KSBA moduli spaces of log CY surfaces with a view toward 5d SCFTs

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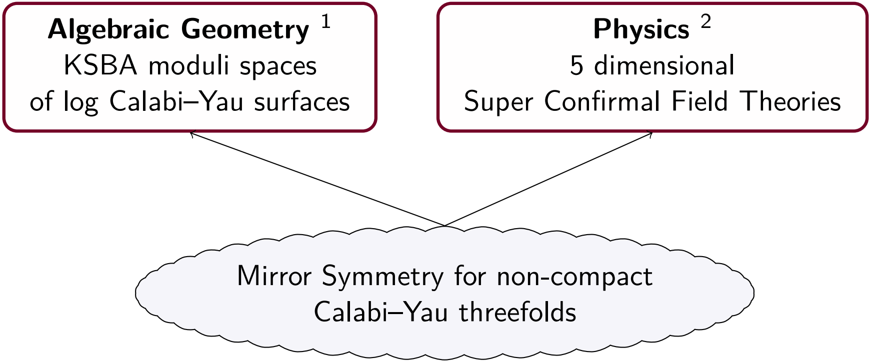
Algebraic Geometry to Physics



[1] Alexeev–Argüz–Bousseau,“The KSBA moduli space of stable log Calabi-Yau surfaces", arXiv:2402.15117.

[2] Argüz–Bousseau, “Non-toric brane webs, Calabi-Yau 3-folds, and 5d SCFTs", in preparation.

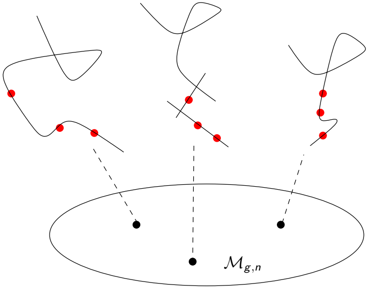
Plan of the talk

- KSBA moduli spaces of toric surfaces
  - Mirror symmetry CY 3-folds in the **“toric set-up”**
- KSBA moduli spaces of log Calabi–Yau surfaces
  - Mirror symmetry for CY 3-folds in the **“non-toric set-up”**

> The KSBA moduli space of stable log Calabi–Yau surfaces, up to a finite cover, is a toric variety.

- *Toward 5d SCFTs...*

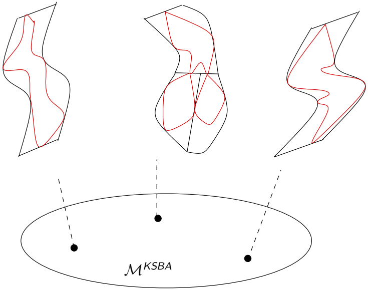
- $\overline{\mathcal{M}}_{g,n}$: Deligne–Mumford "moduli space of stable curves" $(\mathcal{C}, p_1, \ldots p_n)$,
  - $\mathcal{C}$: reduced curve with arithmetic genus g has at worst nodal singularities
  - $p_1, \cdots p_n$: distinct smooth points on $\mathcal{C}$
  - Stability condition: $\mathrm{Aut}(\mathcal{C}, p_1, \ldots p_n) < \infty \iff K_{\mathcal{C}} + \sum_i p_i$ is ample



- $\overline{\mathcal{M}}_{g,n}$ is a proper Deligne–Mumford stack of dimension $3g - 3 + n$

The KSBA moduli space

- $\mathcal{M}^{KSBA}$: "moduli space of stable pairs" (Y, B), where Y is a projective variety, B is a $\mathbb{Q}$-divisor.
  - (Y, B) has semi-log-canonical (slc) singularities
  - Stability condition: $K_Y + B$ is $\mathbb{Q}$-Cartier and ample



- Connected components of are proper Deligne-Mumford stacks
  - Not generally irreducible
    $\rightsquigarrow$ Describe the geometry of irreducible components of $\mathcal{M}^{KSBA}$.

