Equivariant Localization in Supergravity

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Introduction

- Equivariant localization:
  Powerful new method to compute supersymmetric observables in supergravity **without solving the supergravity equations of motion!**

- Several applications in supersymmetric holography:

  - Susy $AdS_D \times M$ solutions of D=10/11 SUGRA, dual to $d = D - 1$ SCFTs; compute central charge, Δ of some op's.

  - Susy black hole solutions in AdS ; compute S_{BH}

  - SCFTs on M ; compute $I = -\log Z$

Susy solutions of supergravity (general D)

- Equations of motion for bosonic fields

Einstein equations: $R_{\mu\nu} = ...$

Matter/flux equations of motion

- Supersymmetry $\Leftrightarrow$ Killing spinors ϵ

$$\nabla\epsilon + (\text{flux}\cdot\Gamma)\epsilon = 0$$

$$(\text{flux}\cdot\Gamma)\epsilon = 0$$

Susy solutions with an R-symmetry Killing vector:

$$\xi = \bar{\epsilon}\Gamma^{\mu}\epsilon\partial_{\mu} \qquad \mathcal{L}_{\xi}\epsilon \equiv \xi^{\mu}\nabla_{\mu}\epsilon + \frac{1}{8}d\xi_{\mu\nu}\Gamma^{\mu\nu}\epsilon = iq\epsilon$$

Very general

Define equivariant exterior derivative: $d_\xi = d - \xi \lrcorner$

Satisfies $(d_\xi)^2 = -\mathcal{L}_\xi$

There is a corresponding equivariant cohomology

More general spinor bi-linears:

$\Psi_{(r)} = \bar{\epsilon}\Gamma_{(r)}\epsilon$ where $\Gamma_{(r)} \equiv \frac{1}{r!}\Gamma_{\mu_1\dots\mu_r} dx^{\mu_1}\wedge \dots dx^{\mu_r}$

$\tilde{\Psi}_{(r)} = \bar{\epsilon}^c\Gamma_{(r)}\epsilon$

Results from around 20 years ago: [Gauntlett,Pakis 02] [Gauntlett,Martelli,Pakis,Waldram 02]

The Killing spinor ϵ defines a G-structure

$\Leftrightarrow$

Algebraic and Differential relations on $\Psi_{(r)}$, $\tilde{\Psi}_{(r)}$

New Observation:

Construct multi-forms $\Phi = \Phi_{2n} + \Phi_{2n-2} + \ldots \Phi_0$ which are polynomials in the supergravity fields and the bilinears $\Psi_{(r)}$ that are equivariantly closed: $d_\xi \Phi = 0$

Comments:

- Only need to impose subset of the KSEs/equations of motion: Such structures exist partially off-shell
- Typically several Φ exist in a given theory
- Several physical quantities can be computed using the equivariant cohomology of Φ . The topology and R-symmetry is enough information - don't need to solve SUGRA pdes - just need to assume solution exists

Berline-Vergne-Atiyah-Bott theorem

Let M be a ξ invariant closed (sub)manifold, 2n-dimensional

$$\int_M \Phi = \sum_{\substack{F \\ \mathrm{codim} F = 2k}} \frac{1}{d_F} \frac{(2\pi)^k}{\prod_{i=1}^k b_i} \int_F \frac{f^* \Phi}{\prod_{i=1}^k \left[1 + \frac{2\pi}{b_i} c_1(L_i)\right]} \, .$$

- $f : F \hookrightarrow M$, where $\xi = 0$
- b_i are the weights of action of ξ on $N_F = \oplus_{i=1}^k L_i$

$$\xi = \sum_{i=1}^k b_i \frac{\partial}{\partial \phi_i}$$

- Normal space can have orbifold singularities $N_F \cong \mathbb{R}^{2k}/\Gamma$ with Γ a finite group of order d_F

