon_0}\sum_{\text{species}} q\delta N \\ \epsilon(\vec{k},\omega) &\doteq 1 - \sum_{\text{species}} \frac{q}{\epsilon_0 k^2}\frac{\delta\hat{N}}{\hat{\phi}} = 0, \end{aligned}$$

Simplest system possible ITG adiabatic electrons

Explicitly performing integral and n=0 limit (long wavelength low frequency)

$$1 + \sum_{\text{species}} \frac{1}{(k\lambda_D)^2} \left\{ 1 + \frac{\omega - \omega_d'}{\omega - \omega_F} \left[W\left(\frac{\omega - \omega_F}{|k_z| v_{th}} \right) - 1 \right] \Lambda_0(\xi) \right\} = 0.$$

One can consider various limiting cases:

- only density gradient and F not dependent on q -> flute instability (like Rayleigh-Taylor)
- only B gradients
- only density gradient -> drift waves, modification of sound waves
- **only temperature gradients -> slab ITG**

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Neglect FLR, expand W, neglect resonances

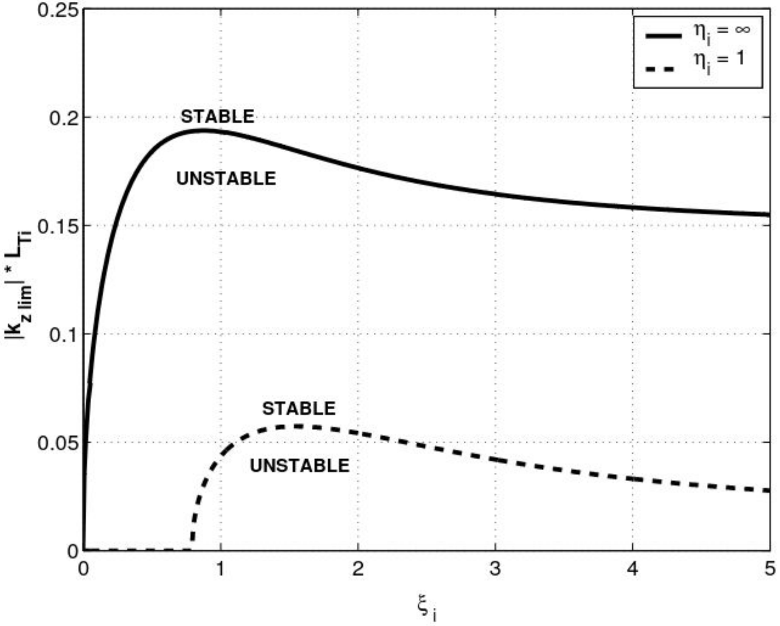
$$1 - \left(\frac{k_z c_s}{\omega} \right)^2 \left(1 - \frac{\omega_{T_i}}{\omega} \right) = 0$$

Simplest system possible ITG adiabatic electrons

In practice we usually have T and n gradients

$$\eta_i \doteq \frac{d \ln T_i}{d \ln N} = \frac{L_N}{L_{T_i}}$$

Proceed same way, but solve numerically now



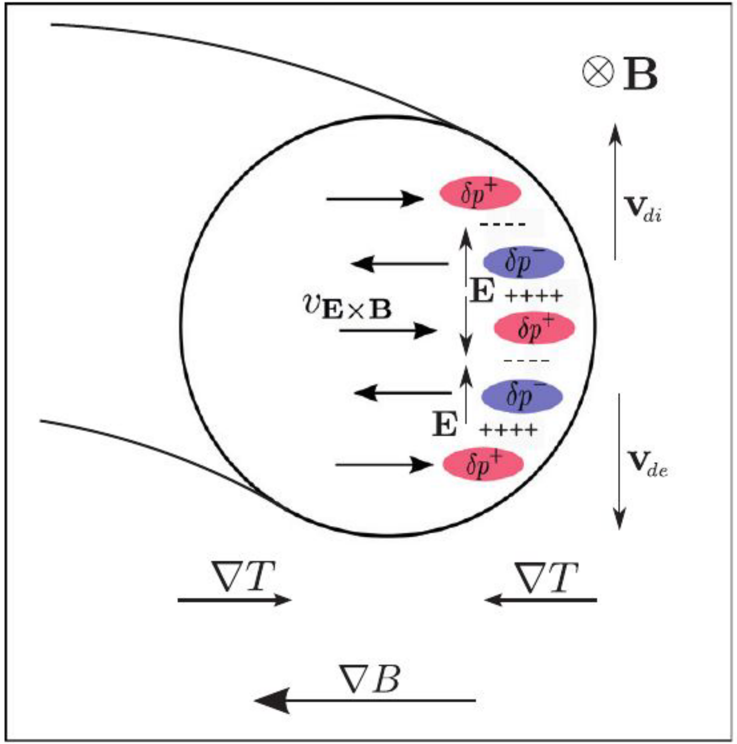
Simplest system possible ITG adiabatic electrons

Why it is so relevant? Understood as one of the most important turbulence drive mechanisms

$$\omega = \pm\sqrt{-2\frac{T_e}{T_i}(k_\theta \rho_i)^2 \nabla \log T \cdot \nabla \log B}$$

Unstable at the outboard midplane.

Electron contribution is not necessarily negligible. They are responsible for the TEM, the other major instability in a Tokamak.

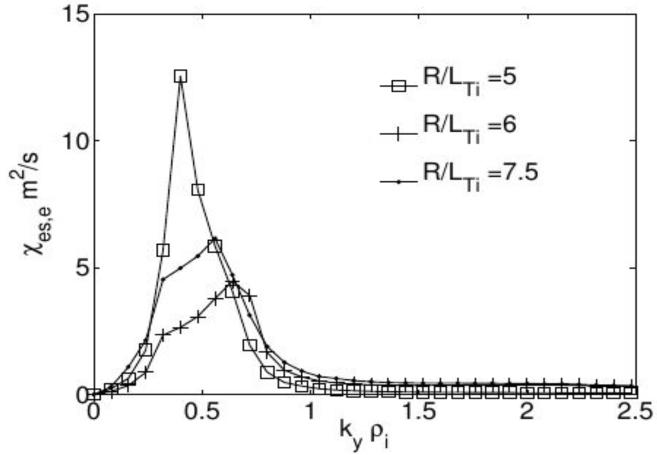


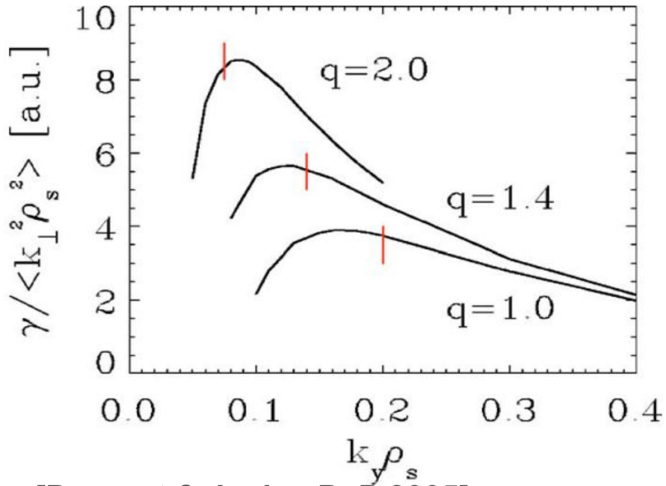
Mixing length estimates

Mixing length argument, balance linear growth and nonlinear convection $$D_{mix} = \chi \sim \gamma / k_\perp^2$$

Variations incorporating the eigenfuction work better

$$\langle k_\perp^2 \rangle (k_y) = \sum_{k_x} \int k_\perp^2 (k_x, k_y, z) \, |\hat{\phi}(k_x, k_y, z)|^2 \times J(z)\, dz \Big/ \sum_{k_x} \int |\hat{\phi}(k_x, k_y, z)|^2 J(z)\, dz$$

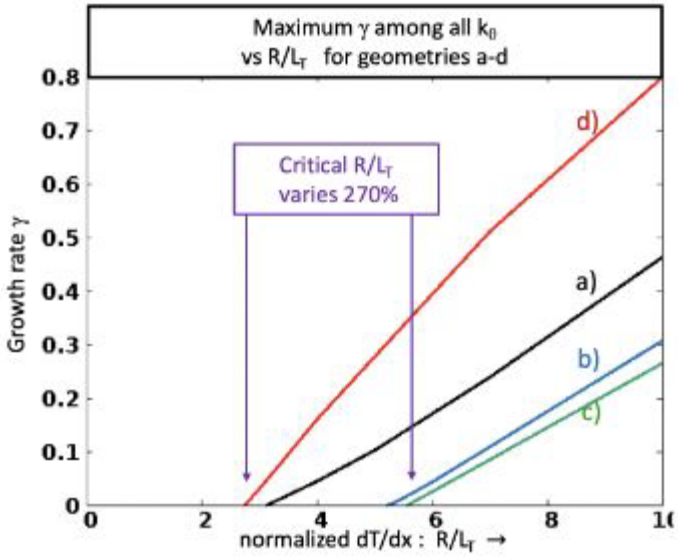




[Dannert & Jenko, PoP 2005]

Transport barriers

Let's consider a series of variations around nominal parameter sets representative of transport barrier conditions