The toric set-up

- P: lattice polytope in $\mathbb{R}^n \rightsquigarrow$ polarized projective toric variety
  - $Y = \mathrm{Proj}\ \mathbb{C}[C(P)_{\mathbb{Z}}]$
  - $D = -K_Y$: Toric boundary divisor
  - L: ample line bundle, $H^0(Y, L) \cong \bigoplus_{p \in P_{\mathbb{Z}}} \mathbb{C}z^p$

Example

Polarized toric varieties (Y, D, L) from "momentum polytopes" P

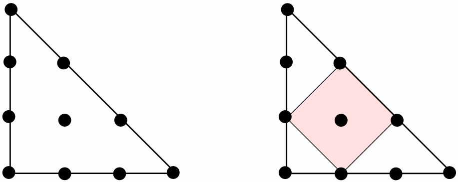
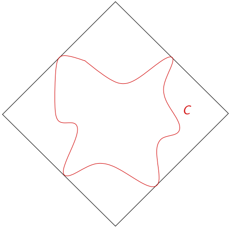


Figure: LHS: $(\mathbb{P}^2, \mathcal{O}(3))$, RHS:$(dP_4, -K_{dP_4})$

The KSBA moduli space of stable toric varieties

- $\mathcal{M}_{(Y,D,L)}$: closure in $\mathcal{M}^{KSBA}$ of the locus of stable pairs $(Y, D + \epsilon C)$, where $C \in |L|$ and $0 < \epsilon << 1$.
  - C is *torically transverse* in Y $\leftrightsquigarrow$ doesn't intersect 0 dim'l strata of D



Theorem (Alexeev, 2002)

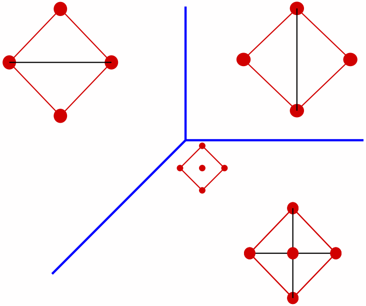
The (normalization of the) moduli space $\mathcal{M}_{(Y,D,L)}$ is a toric variety with associated fan given by the secondary fan[1] of the momentum polytope of (Y, D, L).

[1] Gelfand–Kapranov–Zelevinsky, "Discriminants, resultants, and multidimensional determinants"

- Maximal cones of the "secondary fan": "regular triangulations of P"

Example

The KSBA moduli space for $(dP_4, -K_{dP_4})$ with $C \in |-K_{dP_4}|$ is $\mathbb{P}^2$

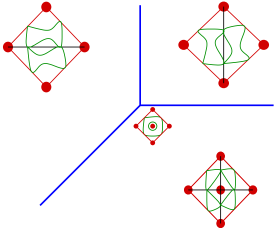


KSBA moduli spaces of stable toric varieties

- Consider all divisors

$$C = \{ \sum_{p \in P_{\mathbb{Z}}} a_p z^p = 0\} \in |L|, \ \text{with} \ a_p \in \mathbb{C}^\star$$

  - All such C are "torically transverse" $\rightsquigarrow$ $(Y, D + \epsilon C)$ is "KSBA stable"
  - The space of all such divisors is the torus $(\mathbb{C}^\star)^{|P_{\mathbb{Z}}| - \dim(Y) - 1}$
- Regular triangulation $\rightsquigarrow$ degeneration of $(Y, D + \epsilon H)$
- Torically transverse curves degenerate to torically transverse ones in the central fiber $\rightsquigarrow$ the KSBA compactification of $(\mathbb{C}^\star)^{|P_{\mathbb{Z}}| - \dim(Y) - 1}$ is a toric variety with fan the secondary fan for the momentum polytope of (Y, D, L).