Berline-Vergne-Atiyah-Bott theorem

$$\int_M \Phi = \sum_{\text{dim } 0} \frac{1}{d_{F_0}} \frac{(2\pi)^n}{b_1 \cdots b_n} \Phi_0$$

$$+ \sum_{\text{dim } 2} \frac{1}{d_{F_2}} \frac{(2\pi)^{n-1}}{b_1 \cdots b_{n-1}} \int \left[\Phi_2 - \Phi_0 \sum_{1 \le i \le n-1} \frac{2\pi}{b_i} c_1(L_i) \right]$$

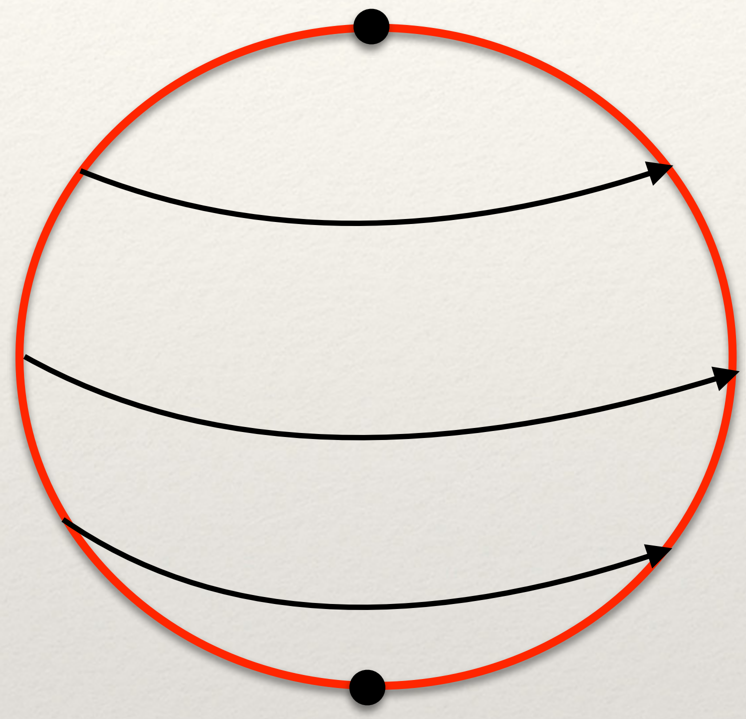
$$+ \dots$$

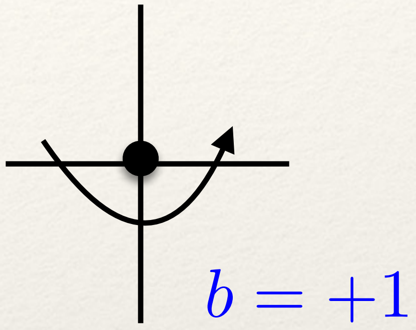
Simple example: S^2

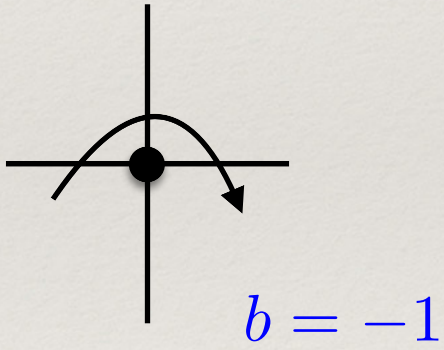
$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \qquad \xi = \partial_\phi$$

$$\Phi = \sin\theta d\theta \wedge d\phi + \cos\theta$$

$$d_\xi \Phi = 0$$







$$\int_{S^2} \mathrm{vol} = \int_{S^2} \Phi$$

$$= 2\pi \left(\frac{\cos\theta}{b}\bigg|_N + \frac{\cos\theta}{b}\bigg|_S \right)$$

$$= 2\pi \left(\frac{1}{1} + \frac{-1}{-1} \right) = 4\pi$$

Example 1: $AdS_5 \times M_6$ [Gauntlett,Martelli,Sparks,Waldram 04]