Very strong variation of critical gradients with plasma parameters. Best is to use F_p the fraction of pressure gradient in density:

$$F_P = (T dn/dx)/[d(nT)/dx] = \frac{1/L_n}{(1/L_n + 1/L_T)}$$

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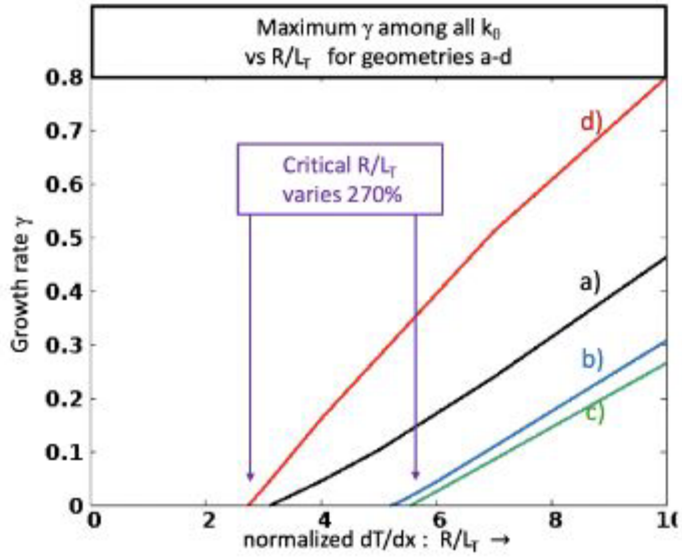
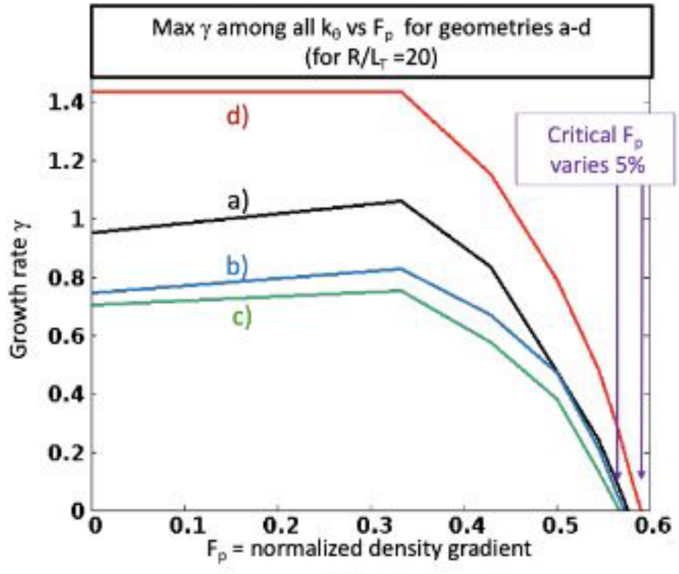
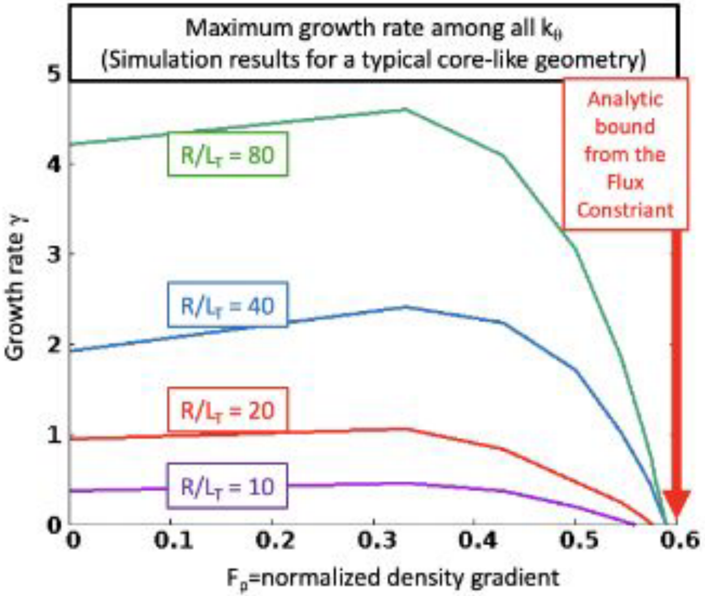


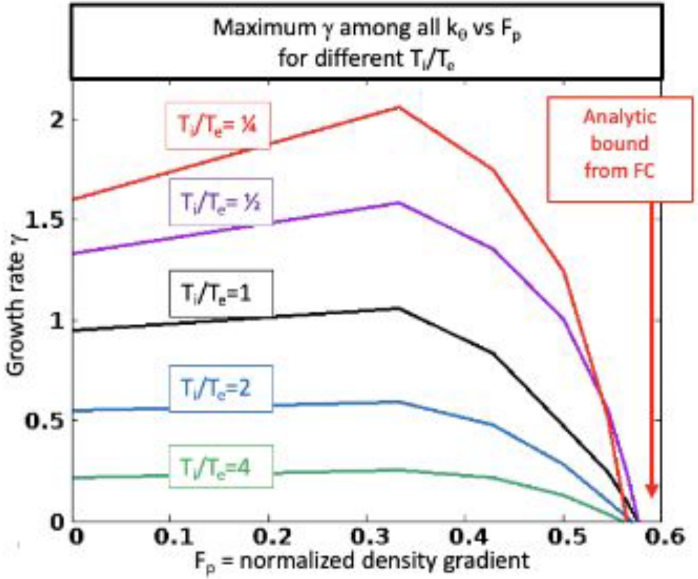


F_p is much more robust indicator of the intrinsic system behavior

Transport barriers

For a lot of variations





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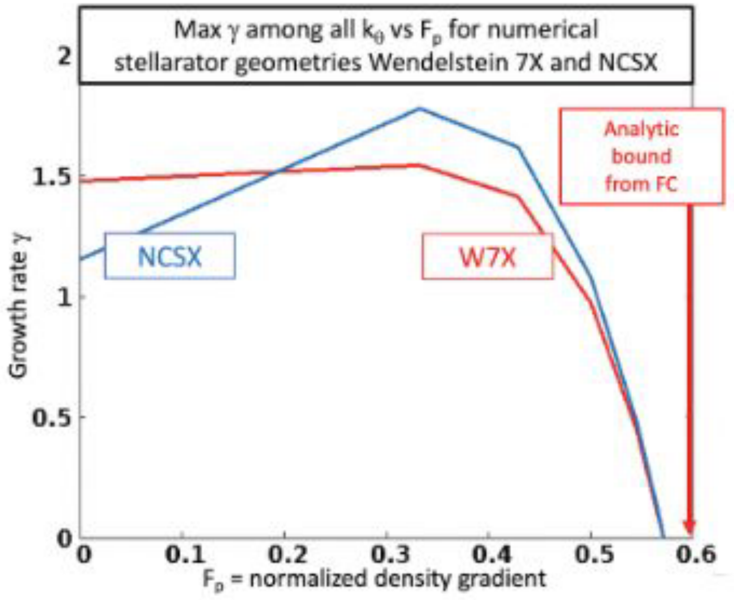

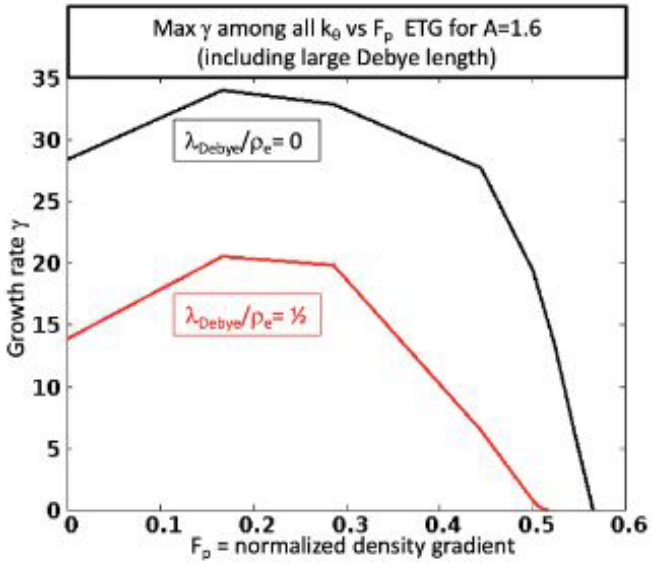

Regardless of the completely different parameters, the system follows the same behavior, why?

The charge Flux Constraint - FC

The reason of such a robust behavior stems from a very simple constraint

$$\sum_s q_s \Gamma_{rs} = 0$$

The charge weighted flux must be zero.

