Anomaly cancellation in string theory using homotopy theory

Arun Debray — June 10, 2024

based on joint work with Ivano Basile, Matilda Delgado, Markus Dierigl, Jonathan J. Heckman, Miguel Montero, and Matthew Yu

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- Potential things one could get out of this talk
  - What is bordism, and why does it have anything to do with string theory?
  - How can one extract a concrete mathematical question out of the physics question of calculating an anomaly?
  - What is the lay of the land for these computations for various theories? What makes a given example tractable or difficult?

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- "The anomaly of a string theory" is mathematically contentious for a couple of reasons
    - That phrase suggests a settled mathematical formulation of string theory, which is not true, so **the mathematically correct thing to discuss is anomalies of supergravity theories**, which are expected to be low-energy limits of string theories
    - In order to have a quantity that can be evaluated on a spacetime background, the whole discussion takes place before performing the sum over such backgrounds

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- In quantum mechanics, the space of states is the projectivization of a Hilbert space $\mathcal{H}$, but one usually prefers to work directly with $\mathcal{H}$
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    - And indeed, some constructions *require* working with $\mathcal{H}$ rather than $\mathbb{P}(\mathcal{H})$, such as gauging a symmetry
- In quantum mechanics, the **anomaly** of a G-action on $\mathbb{P}(\mathcal{H})$ is defined to be the obstruction to lifting to a linear action of G on $\mathcal{H}$

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- Following that line of logic leads to the conclusion that the anomaly itself *is* a field theory α, but in one dimension higher (Freed-Teleman '12); related to **anomaly inflow**
- In addition, the state spaces of α must be one-dimensional: α is an **invertible field theory** (Freed-Moore '04)

Reflection-positive invertible field theories (IFTs)

Theorem (Freed-Hopkins '16, Grady '23)

Let IFT^n_ξ *denote the abelian group of* n*-dimensional reflection-positive IFTs on manifolds with* ξ*-structure. There is a short exact sequence*

$$0 \longrightarrow \mathrm{Hom}(\Omega^\xi_n, \mathbb{C}^\times)_{\mathrm{tors}} \longrightarrow \mathrm{IFT}^n_\xi \longrightarrow \mathrm{Hom}(\Omega^\xi_{n+1}, \mathbb{Z}) \longrightarrow 0.$$

Tangential structure

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- "ξ" appears to encode precisely the topological part of that data: a **tangential structure** (Lashof '63)
- This is anything like an orientation, spin structure, principal bundle, ... but no metric or connections!
- The point is, to specify a field theory, and therefore define IFT^n_ξ, you need to know the dimension and the tangential structure of the possible spacetimes of the theory

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- This group is called the **n-dimensional ξ-bordism group**

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Let IFT_ξ^{n+1} *denote the abelian group of* $(n+1)$*-dimensional reflection-positive IFTs on manifolds with* ξ*-structure. There is a short exact sequence*

$$0 \longrightarrow \mathrm{Hom}(\Omega_{n+1}^{\xi}, \mathbb{C}^\times)_{\mathrm{tors}} \longrightarrow \mathrm{IFT}_\xi^{n+1} \longrightarrow \mathrm{Hom}(\Omega_{n+2}^{\xi}, \mathbb{Z}) \longrightarrow 0.$$

Interpretation for an anomaly of an n-dimensional theory:

- The quotient is a free group of characteristic classes in dimension $(n+2)$, namely the **anomaly polynomial** or **local anomaly**, visible perturbatively
- The sub is the topological anomaly theories, and is not seen perturbatively. Sometimes called the **global anomaly**

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- The first step, then, is to determine the isomorphism type of this bordism group and find manifolds which represent a generating set

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- E.g. $n + 1 = 2$, $\xi =$ orientation: all closed, oriented surfaces are disjoint unions of many-holed tori and can be "filled in"
- We saw that the anomaly vanishes on manifolds which are boundaries, so in this setting *the anomaly must be trivial!*

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  - λ is the "$\frac{1}{2}p_1$" class on spin manifolds
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- This trivialization is an example of a **twisted string structure**

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  - Spin: usually more complicated than ordinary homology (Anderson-Brown-Peterson '67)
  - String: not known above dimension 49 (Giambalvo '71, Hovey-Ravenel '95, Mahowald-Gorbounov '95); difficult even in string-theoretic dimensions

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- Conclusion: *the $E_8 \times E_8$ heterotic string is anomaly-free*

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- Freed-Hopkins '21 (M-theory with parity symmetry), D.-Yu '22 (a 4d U-duality symmetry), D. '23, Basile-D.-Delgado-Montero '23 ($E_8 \times E_8$ heterotic string theory)

$E_8 \times E_8$ heterotic string theory, again

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- Witten's argument breaks the $\mathbb{Z}/2$ symmetry, so does not apply here

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Theorem (D., '23, Basile-D.-Delgado-Montero, '23)

Let ξ denote the tangential structure for the $E_8 \times E_8$ heterotic string with its $\mathbb{Z}/2$ symmetry.