- Consider most general solutions of D=11 supergravity that are dual to $\mathcal{N}=1$ SCFTs in d=4

$$ds^2 = e^{2\lambda}\left[ds^2(AdS_5) + ds^2(M_6)\right]$$

$G_{(4)}$ a four-form on M_6 ; flux quantisation $cG_{(4)} \in H^4(M_6, \mathbb{Z})$

- Susy: D=11 spinor $\varepsilon = \psi \otimes \epsilon$

  where ψ a Killing spinor on AdS_5 and ϵ on M_6 satisfies the KSEs:

$$\left[\nabla_m + \frac{i}{2}\gamma_m\gamma_* - \frac{1}{4}(e^{-3\lambda}G\cdot\gamma)_m\right]\epsilon = 0$$

$$\left[\gamma\cdot d\lambda - i\gamma_* + \tfrac{1}{3!}(e^{-3\lambda}G\cdot\gamma)\right]\epsilon = 0$$

- Construct bi-linears

$$1 = \bar{\epsilon}\epsilon \qquad \sin\zeta = -i\bar{\epsilon}\gamma_* \epsilon$$

$$K = \bar{\epsilon}\gamma_{(1)}\epsilon \qquad \xi^\flat = \frac{1}{3}\bar{\epsilon}\gamma_{(1)}\gamma_*\epsilon$$

$$Y = -i\bar{\epsilon}\gamma_{(2)}\epsilon \qquad Y' = \bar{\epsilon}\gamma_{(2)}\gamma_*\epsilon$$

Comments:

- $\xi^\flat$ is a one-form dual to an R-symmetry Killing vector - dual to the R-symmetry of the dual d=4 SCFT

- More bilinears exist using ϵ^c, which we won't need (all were used in the G-structure analysis of [Gauntlett,Martelli, Sparks,Waldram 04])

$\Rightarrow$ Using only a subset of information in KSEs/equations of motion

- Convenient definition: $y = \frac{1}{2} e^{3\lambda} \sin\zeta$ $\qquad dy = e^{3\lambda} K$

- Equivariantly closed forms:

$$\boxed{\Phi = e^{9\lambda} \mathrm{vol}_6 + \frac{1}{12} e^{9\lambda} * Y - \frac{1}{36} y e^{6\lambda} Y - \frac{1}{162} y^3} \qquad d_\xi \Phi = 0$$

$$a \propto \int_{M_6} \Phi$$

i.e. the a central charge localizes!

$$\boxed{\Phi^G = G_{(4)} - \frac{1}{3} e^{3\lambda} Y' + \frac{1}{9} y} \qquad d_\xi \Phi^G = 0$$

Flux quantization can be implemented via localization!

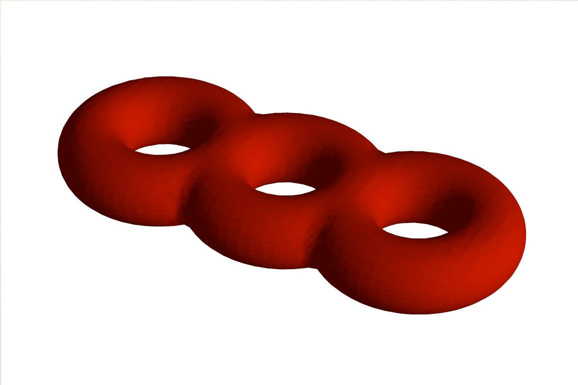
$$c \int_{C_4} G_{(4)} \in \mathbb{Z} \qquad C_4 \in H_4(M_6, \mathbb{Z})$$

Putting these tools to work ("computing a without trying")