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This acts together with the energetics dynamics of the system

$$\frac{\partial}{\partial t} \sum_s [\frac{T_s}{2} \int d\boldsymbol{x} d\boldsymbol{v} \frac{\delta f_s^2}{F_M} + \frac{m_s n_s \delta V_{E\times B}^2}{2} + \frac{\delta E^2}{8\pi} + \frac{\delta B^2}{8\pi}] =$$

$$\sum_s [(Q_s \frac{1}{T_s} \frac{dT_s}{dx} + T_s \Gamma_s \frac{1}{n_s} \frac{dn_s}{dx})] - \sum_s \int d\boldsymbol{x} d\boldsymbol{v} \frac{\delta f_s}{F_M} \boldsymbol{C}(\delta f_s)$$

Free energy in fluctuations grows because of fluxes (minus entropy)

A mean field theory - SKiM

Gyrokinetics is extremely complicated, many degrees of freedom. Simplify it with a mean field theory (Simplified Kinetic Model - SKiM), constructed by taking an average along the field line:

$$< k_{||} >^2 = \frac{\int \mathrm{d}l |\frac{\partial \phi}{\partial l}|^2}{\int \mathrm{d}l |\phi|^2|}$$

$$< k_{\perp} >^2 = \frac{\int \mathrm{d}l |\phi|^2 k_{\perp}^2}{\int \mathrm{d}l |\phi|^2},$$

$$< \omega_{di} > = \frac{\int \mathrm{d}l |\phi|^2 \omega_{di}}{\int \mathrm{d}l |\phi|^2}$$

Eigenfunction averaged wave vectors and drift frequencies

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Non-adiabatic part of distribution function h

$$h_s = \frac{\frac{q_s \phi}{T_s} J_0(< k_{\perp} > \rho_s)(\omega - \omega_s^{\star}) f_{Ms}}{(\omega - < \omega_{ds} >) - v_{||} < k_{||} > + iC}$$

Yielding with QN the dispersion relation

$$\sum_{ion\ species} \left(\int \mathrm{d}\vec{v} \frac{F_{Ms} J_0(< k_{\perp} > \rho_i)^2 (\omega - \omega_s^{\star})}{\omega - < k_{||} > v_{||} - < \omega_{ds} >} - 1 \right) \frac{q_s^2}{T_s} - \frac{n_e e^2}{T_e} = 0$$

Eigenfunction averaged wave vectors and drift frequencies

One can solve SKiM and compare to “standard” dispersion relation, but there is some basic information one can extract

A mean field theory - SKiM

Compute the particle flux resulting from h gives

$$\int d\boldsymbol{v} f_M J_0(< k_\perp > \rho_i)^2 \left[\frac{\gamma}{\gamma^2 + (\omega_r - < k_\| > v_\| - < \omega_d >)^2}\right]$$
$$[1/L_n + (m_s v^2/2T_s - 3/2)/L_T] = 0$$

Cannot be solved if

$$1/L_n > 3/(2L_T)$$

or

$$F_P > 0.6$$

Simple prediction, no need to perform any complex GK simulation, valid for any fluctuation!

The violation of the charge flux constraint is extremely robust and fundamental.

FC vs energetics - a graphical picture

The crucial role of the FC can be more easily understood visually

Positive flux

Negative flux

$$D(\omega) = \sum_{ion\ species} \left(\int d\vec{v} \frac{F_{Ms} J_0(<k_\perp> \rho_i)^2 (\omega - \omega_s^\star)}{\omega - <k_\parallel> v_\parallel - <\omega_{ds}>} - 1 \right) \frac{q_s^2}{T_s} - \frac{n_e e^2}{T_e} = 0$$

Solved only if

$$Im(D(\omega)) = 0$$

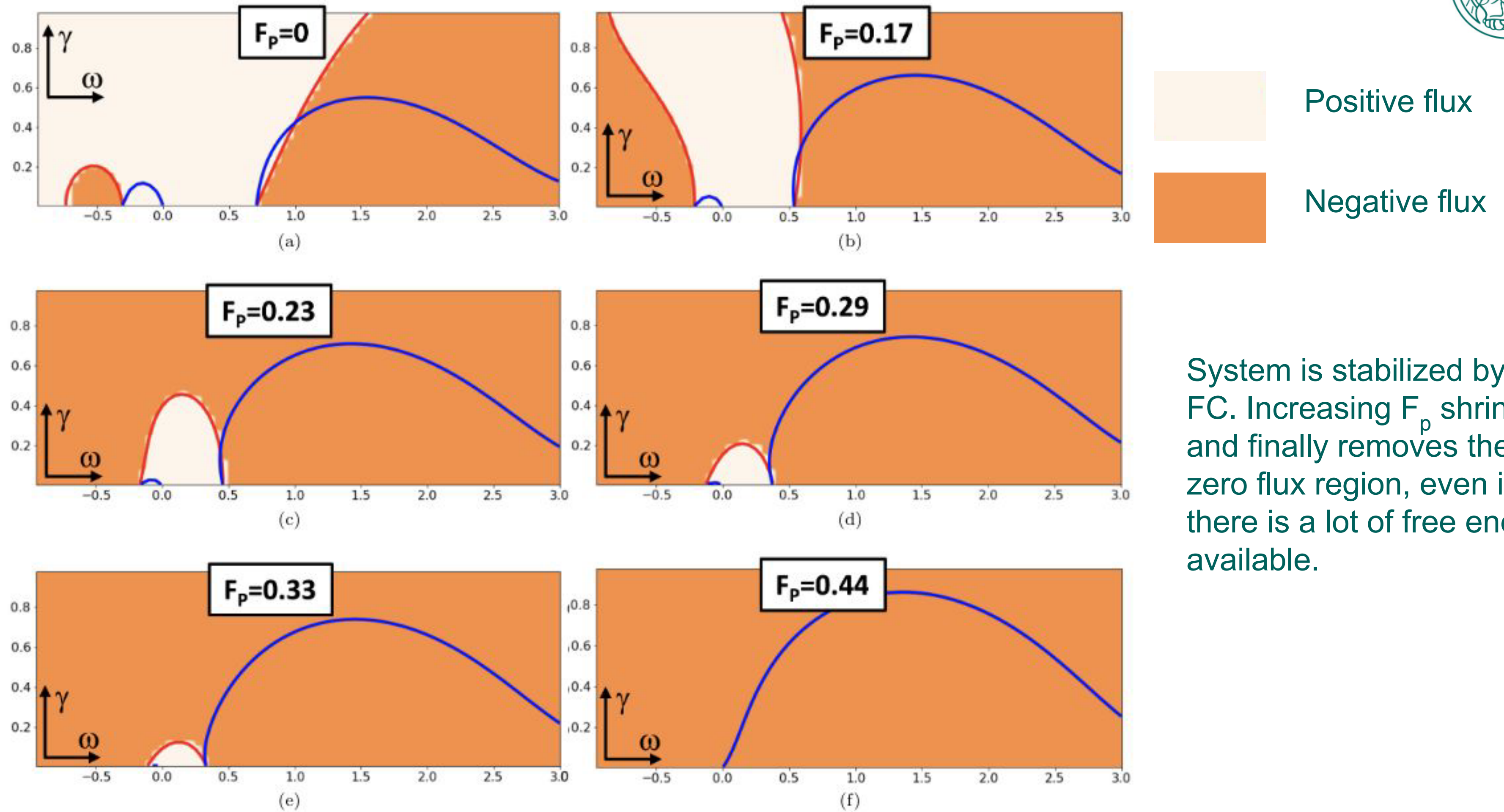
Flux constraint

$$Re(i\omega D(\omega)) = 0$$

Free energy

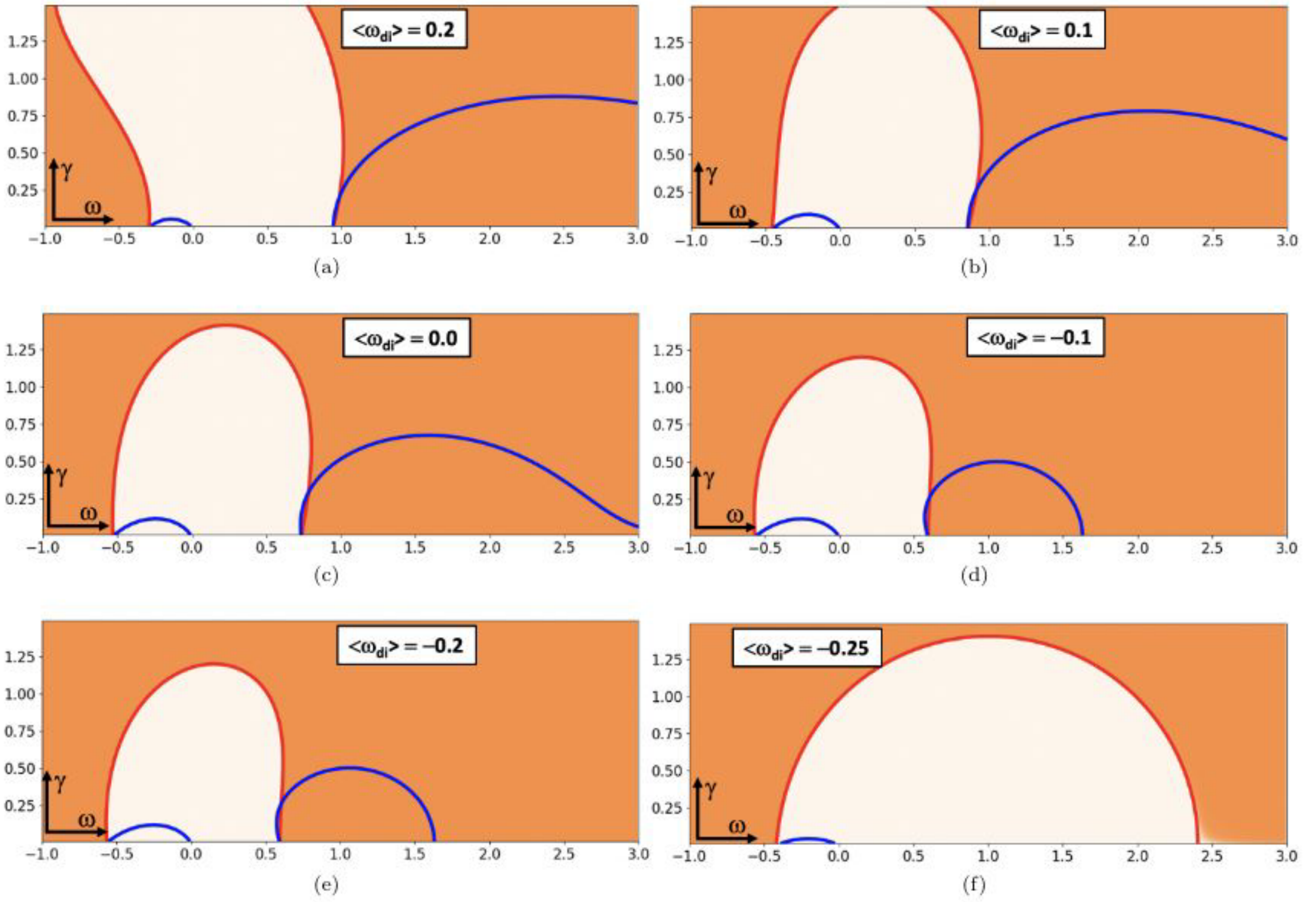
Solution of the system is the intersection of the two curves in the complex plane.
Both are important and provide two different dynamics to the system evolution

FC vs energetics - a graphical picture



System is stabilized by the FC. Increasing F_p shrinks and finally removes the zero flux region, even if there is a lot of free energy available.