KSBA moduli space as the complex moduli of a CY 3-fold

- $\mathcal{Z} \subset \mathbb{C}^2_{u,v} \times (\mathbb{C}^*)^2_{x,y}$ given by $uv = f(x,y)$ for $f(x,y) \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$
  - P: Newton polytope of $f(x,y) \leftrightsquigarrow$ polarized toric surface (Y, D, L)
- $C^0 = \{f(x,y) = 0\} \subset (\mathbb{C}^*)^2_{x,y} \cong Y \setminus D$
  (**Seiberg–Witten curve** for the 4d $\mathcal{N} = 2$ theory obtained by compactifying type IIB string theory on $\mathcal{Z}$)

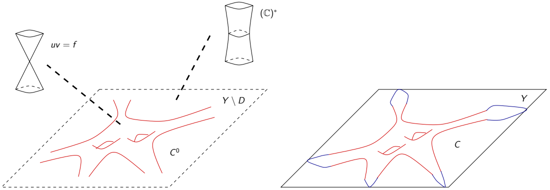
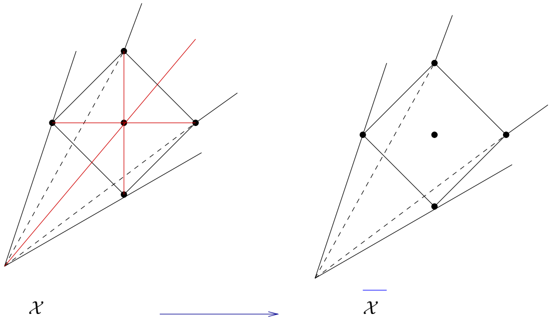


Figure: $Z := \{uv = f(x,y)\} \subset \mathrm{Tot}(L|_{Y\setminus D} \oplus \mathcal{O}_{Y\setminus D}) \to Y \setminus D$

- The KSBA moduli space $\mathcal{M}_{(Y,D,L)}$: complex moduli space of $\mathcal{Z}$.
  - Moduli of curves $C \in |L|$, obtained by varying coefficients of $f(x,y)$.
- Mirror symmetry: $\mathcal{M}_{(Y,D,L)}$ should be the complexified Kähler moduli space of "the mirror to $\mathcal{Z}$" or "the mirror family to (Y, D, L)".

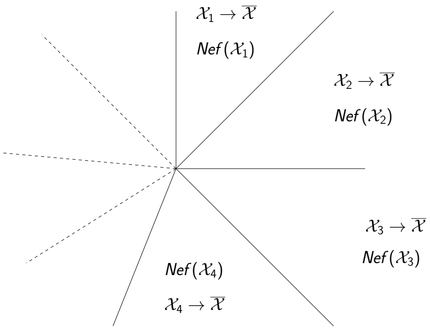
The mirror family $\mathcal{X}$ to a max degeneration of (Y, D, L)

- Question: What is the "mirror" CY 3-fold $\overline{\mathcal{X}} = \mathcal{Z}^\vee$
  – or the "mirror family" to (Y, D, L) ?
- $\overline{\mathcal{X}}$: (singular) toric variety $\overline{\mathcal{X}} \to \mathbb{A}^1$ with fan $\mathrm{Cone}(P \times \{1\}) \subset \mathbb{R}^3$
- The mirror family to a "maximal degeneration" of (Y, D, L):
  - Crepant resolution: $\mathcal{X} = K_{\mathbb{P}^1 \times \mathbb{P}^1} \to \overline{\mathcal{X}}$



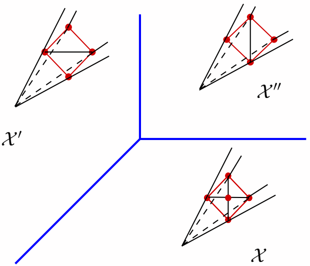
KSBA moduli spaces and mirror symmetry

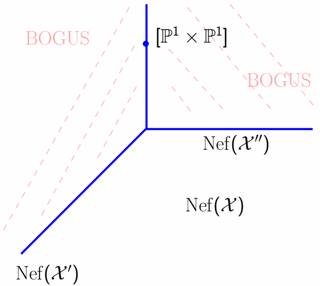
- Expectation from mirror symmetry: $\mathcal{M}_{(Y,D,L)}$ is a toric variety with fan the union of the Kähler cones of all crepant resolutions of $\overline{\mathcal{X}}$.
  - Algebro-geometrically: closure of Kähler cones are Nef cones, that is cones of Nef divisors (divisors which have non-negative intersection with every effective curve class).
- Sean Keel: for a Mori dream space define the MoriFan$(\mathcal{X})$
  - complete fan with maximal cones all Nef cones of all toric crepant resolutions of $\overline{\mathcal{X}}$ and "Bogus cones".