- Specify topology of M_6 and action of R-symmetry
- Compute integrals via BVAB

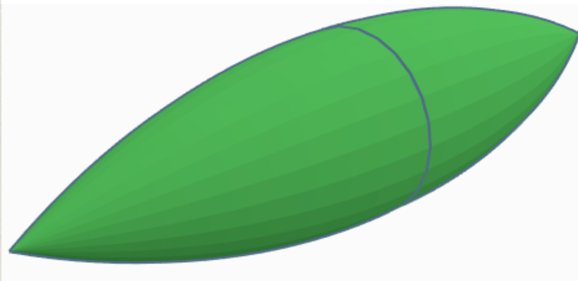
- Solutions arising from wrapping M5-branes $\Sigma \subset CY_3$ with

a) $\Sigma = \Sigma_g$ a genus g Riemann surface



[Maldacena,Nunez 00][Bah,Beem,Bobev,Wecht 12]

b) $\Sigma = \Sigma(n_+, n_-)$ a spindle



[Ferrero,Gauntlett,Martelli,Sparks 21]

The CY can be viewed as the total space of

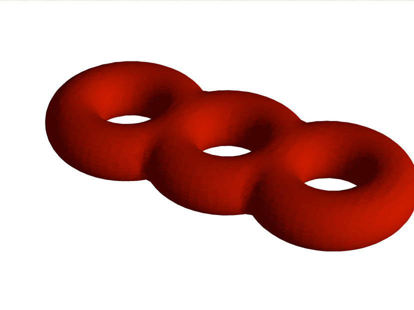
$\mathcal{O}(-p_1) \oplus \mathcal{O}(-p_2) \to \Sigma$ with $p_1 + p_2 = \chi(\Sigma)$

IF this gives rise to a d=4 CFT there will be a $AdS_5 \times M_6$ solution with $S^4 \rightarrow M_6 \rightarrow \Sigma$ and the fibration determined by

$S^4 \subset \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \mathbb{R}$ with $\mathbb{C}_i$ twisted using $\mathcal{O}(-p_i)$

And the solution will have an R-symmetry Killing vector

a) $\Sigma = \Sigma_g$: M5-branes wrapping a Riemann surface



Let ∂_{φ_i} rotate the $\mathbb{C}_i$ and hence give KVs acting S^4

Write $\xi = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2}$

For $b_i \neq 0$ the KV has fixed points at N and S poles of S^4 and the fixed point set is Σ_g

Spinor at the poles is chiral $\Rightarrow b_1 + b_2 = 1$

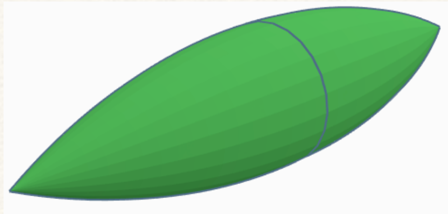
Now use BVAB/localisation:

- Flux quantisation $$c \int_{S^4} G_{(4)} = N$$
- Central charge $$a = -\frac{9}{8} b_1 b_2 (b_1 p_2 + b_2 p_1) N^3$$

Comments:

- This is an off-shell result. On-shell result obtained by extremizing $a(b_i)$ subject to $b_1 + b_2 = 1$ and agrees with known sugra solutions
- Result agrees with off-shell computation in field theory
- SUGRA computation using BVAB is much simpler than constructing sugra solution

b) $\Sigma = \Sigma(n_+, n_-)$: M5-branes wrapping a spindle



Now write $\xi = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} + b_0 \partial_{\varphi_0}$

where ∂_{φ_0} is the KV which rotates the spindle

For $b_i \neq 0$ the KV now has 4 isolated fixed points located at the N and S poles of the S^4 and the + and - poles of the spindle. Now find:

$$a = \frac{9}{16 b_0}[(b_1^+ b_2^+)^2 - (b_1^- b_2^-)^2] N^3$$

with constraints: $b_1^\pm + b_2^\pm = 1 \mp \frac{b_0}{n_\pm}$ $\qquad b_i^+ - b_i^- = -\frac{p_i}{n_+ n_-}$

- After extremizing $a(b_0, b_i)$ one obtains the result for the explicit SUGRA solution
- Result agrees with off-shell computation in field theory [Ferrero,Gauntlett,Martelli,Sparks 21] (the form is a "gravitational block" [Hosseini,Hristov,Zaffaroni 19])