FC vs energetics - a graphical picture



Positive flux

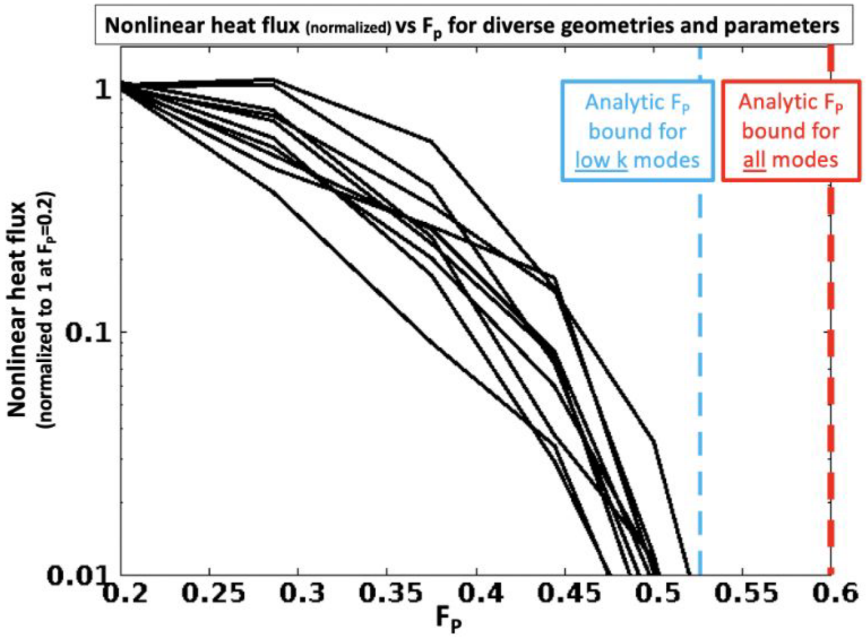
Negative flux

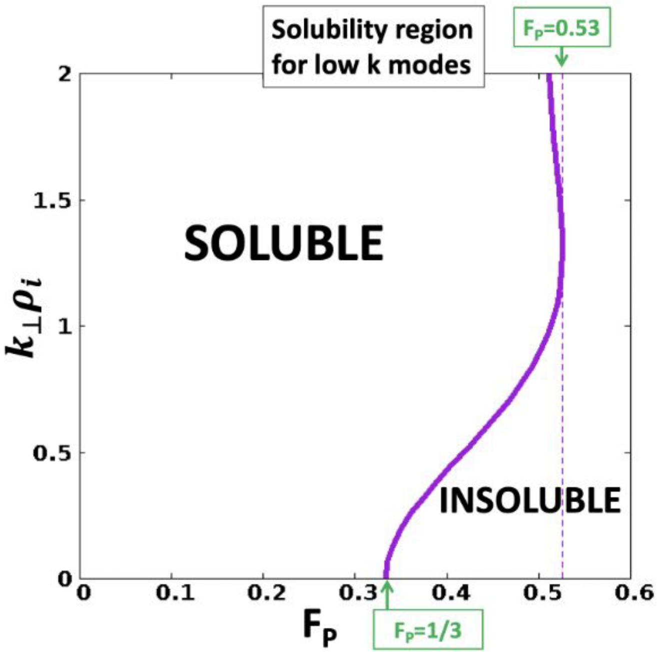
Scan in curvature, from positive to very negative

In this case the stabilization is via energetics, the FC can always be satisfied

What about nonlinear behavior?

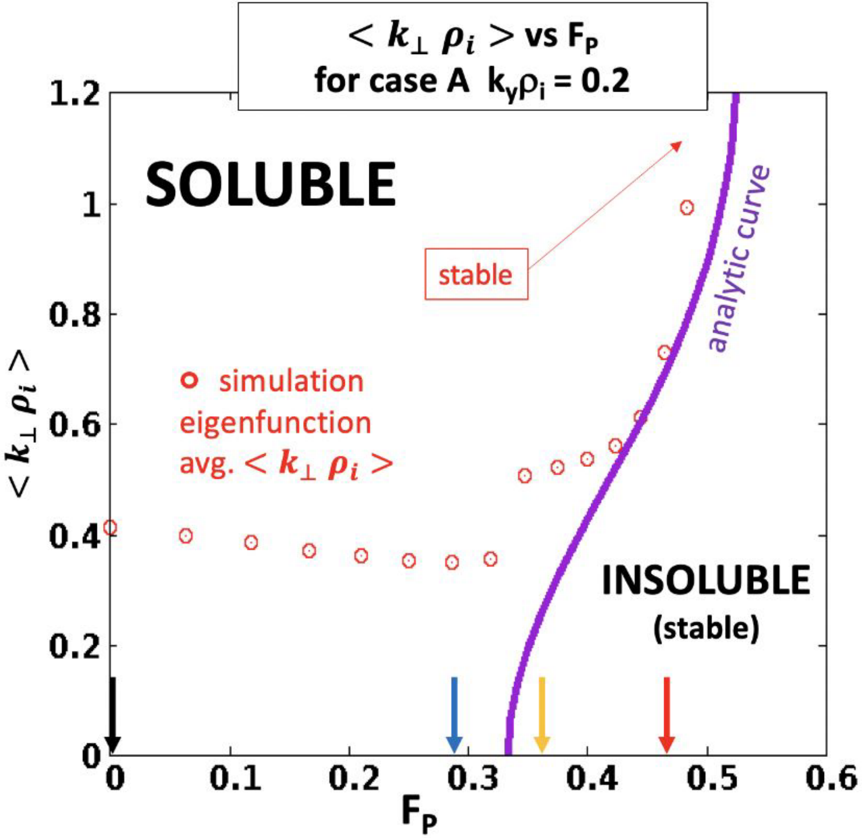
Nonlinear transport is dominated by low k_y modes & follows similar trend

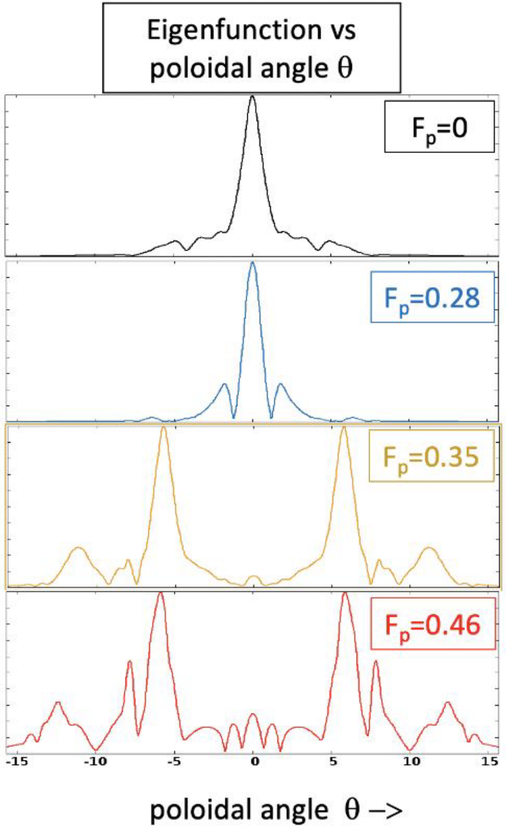




Dependence of F_p on details of the eigenfunction, max F_p~0.53 & k~1, but for low k_y dramatic changes once near 1/3