Example

The MoriFan for $\mathcal{X} = K_{\mathbb{P}^1 \times \mathbb{P}^1} \to \overline{\mathcal{X}}$ is $\mathbb{P}^2$.





- Birkar–Cascini–Hacon–McKernan: the crepant resolution $\mathcal{X} \to \overline{\mathcal{X}}$ of a canonical singularity is a Mori dream space.
- Secondary fan for P = MoriFan of the mirror $\mathcal{X} \to \overline{\mathcal{X}}$ to (Y, D, L)

The non-toric set-up: polarized log Calabi–Yau surfaces

- Fix (Y, D, L): polarized log Calabi–Yau surface
  - Y: projective surface, $D \subset Y$: reduced, anticanonical divisor
  - (Y, D): log canonical (for instance Y: smooth, D: normal crossing)
  - L: ample line bundle on Y

Assume (Y, D) is **maximal**: $D \neq \emptyset$ and admits a 0-dimensional strata.

Example

- Y: projective toric variety, D: toric boundary divisor

Example

- $(\mathrm{Bl}_p\mathbb{P}^2, D)$, where p is a point and D is the strict transform of the toric boundary in $\mathbb{P}^2$.

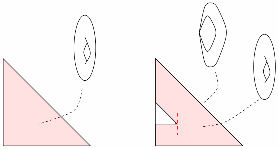


Figure: RHS: "Symington polytope" for $\mathrm{Bl}_p\mathbb{P}^2$ at a non-toric point

Further examples of log CY surfaces

Example

- Another non-toric example: $Y = \mathbb{P}^2$, $D=$ line $\cup$ conic

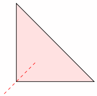
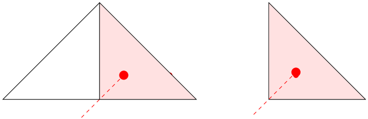


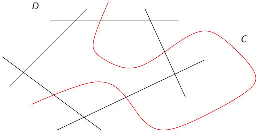
Figure: A Symington polytope for ($\mathbb{P}^2$, line $\cup$ conic)

- Gross–Hacking–Keel: Any log Calabi–Yau surface admits a toric model, i.e. it can be obtained by a blow-up of a toric surface along smooth points of the toric boundary (up to (corner) blow-ups with centers on 0-dimensional strata of D).



The KSBA moduli of polarized log Calabi–Yau surfaces

- $\mathcal{M}_{(Y,D,L)}$: closure in $\mathcal{M}^{KSBA}$ of the locus of stable pairs "deformation equivalent" to stable pairs $(Y, D + \epsilon C)$, where $C \in |L|$ and $0 < \epsilon << 1$.



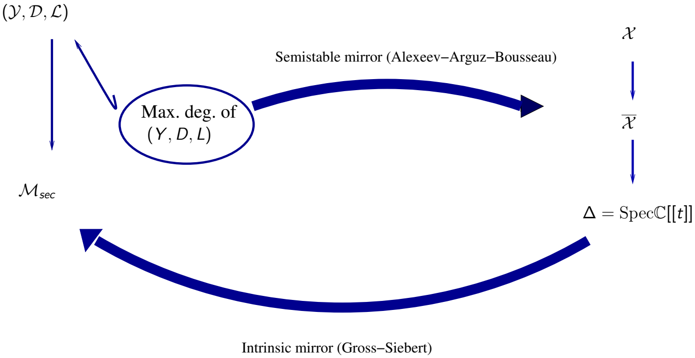
- $(Y, D + \epsilon C)$: C does not contain strata of D
- $K_Y + D + \epsilon C$ is ample for $\epsilon > 0$

Theorem (Alexeev–A.–Bousseau., 2024)

The moduli space $\mathcal{M}_{(Y,D,L)}$ of polarized log Calabi–Yau surfaces, up to a finite cover, is a toric variety.