1. Ω_{11}^{ξ} *is isomorphic to either* $\mathbb{Z}/8 \oplus \mathbb{Z}/8$, $\mathbb{Z}/16 \oplus \mathbb{Z}/4$, $\mathbb{Z}/32 \oplus \mathbb{Z}/2$, *or* $\mathbb{Z}/64$. *A generating set of manifolds is* $\mathrm{Bott} \times \mathbb{RP}^3$ *and a certain* $(S^4 \times S^4)$*-bundle over* $\mathbb{RP}^3$.

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2. *The anomaly theory* $\alpha_{\mathrm{HE}} \colon \Omega_{11}^{\xi} \to \mathbb{C}^\times$ *vanishes.*

Note: Tachikawa-Yamashita '22 provide a different (and very cool!) argument for anomaly vanishing, which has not yet been shown to be mathematically equivalent to ours

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- The **Pontrjagin-Thom theorem** identifies Ω_*^ξ with the homotopy groups of an object $MT\xi$ called a **Thom spectrum**
- Everything in homotopy theory has an enormous amount of algebraic data associated to it, e.g. cohomology groups, Steenrod operations, ...

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- Now **turn it around — use algebraic information to constrain topology**
- In general the relationship is subtle, so rather than make these inferences “by hand,” work with well-studied machines that organize the flow of information back and forth

Spectral sequences

- Common technique: a **spectral sequence**
- “It has been suggested that the name ‘spectral’ was given because, like spectres, spectral sequences are terrifying, evil, and dangerous. I have heard no one disagree with this interpretation, which is perhaps not surprising since I just made it up.” – Ravi Vakil

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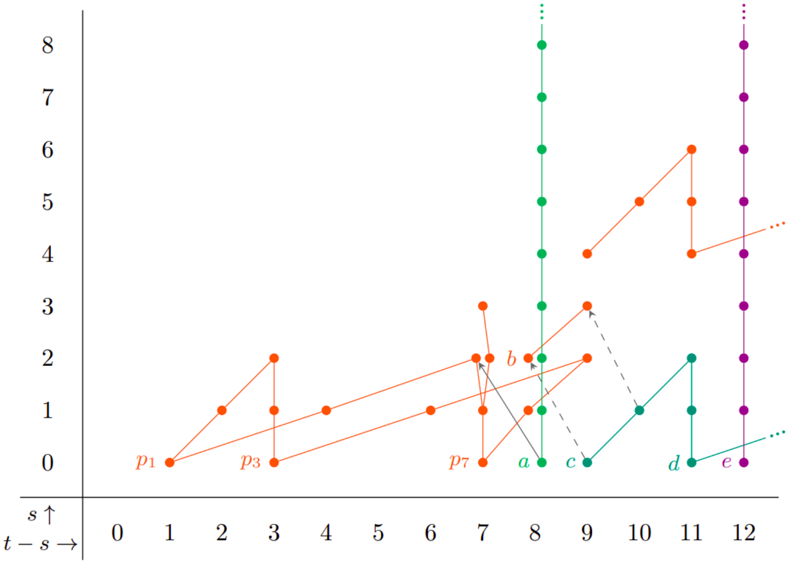
- In this computation we use the **Adams spectral sequence**

The E_2-page



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- Upshot: in many situations of interest, physical expectation is that the anomaly vanishes
- And yet, sometimes things go wrong!

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- So the tangential structure is $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{Mp}_2(\mathbb{Z})$
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  - We have $\mathrm{Spin} \times \mathrm{Mp}_2(\mathbb{Z})$, but the two fermion parity operators act identically so we can identify them
- In fact, including worldsheet orientation reversal (Tachikawa-Yonekura '18) enlarges the duality group to $\mathrm{GL}_2(\mathbb{Z})$ and the tangential structure to $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{GL}_2^+(\mathbb{Z})$

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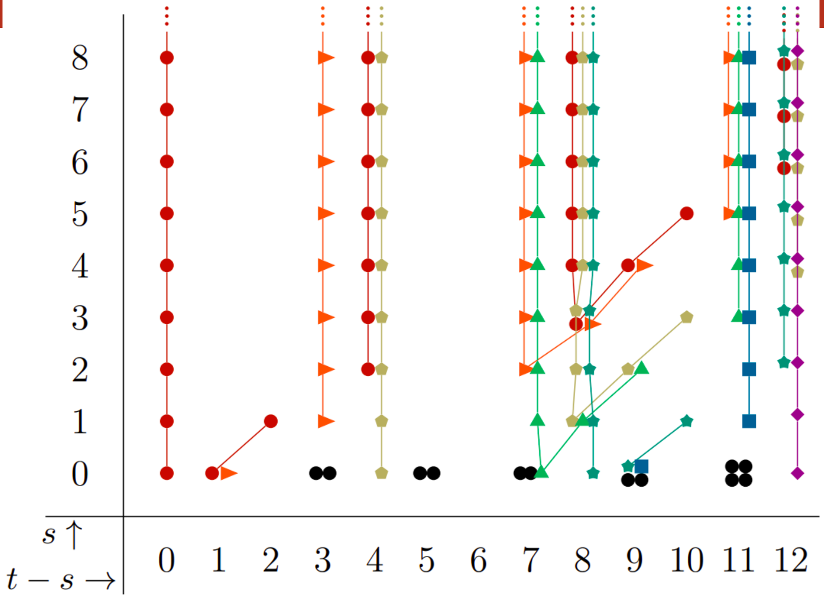
- Many examples of twisted spin bordism computations in the literature (see in particular Beaudry-Campbell '18)
- Comes down to: **how complicated is the classifying space of $\mathrm{GL}_2(\mathbb{Z})$?**
- Turns out, not complicated: **you can replace $\mathrm{GL}_2(\mathbb{Z})$ with the dihedral group D_{24}**