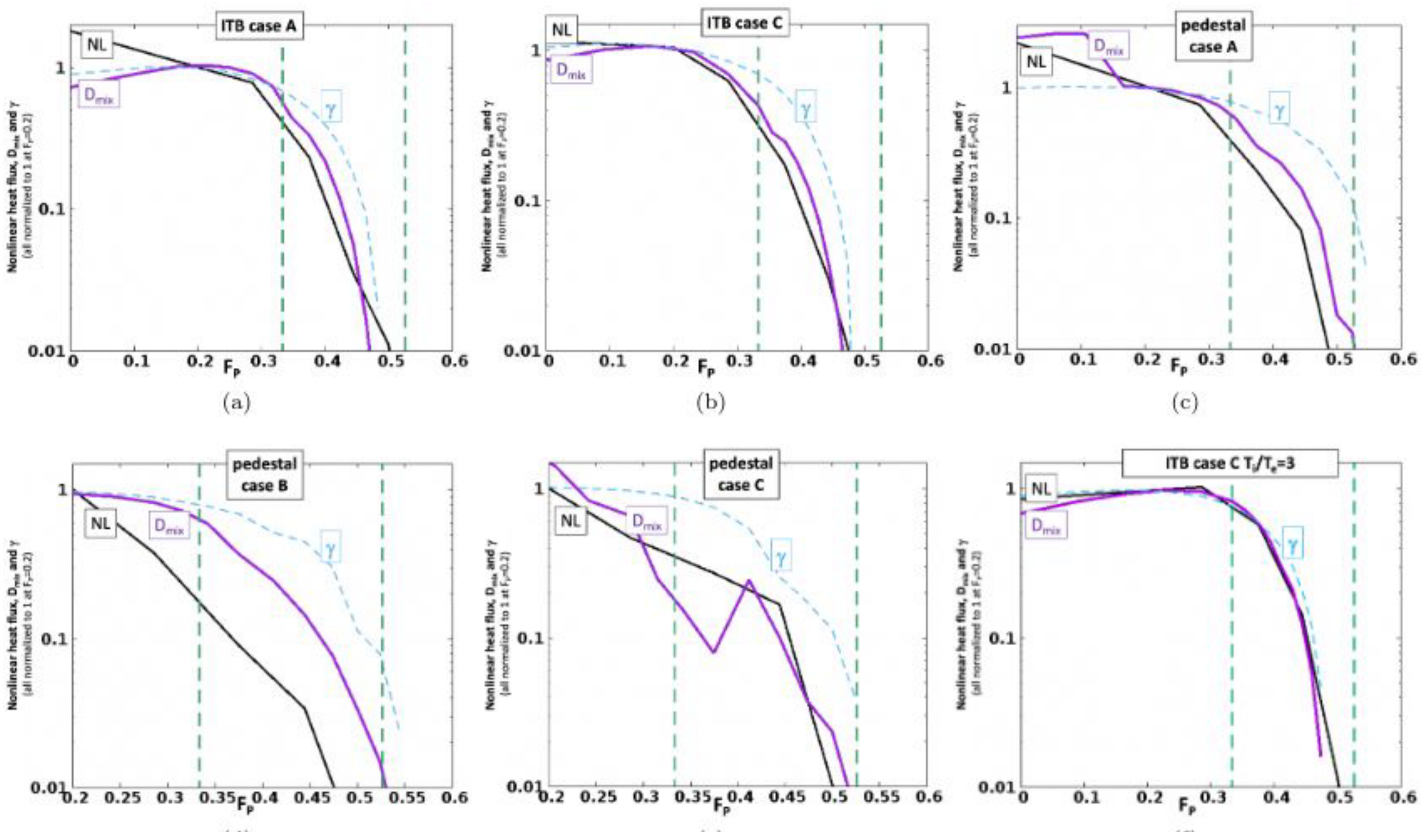
What about nonlinear behavior?





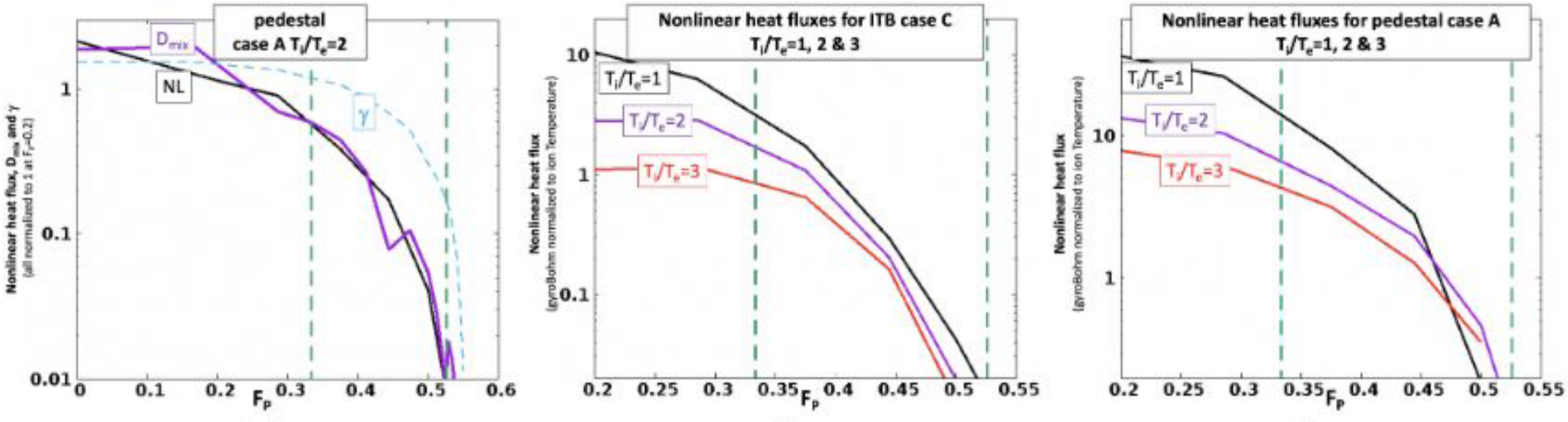
As F_p is increased and the solubility is approached, the system adapts with major changes in the eigenfunction

What about nonlinear behavior?



$k_\perp$ matters, adaptivity

What about nonlinear behavior?



Q_{NL} doesn't

Inclusion of kinetic electrons

Non-adiabatic electron response often plays a crucial role in gyrokinetic instabilities. For trapped electrons, it can lead to the TEM.

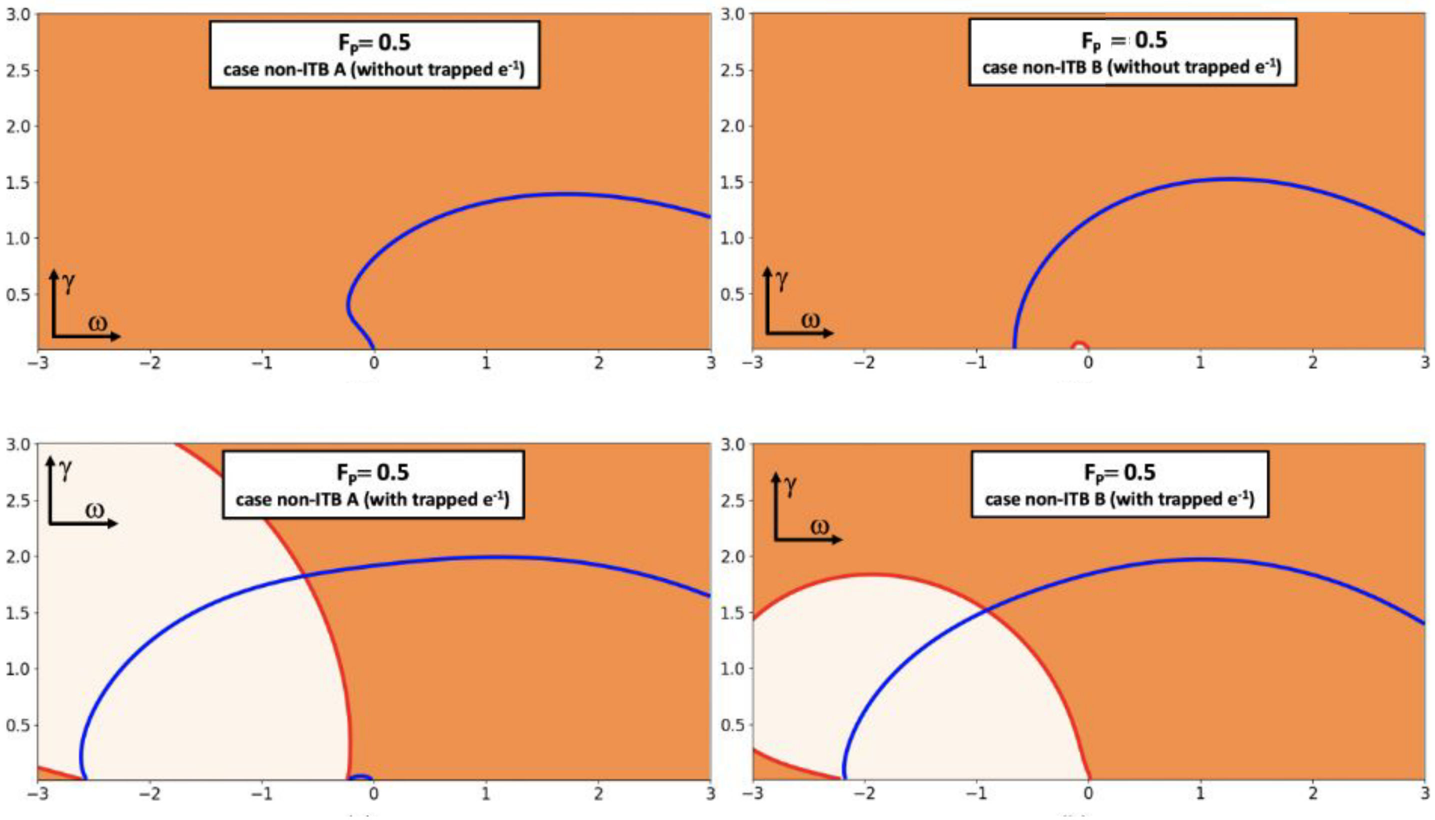
Since non-adiabatic electrons contribute charge flux in the opposite direction violation of FC becomes more difficult.