- This is a conjecture of Hacking–Keel–Yu in any dimension.
- Previously known cases:
  - Alexeev (2002): Toric varieties
  - Hacking–Keel–Yu (2000): for Y a del Pezzo surfaces of degree n where $1 \leq n \leq 6$, and D is a cycle of n many (-1)-curves, $L = -K_Y$.

Strategy



- **Step** 1: Construct the mirror $\mathcal{X} \to \overline{\mathcal{X}} \to \Delta$ to (*max deg of*) (Y, D, L)
  $\rightsquigarrow$ Let $\mathcal{M}_{sec}$: toric variety with fan MoriFan$(\mathcal{X}/\overline{\mathcal{X}})$
- **Step** 2: Construct the "double mirror": $(\mathcal{Y}, \mathcal{D} + \epsilon\mathcal{C}) \to \mathcal{M}_{sec}$
- **Step** 3: Prove $(\mathcal{Y}, \mathcal{D} + \epsilon\mathcal{C}) \to \mathcal{M}_{sec}$ is a KSBA stable family and the general fiber is deformation equivalent to (Y, D, L).
- **Step** 4: Show $\mathcal{M}_{sec} \to \mathcal{M}_{(Y,D,L)}$ is finite and surjective.

Step 1: overview

Maximal degeneration of (Y, D, L) into $Y_0 = \cup \mathbb{P}^2$

↕

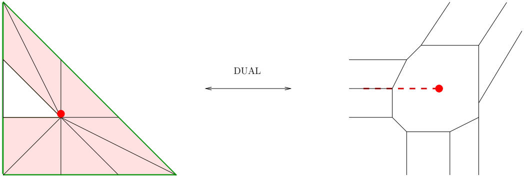
Regular triangulation of the Symington polytope of Y ⟷ Intersection complex of Y_0

↕

Dual intersection complex of a normal crossing surface X_0

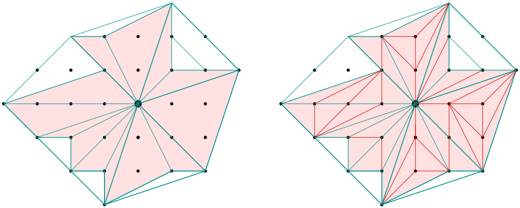
↓

The mirror $\mathcal{X}$ to the max. deg of (Y, D, L) is constructed by a smoothing of X_0



Step 1: the normal crossing surface $\mathcal{X}_0$

- Engel–Friedman: Any (Y, D, L) has a "nice" Symington polytope P with a central point.



Theorem (Alexeev–A.–Bousseau)

There is a regular triangulation T of P and a maximal degeneration of (Y, D, L) with special fiber Y_0 whose intersection complex is (P, T).

- $\mathcal{X}_0$: normal crossing surface with "dual intersection complex" (P, T).
  - irreducible components X_v of $\mathcal{X}_0$: vertices $v \in P$
  - v: non-singular point $\implies$ X_v is smooth toric (non-compact if $v \in \partial P$)
  - v: singular point $\implies$ X_v is a smooth log CY surface, which we obtain by a $\mathbb{Q}$-Gorenstein smoothing of a toric surface with quotient Wahl singularities locally of the form $\mathbb{A}^2/\frac{1}{n^2}(1, an-1)$.

Step 1: The normal crossing surface $\mathcal{X}_0$

Example

For $(\mathbb{P}^2, \mathrm{line} \cup \mathrm{conic})$: $\mathcal{X}_0$ is a union of 2 copies of $\mathbb{A}^2$ and a $\mathbb{P}^1 \times \mathbb{A}^1$ (with a non-toric boundary).