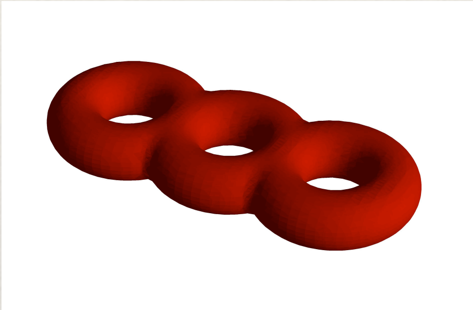
Example 2: D=4 Gauged Supergravity and black hole entropy

- Consider D=4 $\mathcal{N}=2$ gauged SUGRA coupled to matter

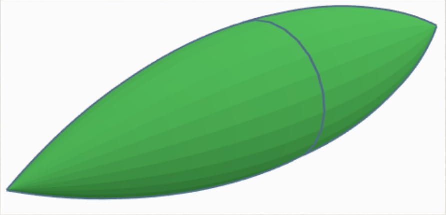
- Focus on $AdS_2 \times M_2$ solutions $F^I = p^I \mathrm{vol}(M_2)$

- These arise as the near horizon limit of susy magnetically charged black holes with event horizons:

$M_2 = \Sigma_g$



$M_2 = \Sigma(n_+, n_-)$



black holes are accelerating

- Find that the black entropy can be evaluated using localisation. New version of attractor mechanism.

- E.g. $\mathcal{N}=2$ gauged SUGRA coupled to n vector multiplets

Metric: $g_{\mu\nu}$

$n+1$ gauge fields: A^I $I=0,1,\ldots,n$

n complex scalars: $X^I(z^i)$ $i=1,\ldots,n$

parametrize special Kähler manifold $\mathcal{K}$

Holomorphic prepotential $\mathcal{F}(X^I)$

FI gauge parameters ξ^I

- Construct equivariantly closed forms using bilinears and then use BVAB…focus on the spindle case, which has fixed points at poles

- Result:

"Dressed scalars": $x^I \equiv e^\lambda e^{\mathcal{K}/2} X^I P$

$$S_{BH} = -\frac{\pi}{2G}\frac{1}{2b_0}[\mathcal{F}(x^I_+) - \sigma\mathcal{F}(x^I_-)]$$

$$\xi_I x^I_+ = 2 + \frac{2b_0}{n_+} \qquad \xi_I x^I_- = 2 - \sigma\frac{2b_0}{n_-}$$

$$p^I = \frac{1}{2b_0}(x^I_+ - x^I_-)$$

$\sigma = +1$ "twist"

$\sigma = -1$ "anti-twist"

- Off-shell result; need to extremize over $b_0, x^I_\pm$

- Agrees with results for known $AdS_2 \times \Sigma(n_+, n_-)$ sugra solutions

Comments:

- For S^2 : set $n_\pm = 1$ and $b_0 \to 0$

$$S_{BH} = -\frac{\pi}{2G} p^I \partial_I \mathcal{F}(x^I)$$

$$\xi_I x^I = 2 \qquad\qquad \xi_I p^I = 2$$

Attractor result

[Cacciatori,Klemm 09]
[Dall'Agata,Gnecchi 10]

- For STU model, uplift solutions on S^7

$$AdS_2 \times Y_9 \qquad\qquad S^7 \to Y_9 \to \Sigma(n_+, n_-)$$

has a GK geometry

[Kim,Park 06][Gauntlett,Kim 07]

[Couzens,Gauntlett,Martelli,Sparks 18]
[Boido,Gauntlett,Martelli,Sparks 22][…]

- Can add hypermultiplets too….