It is possible that when the non-adiabatic electron charge flux is large enough, density gradients do not, necessarily, lead to insolubility. This situation occurs for many typical tokamak geometries, in fact.

In these cases, the stabilization is greatly reduced because the FC remains soluble for much larger density gradients. But the flux constraint may still be potent enough to suppress instabilities and allow TB formation. Negative shear and/or large Shafranov shifts are experimental cases where TB for without velocity shear

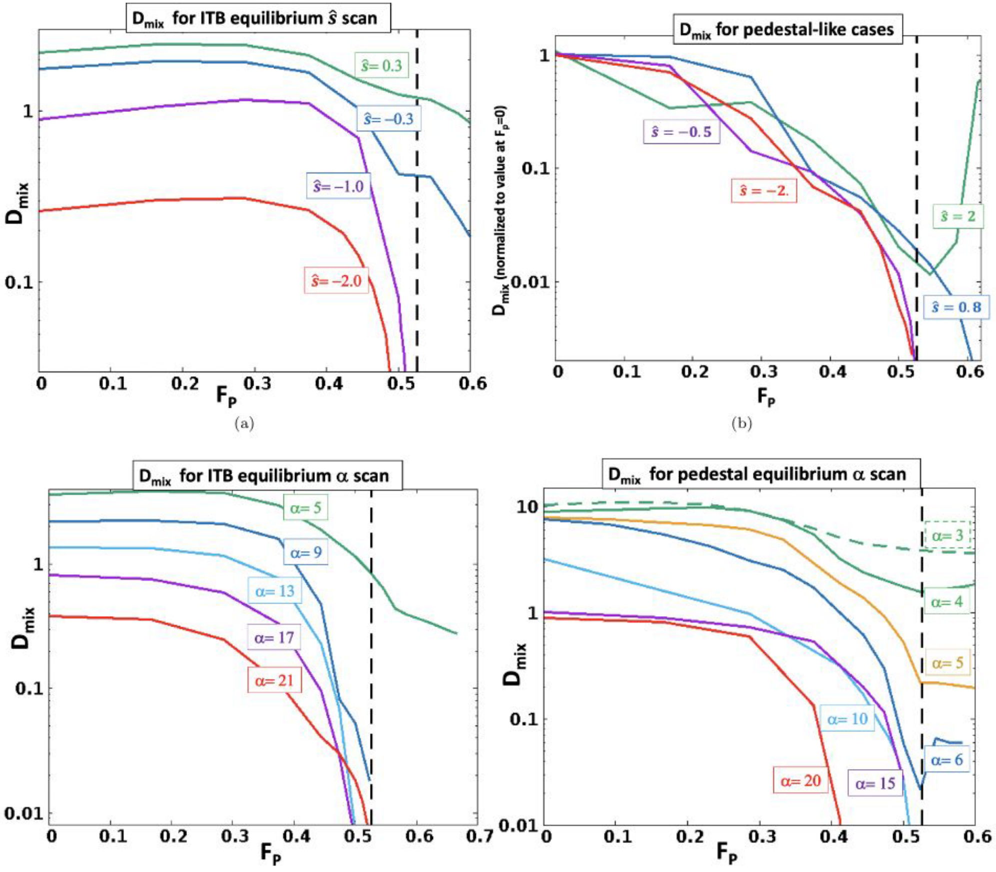
The FC still applies, and can still be used to understand how to drive instabilities!

Inclusion of kinetic electrons



There is more energetics from trapped particles (not really necessary) and FC is soluble -> instability appear

Inclusion of kinetic electrons

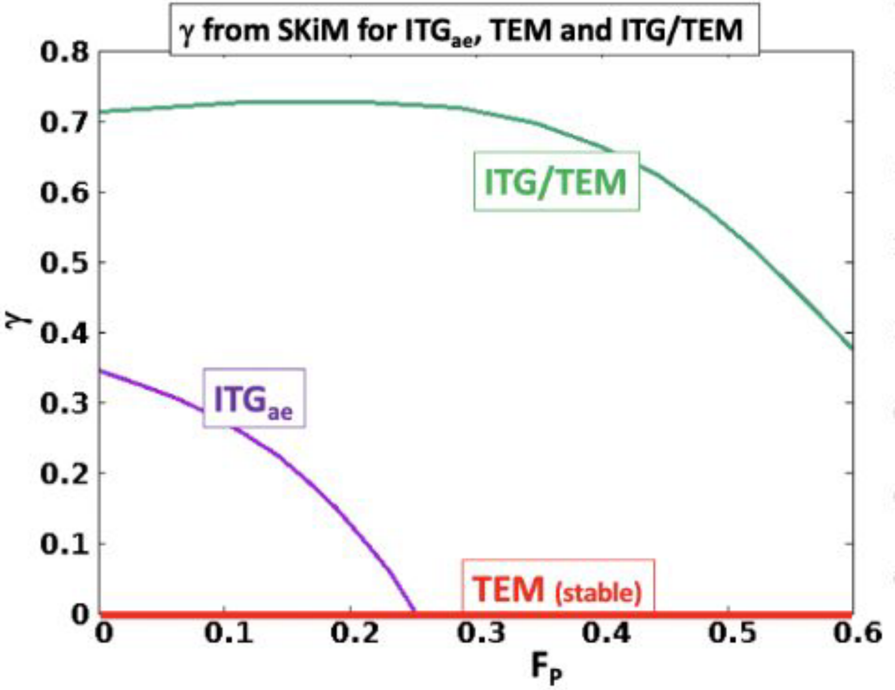


The behavior is however quite similar to what happened with adiabatic electrons.

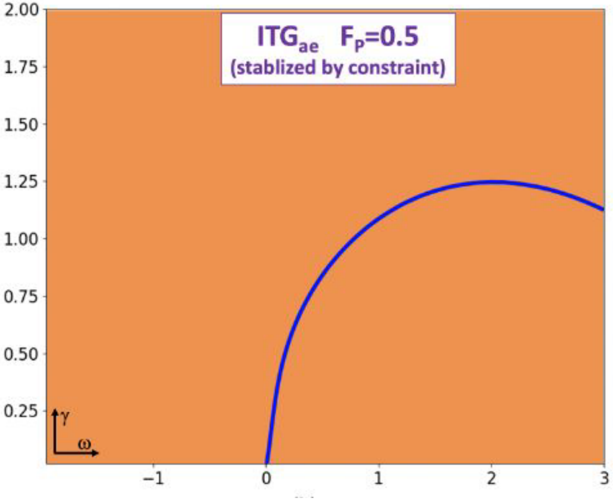
The system adapts to stabilizing effect (e.g. curvature by getting closer to adiabatic e)

Inclusion of kinetic electrons

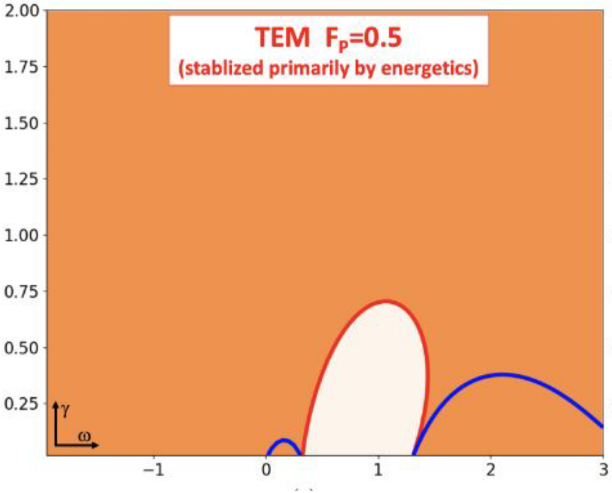
Consider a single case where we vary the gradients to (de)-stabilized a mode