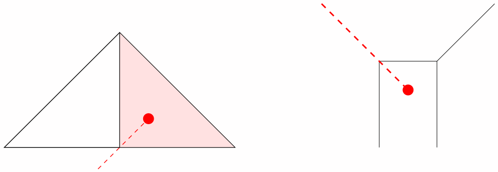


Figure: The intersection and dual intersection complexes for $\mathcal{X}_0$

Step 1: the mirror $\mathcal{X}$

Theorem (Alexeev–A.–Bousseau)

There exists a d-semistable gluing of X_v's to $\mathcal{X}_0$ and a smoothing $\mathcal{X} \to \Delta = \operatorname{Spec} \mathbb{C}[[t]]$ of it such that

- *The total space $\mathcal{X}$ is smooth and $\mathcal{X}_0 \subset \mathcal{X}$ is a reduced normal crossing divisor (i.e. $\mathcal{X} \to \Delta$ is a semistable degeneration).*
- *$\mathcal{X}$ is quasi-projective and $K_{\mathcal{X}} = 0$.*
- *There exist a contraction $\mathcal{X} \to \overline{\mathcal{X}}$, such that $\overline{\mathcal{X}}$ is affine with canonical singularities and $\mathcal{X} \to \overline{\mathcal{X}}$ is a projective crepant resolution $\rightsquigarrow \mathcal{X}$ is a Mori dream space.*

- Proof uses log smooth deformation theory

Step 2: the intrinsic mirror to the mirror $\mathcal{X} \to \Delta$

- Gross–Siebert: given a semistable (projective) family of Calabi–Yau's $\mathcal{X} \to \Delta$, construct the intrinsic mirror

$$\mathcal{Y}_{\mathcal{X}} = \mathrm{Proj}\left(\bigoplus_{p \in C(P)_{\mathbb{Z}}} \mathrm{Spf}\,\mathbb{C}[[NE(\mathcal{X})]]\vartheta_p \right)$$

  $NE(\mathcal{X})$: Mori cone of effective curve classes (dual to $Nef(\mathcal{X}/\overline{\mathcal{X}})$).
  - Structure constants: log Gromov–Witten invariants of ACGS.
- Subtlety: the semi-stable mirror $\mathcal{X} \to \Delta$ is not projective – but we can generalize the intrinsic mirror construction.

Theorem (Alexeev–A.–Bousseau)

The intrinsic mirror extends over the affine toric variety $\mathrm{Spec}\,\mathbb{C}[NE(\mathcal{X})]$.

- Proof: uses the birational geometry of the crepant resolution $\mathcal{X} \to \overline{\mathcal{X}}$ to constrain the curve classes appearing in the intrinsic mirror construction.

Gluing intrinsic mirrors to all crepant resolutions of $\overline{\mathcal{X}}$

- Show ϑ_p, for $p \in \operatorname{Int} C(P)_{\mathbb{Z}}$ define an ideal $\rightsquigarrow$ divisor $\mathcal{D}_{\mathcal{X}} \subset \mathcal{Y}_{\mathcal{X}}$.
- There is a natural line bundle $\mathcal{L}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}_{\mathcal{X}}}(1)$ with sections ϑ_p, for $p \in P_{\mathbb{Z}}$. Define the divisor

$$\mathcal{C}_{\mathcal{X}} = \left\{ \sum_{p \in P_{\mathbb{Z}}} \vartheta_p = 0 \right\} \in |\mathcal{L}_{\mathcal{X}}|.$$

- Run the intrinsic mirror construction for every projective crepant resolution $\mathcal{X}' \to \overline{\mathcal{X}}$.

Theorem (Alexeev–A.–Bousseau)

The families $(\mathcal{Y}_{\mathcal{X}'}, \mathcal{D}_{\mathcal{X}'} + \epsilon \mathcal{C}_{\mathcal{X}'})$ glue and naturally extend into a family

$$(\mathcal{Y}, \mathcal{D} + \epsilon \mathcal{C}) \longrightarrow \mathcal{M}_{sec},$$

of surfaces over the toric variety $\mathcal{M}_{sec}$ with fan the Morifan$(\mathcal{X}/\overline{\mathcal{X}})$.