Example 3: D=4 Gauged Supergravity and on-shell action

- Minimal D=4 N=2 gauged supergravity Euclidean

$$I = -\frac{1}{16\pi G}\int_M (R_g + 6 - F^2)\mathrm{vol}_g + I_{\partial M}$$

$$I_{\partial M} = -\frac{1}{16\pi G}\int_{\partial M}(2K - 4 - R_h)\mathrm{vol}_h$$

$$(\nabla_\mu - iA_\mu + \tfrac{1}{2}\Gamma_\mu + \frac{i}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu)\epsilon = 0$$

- Solutions can be uplifted on SE_7 to get solutions of D=11 SUGRA dual to $d = 3, \mathcal{N} = 2$ SCFTs on ∂M

[Gauntlett, Varela 07]

- Prediction for dual SCFTs $I = -\log Z^{SCFT}(\partial M)$

On-shell action:

$$I = \left[\frac{1}{(2\pi)^2}\int_M \Phi\right]\frac{\pi}{2G_4} + I_{\partial M} \qquad\qquad d_\xi \Phi = 0$$

- Can't use BVAB directly:

  Assume ξ has fixed points only in the bulk

  Then bulk integral will give a fixed point contribution plus a divergent boundary piece. In fact, this boundary piece exactly cancels the terms in $I_{\partial M}$

- Result:

  The fixed points can either be "nuts" (fixed points) or "bolts" (fixed surfaces) [Gibbons,Hawking 77]

  The spinor has fixed chirality on the fixed point set

- On-shell action:

$$I = \left\{ \sum_{\substack{\text{fixed} \\ \text{points}_\pm}} \mp \frac{(b_1 \mp b_2)^2}{4b_1 b_2} + \sum_{\substack{\text{fixed} \\ \Sigma_\pm}} \int_{\Sigma_\pm} \left[\frac{1}{2} c_1(T\Sigma_\pm) \mp \frac{1}{4} c_1(L) \right] \right\} \frac{\pi}{2G_4}$$

[Benetti-Genolini, Perez-Ipina, Sparks 19]

- Comments:
  - Recovers results for known SUGRA solutions
  - Results for many new solutions (provided they exist)
  - Agrees with many field theory computations using localization

- Generalizes - include arbitrary vector multiplets

$$I = \frac{\pi}{G_4}\left[\sum_{\text{nuts}_\pm} \mp \frac{1}{d_\pm}\frac{(b_1 \mp b_2)^2}{b_1 b_2} i\mathcal{F}(u_\pm^J) + \sum_{\text{bolts}_\pm}\left(-\partial_I i\mathcal{F}(u_\pm^J)\mathfrak{p}_\pm^I \pm i\mathcal{F}(u_\pm^J)\int_{\Sigma_\pm} c_1(L)\right)\right]$$

$$u_+^I \equiv \frac{\widetilde{X}^I}{\zeta_J \widetilde{X}^J}\bigg|_+ , \quad u_-^I \equiv \frac{X^I}{\zeta_J X^J}\bigg|_-$$

$$\mathfrak{p}_R \equiv \frac{1}{2}\zeta_I \mathfrak{p}_\pm^I = \frac{1}{2}\int_{\Sigma_\pm} [\pm c_1(L) - c_1(T\Sigma_\pm)]$$

- For STU model we recover sugra results of
  - ABJM on S^3 [Freedman, Pufu 13]
  - ABJM on $S^1 \times \Sigma_g$ (black saddle solutions) [Bobev, Charles, Min 20]
- ABJM on M_3 with $S^1 \to M_3 \to \Sigma_g$: sugra solutions not known but find exact agreement with field theory computation [Toldo, Willet 17]

Final Comments

- New technique for computing BPS observables for supersymmetric solutions in SUGRA.

- Illustrated using diverse examples and easily recover many results for known solutions. Also gives results for solutions that (probably) can't be constructed in closed form.

- Other cases: $AdS_3 \times M_8$ of D=11 SUGRA

  $AdS_3 \times M_2$ of D=5 gauged SUGRA + matter

  D=6 Euclidean SUGRA

- Many other cases to consider e.g odd dimensions

- Why does it work?

- Technique is predicated on solutions existing. Physics gives some constraints e.g. negative central charge. Is it possible that in some cases there are sufficient constraints?