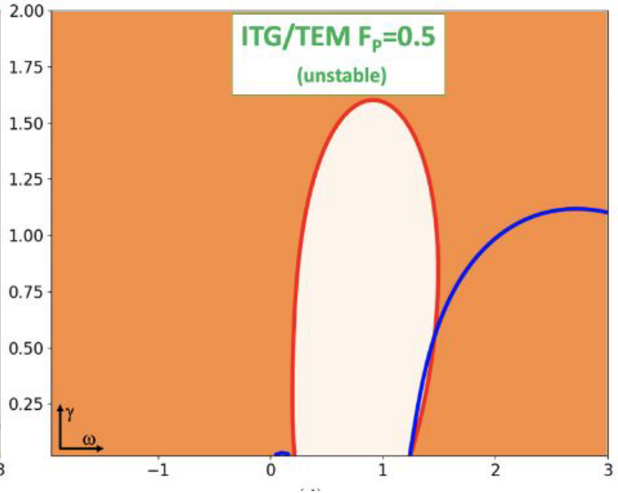
Inclusion of kinetic electrons



Stabilized by FC, as before



Free energy is too small, but FC can be satisfied

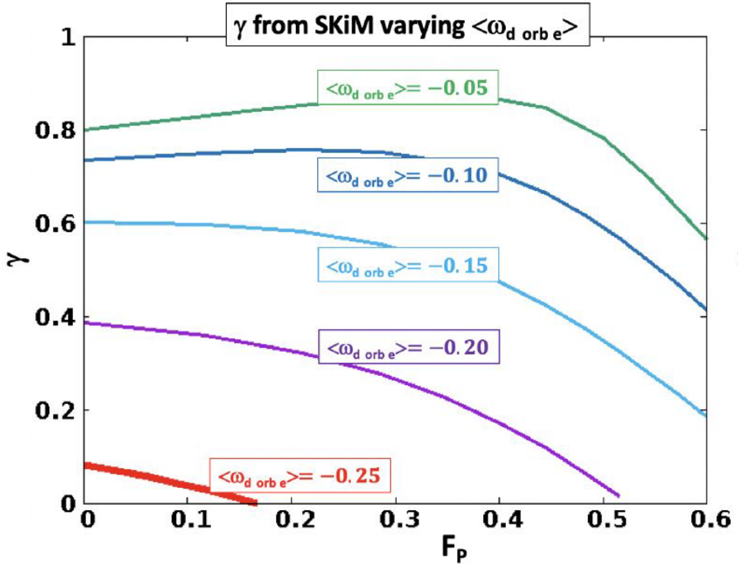


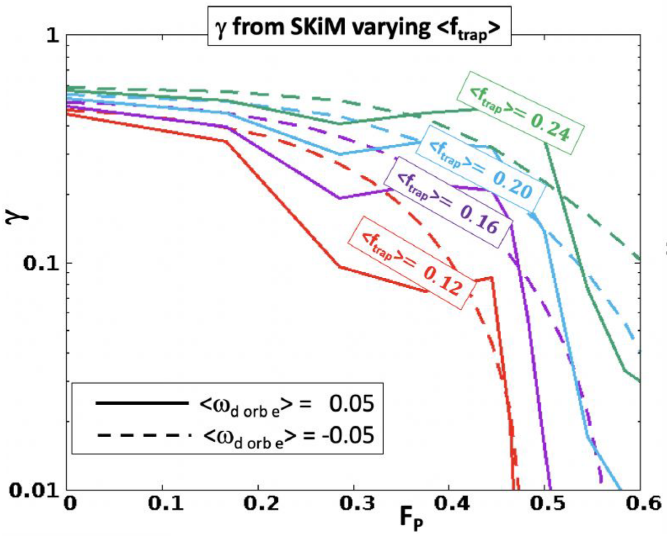
TEM provide FC, ITG free energy at the cost of lower growth rates

ITG and TEM are strongly coupled, their fate goes together

Inclusion of kinetic electrons

SKiM can be generalized to include kinetic electrons and used to understand how the system behaves.

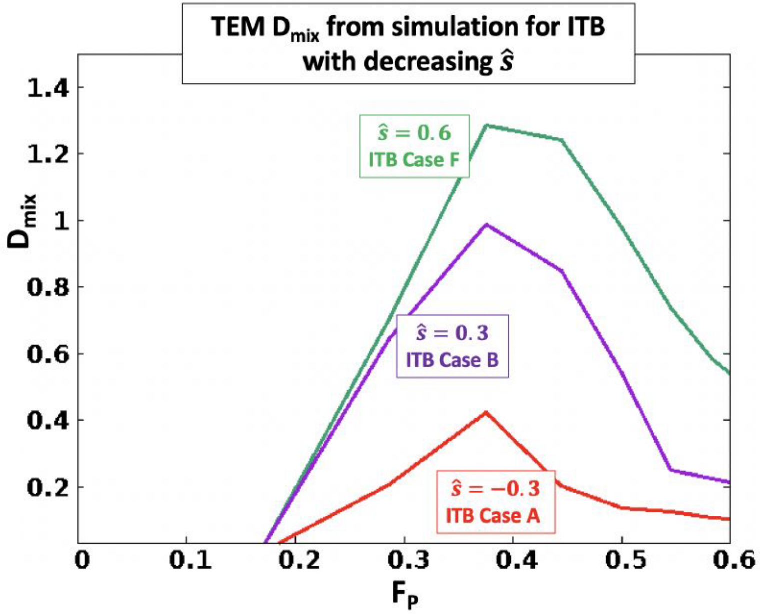




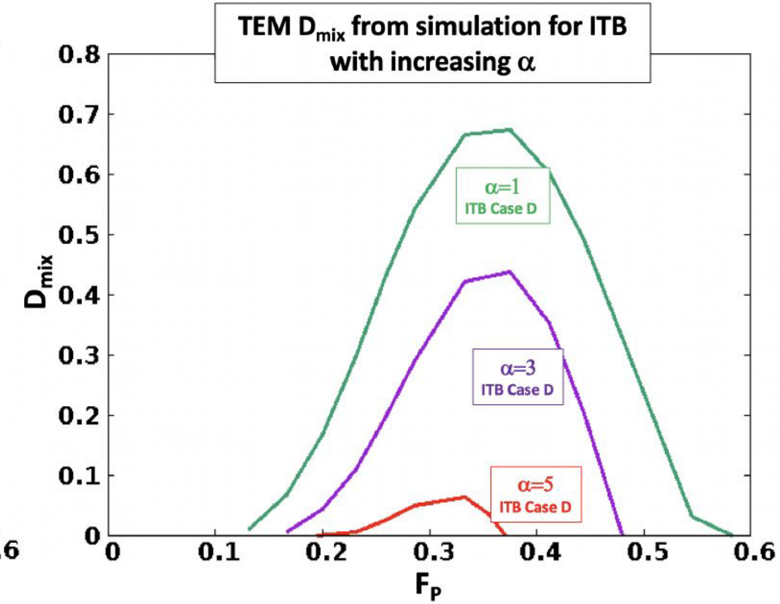
Stabilization is in fact the effect of a reduced effective trapped fraction.
The eigenfunction adapts to avoid curvature stabilization, and in doing so decouples from trapped electrons -> flux decrease

Inclusion of kinetic electrons

This is the regime we want, adaptation makes non-adiabatic response small, hence the FC will be dominating stability



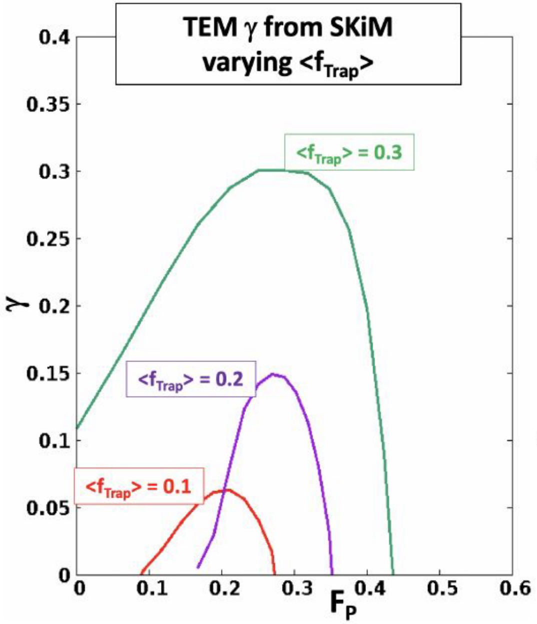
Reduce <f>$_{trap}$

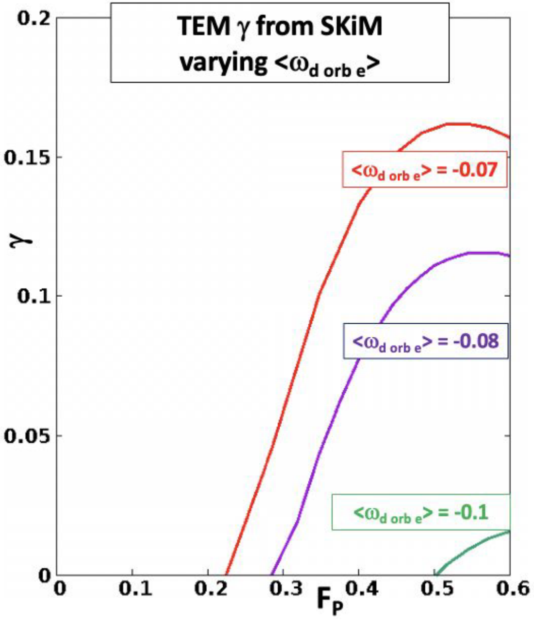


Reduce curvature, FC via adaptation

Inclusion of kinetic electrons

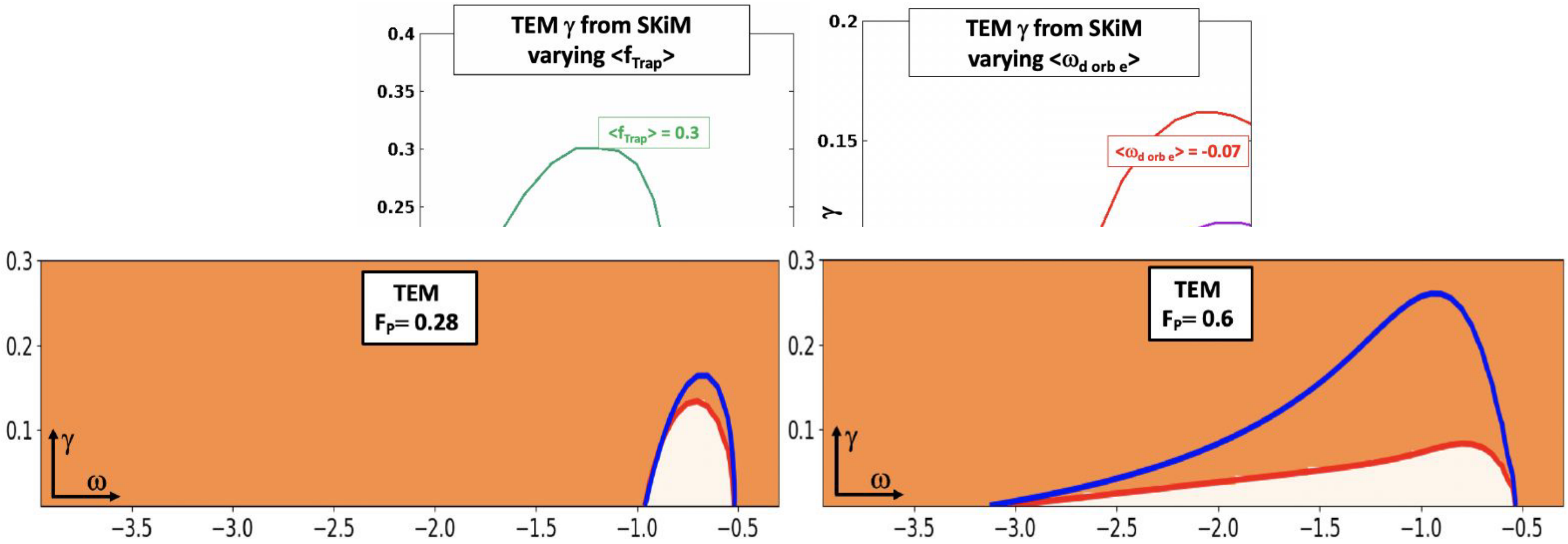
SKiM confirms that the stabilization is due to <f_{trap}> and FC actively working