Step 3: KSBA stability of the mirror family

Theorem (Alexeev–A.–Bousseau)

The double mirror $(\mathcal{Y}, \mathcal{D} + \epsilon\mathcal{C}) \to \mathcal{M}_{sec}$ is a family of KSBA stable log Calabi–Yau surfaces, with general fiber deformation equivalent to (Y, D, L).

To prove the KSBA stability:

- Over the dense torus in $\mathcal{M}_{sec}$: any fiber (Y_t, D_t, L_t) of $(\mathcal{Y}, \mathcal{D}, \mathcal{L})$ is irreducible, by canonical scattering (Gross–Siebert).
  - $C_t \in |L_t|$ does not pass through the 0 dim'l strata of D_t: analysing the restriction of the ϑ-functions defining C_t to D_t.
- On the boundary of $\mathcal{M}_{sec}$: fibers $(Y_t, D_t, L_t) = \cup_i (Y_i, D_i, L_i)$, where each (Y_i, D_i, L_i) is the mirror to a crepant resolution $\mathcal{X}_i \to \overline{\mathcal{X}_i}$ then apply induction.

To prove the general fiber (Y_t, D_t, L_t) is def. equivalent to (Y, D, L):

- Show (Y_t, D_t, L_t) is diffeomorphic to (Y, D, L), then apply previous results of Friedman.

Step 4: The KSBA moduli space as a finite cover of $\mathcal{M}_{sec}$

- By universal property of $\mathcal{M}_{(Y,D,L)}$, obtain a map $\mathcal{M}_{sec} \to \mathcal{M}_{(Y,D,L)}$.

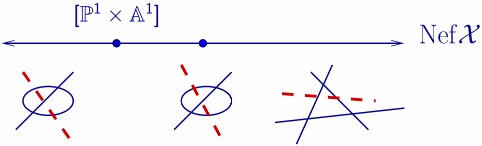
Theorem (Alexeev–A.–Bousseau)

$\mathcal{M}_{sec} \to \mathcal{M}_{(Y,D,L)}$ *is finite and surjective.*

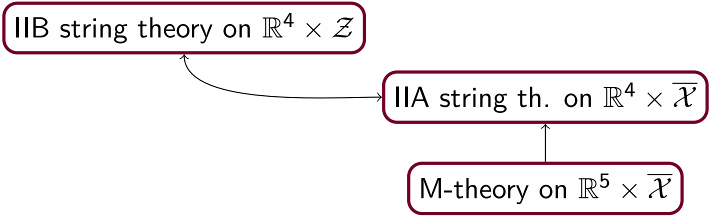
- Proof of finiteness: the family $(\mathcal{Y}, \mathcal{D}, \mathcal{L})$ is non-constant in restriction to one-dimensional torus strata of $\mathcal{M}_{sec}$,
- Proof of surjectivity: we use tropical period computations following [Lai-Zhou, Ruddat-Siebert].

Example

- For ($\mathbb{P}^2$, line $\cup$ conic), we have $\mathcal{M}^{KSBA} = \mathcal{M}_{sec} = \mathbb{P}^1$.



Toward 5d SCFTs



- Webs of 5-branes in IIB string theory $\rightsquigarrow$ 5d $\mathcal{N} = 1$ SCFT
  - M-theory on $\mathbb{R}^5 \times \overline{\mathcal{X}}$, in the toric set-up (for toric $\overline{\mathcal{X}}$)
- Webs of 5 and 7-branes in IIB string theory $\rightsquigarrow$ 5d $\mathcal{N} = 1$ SCFT
  - M-theory on $\mathbb{R}^5 \times \overline{\mathcal{X}}$, where $\mathcal{X} \to \overline{\mathcal{X}}$ is the semi-stable mirror to $\mathcal{Z}$ we described in the non-toric set-up.

For more see Pierrick Bousseau's talk tomorrow!

The end...