The spectral sequence computation



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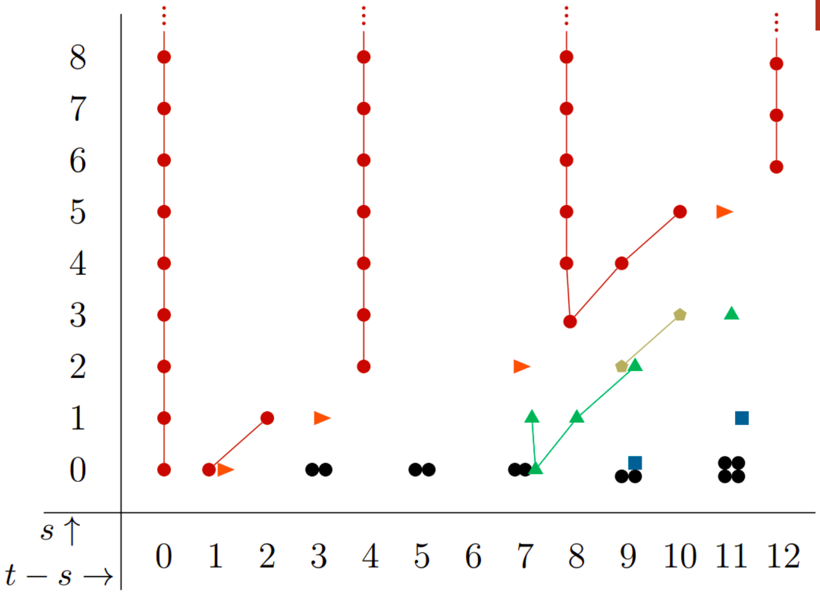

Bordism results

Theorem (D.-Dierigl-Heckan-Montero '21, '23)

1. $\Omega_{11}^{\mathrm{Spin}\times_{\{\pm 1\}}\mathrm{GL}_2^+(\mathbb{Z})} \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 9} \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/3.$

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2. *The following manifolds' bordism classes generate that bordism group:* Q_4^{11}, $\mathbb{HP}^2 \times L_4^3$, $\mathbb{RP}^{11}$, $\mathbb{HP}^2 \times \mathbb{RP}^3$, $S^1 \times$ *Milnor's* X_{10}, *an* $\mathbb{RP}^5$*-bundle over* $\mathbb{RP}^6$, L_3^{11}, *and* $\mathbb{HP}^2 \times L_3^3$.

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- The fermions' contribution are η-invariants
- In general, η-invariants are pretty difficult to compute, but a formula due to Donnelly ' '78 gets us most of the way
- The self-dual field's contribution is a bit fancier (Hsieh-Tachikawa-Yonekura '20):
    - The self-duality structure induces a quadratic refinement q of the torsion linking pairing
    - The value of the anomaly is another η-invariant plus $\mathrm{Arf}(q) - q(c)$, where c is the field strength

The anomaly is nonzero! Oh dear

Theorem (D.-Dierigl-Heckman-Montero, '21)

The anomaly theory described above is nontrivial on L^3_{11}, $\mathbb{HP}^2 \times L^3_3$, Q^4_{11}, *and* $\mathbb{HP}^2 \times L^3_4$; *it vanishes on the other generators, except maybe* X_{11}.

Note: you do not need anything about bordism to make this computation; it just told us what manifolds to look at

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What physical insight does this surprise tell us? A few options:

1. Maybe type IIB string theory is actually anomalous! (**This would be a big problem...** but fortunately there are better paths forward)
2. Insert a torsion correction term into the IIB action to make the problem go away
   - This is what we did
   - At least on spin-$\mathrm{Mp}_2(\mathbb{Z})$ manifolds, this (1) works and (2) only works for the specific anomaly that IIB has, suggesting it is something canonical

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3. Alternatively: wait, shouldn't we have been using K-theory the whole time?
    - It is a difficult open problem to reconcile S-duality in IIB with the K-theoretic description of the RR field
    - But we know how at least some of the elements of $\mathrm{GL}_2^+(\mathbb{Z})$ ought to act on K-theory, so **it would be interesting to try this technique for that subgroup**

Things to study in the future

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1. Anomalies of orbifolds and equivariant bordism
    - String theorists sometimes consider more general backgrounds than manifolds, including "mildly stacky" spaces
    - These are quotients of manifolds by not-necessarily-free group actions

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    - The plan: express data in IIA in terms of (twisted) string^h structures and obtain tractable computations of anomalies