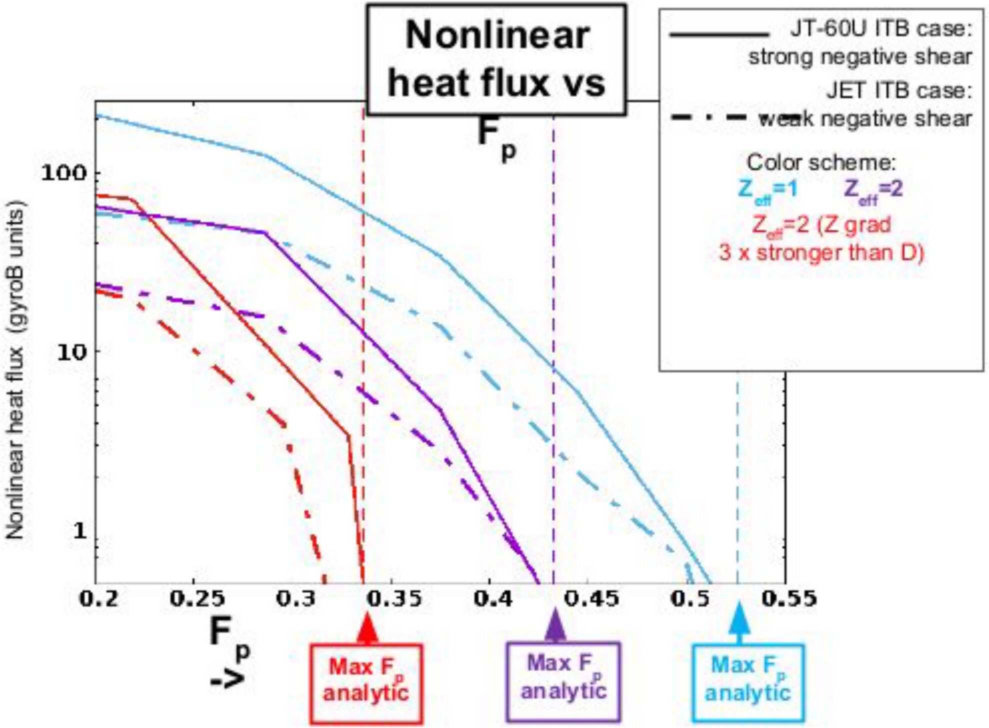


Inclusion of kinetic electrons

SKiM confirms that the stabilization is due to $<f_{trap}>$ and FC actively working

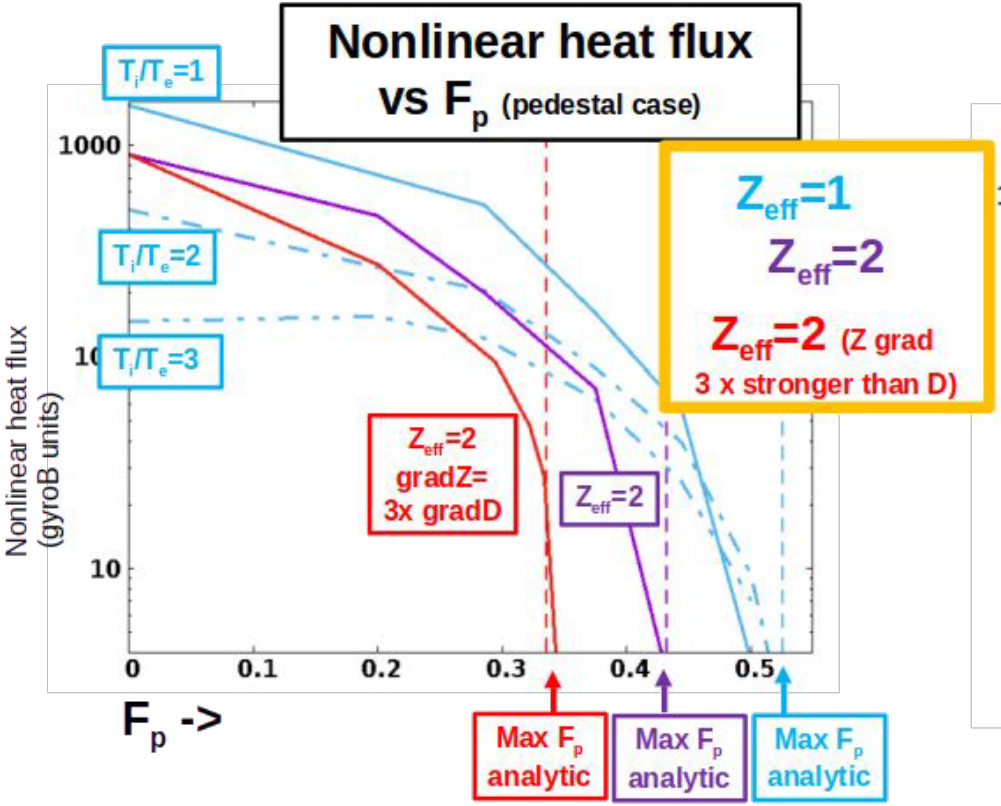


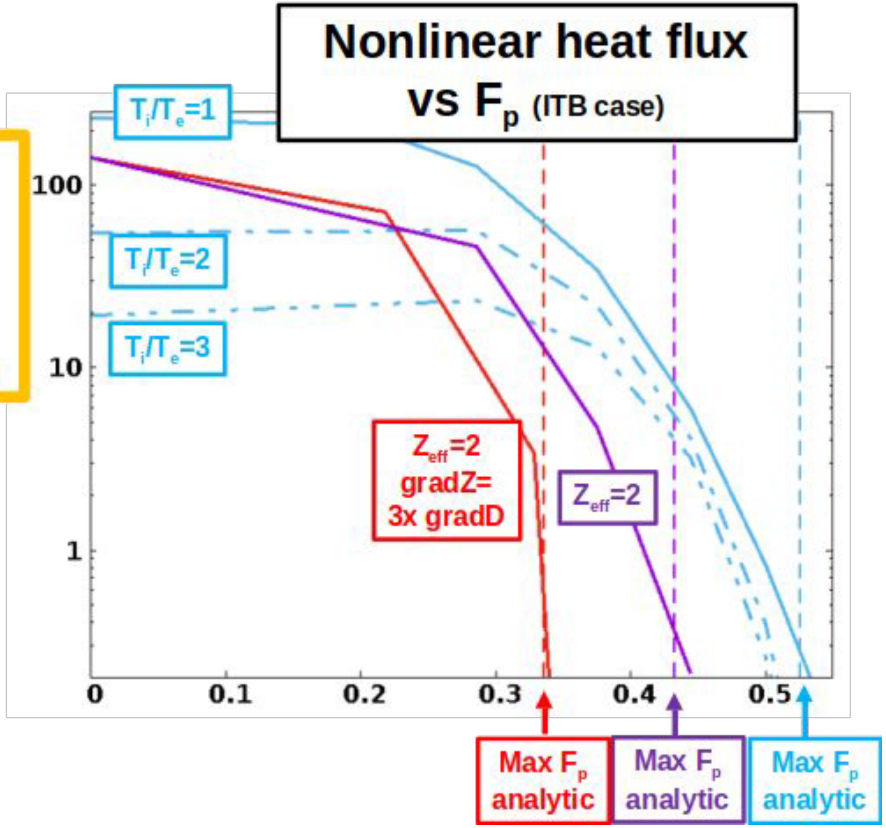
Impurities can amplify the FC effect



As we approach the limit F_p heat flux goes down by ~2 orders of magnitude, enough for a TB

The FC overpowers energetics





Summary

Progress towards understanding the TB formation

There are two basic dynamics that regulate the system: free energy is available for microinstability to grow if they satisfy basic laws: the radial charge flux must be zero.

More energy means more transport but only if the FC is not violated, otherwise the system cannot sustain instabilities.

A way to produce a TB is to drive the system towards conditions where violation of the FC is approached. Not an easy or granted, but there are knobs that one can leverage, i.e. tuning density gradients and impurity content.

FC solubility bounds with impurities