Spontaneous Breaking of (-1)-form U(1) “Symmetries”

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Based on 2402.00117 with Daniel Aloni, Matt Reece and Motoo Suzuki.

Outline

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1. Introduction
2. Spontaneous breaking of U(1) (-1)-form symmetries: a toy model in $2d$.

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3. SU(N) YM, QCD and the Strong CP problem.

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- $\int \star j_0(x) \in \mathbb{Z}$ is the (-1)-form charge.
- Example: $4d$ gauge theory, (-1)-form symmetry charge is the instanton number,

$$\star j_0 = \frac{1}{8\pi^2} \text{tr}(F \wedge F)$$

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$\Longrightarrow$ **In this talk: explore this possibility.**

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Is there a sense in which $U(1)_m^{(-1)}$ is spontaneously broken?

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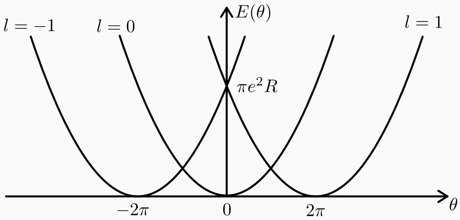
Solved by eigenstates $\phi_l = e^{il\phi}$ with energy,

$$E_l = \pi e^2 R \left(l - \frac{\theta}{2\pi}\right)^2$$

Excited states (not drawn): adding 2 probe particles.

Classically confined.



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We conclude that the gauged (-1)-form U(1) symmetry is in the Higgs phase. We interpret this to mean that the global (-1)-form U(1) symmetry of $2d$ Maxwell is spontaneously broken.

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$\mathcal{X} \neq 0$ related to “masslessness” of $A_\mu(x)$

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- In fact, at large N the IR of SU(N) YM can be described using an effective theory in terms of $F_4 = dC_3$ (Di Vechia, Veneziano, Shifman, Gabadadze, Dvali),

$$\mathcal{L} = -\frac{1}{2\mathcal{X}} F_4 \wedge \star F_4 + \frac{1}{2\pi} \theta F_4$$

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- New solutions? Use $U(1)_m^{(-1)}$ anomalies? Explicit breaking in the UV?

Outlook

- A firmer footing for (-1)-form symmetries. Perhaps using the SymTFT. Or Holography?
- Are (-1)-form symmetries matched under dualities?
- Goldstone Theorem?
- Better understanding of explicit breaking.
- Breaking by monopoles.
- Application to other axion-like fields in particle physics. In particular axion monodromy.

Thanks StringMath 2024!

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MODULARITY OF CERTAIN GROMOV–WITTEN INVARIANTS FROM K3 MIRROR SYMMETRY, IRRATIONALITY OF ZETA VALUES, AND THE GAMMA CONJECTURE

arXiv : 2403.07349

STRING MATH 2024

The Abdus Salam International Centre for Theoretical Physics

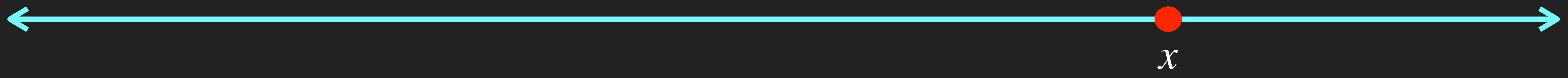
Michael T. Schultz

COLLEGE OF SCIENCE MATHEMATICS VIRGINIA TECH

13 June 2024

IT'S OK TO BE IRRATIONAL SOMETIMES

- Choose a number $x \in \mathbb{R}$ at random



- It is with 100% certainty irrational since $\mathbb{Q}$ has measure zero in $\mathbb{R}$
- But how would you prove this directly for the given number x ?
- For some algebraic numbers, e.g. $x = \sqrt{2}$, proof by contradiction, but this quickly becomes intractable

IT'S STILL OK TO BE IRRATIONAL SOMETIMES

- One tool to analyze irrationality is the Dirichlet irrationality criterion:

Let $x \in \mathbb{R}$. If there exists a $\delta > 0$ and sequences $A_n, B_n \in \mathbb{Q}$ with both $A_n \neq 0$ and $\frac{B_n}{A_n} \neq x$ for all $n \in \mathbb{N}$ such that

$$\left| \frac{B_n}{A_n} - x \right| < \frac{1}{A_n^{\delta+1}},$$

then x is irrational.

$\zeta(3)$ IS IRRATIONAL

- While the values $\zeta(2k), k \geq 1 \in \mathbb{N}$ are known exactly, the nature of $\zeta(3) \approx 1.2020569...$ remained mysterious until Apéry showed

Theorem (Apéry, 1978) : $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ **is irrational.**

- Apéry showed this by utilizing the Dirichlet irrationality criterion with the sequences

$$A_n = 1, 5, 73, 1445, \ \ldots \qquad B_n = 0, 6, \frac{351}{4}, \frac{62531}{36}, \ \ldots$$

$$n^3 u_{n-1} - (2n+1)\left(17n^2 + 17n + 5\right) u_n + (n+1)^3 u_{n+1} = 0$$

$\zeta(3)$ IS IRRATIONAL, CONT.

- These sequences were quite mysterious. Apéry showed that

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \in \mathbb{N},$$

$$B_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left\{ \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right\} \in \mathbb{Q}$$

yet one still would like to know "where these come from".

ANSWER:
THEY “COME FROM ALGEBRAIC GEOMETRY”

SURPRISING GEOMETRY LURKS BENEATH

- Apply the Frobenius method to obtain

$$\omega(t) = \sum_{n=0}^{\infty} A_n t^n \quad , \quad \gamma(t) = \sum_{n=0}^{\infty} B_n t^n$$

$$\mathscr{L}_3 = \theta^3 - t(2\theta+1)\left(17\theta^2 + 17\theta + 5\right) + t^2(\theta+1)^3, \text{ with } \theta = t\frac{d}{dt}$$

$$\mathscr{L}_3(\omega(t)) = 0 \quad , \quad \mathscr{L}_3(\omega(t)) = 6t$$

- The operator $\mathscr{L}_3$ is a CY operator: self-adjoint, MUM at $t = 0$ with $\omega(t)$ the unique holomorphic solution with integral coefficients
- Canonical basis $\omega_0 = \omega, \omega_1, \omega_2$ at $t = 0$ that satisfy $\omega_0\omega_2 = \omega_1^2$
- $\omega(t)$ = holomorphic period of a pencil of K3 surfaces?

SURPRISING GEOMETRY LURKS BENEATH, CONT.

- This was indeed shown by Beukers (1983), and Beukers & Peters (1984) using an integral representation:

$$\omega(t) = \left(\frac{1}{2\pi i}\right)^3 \iiint_S \frac{dXdYdZ}{1-(1-XY)Z-tXYZ(1-X)(1-Y)(1-Z)} = \iint_\Sigma \Omega_t$$

$$\mathcal{X}_t \quad : \quad 1-(1-XY)Z-tXYZ(1-X)(1-Y)(1-Z)=0\,, \quad \Sigma \in H_2(\mathcal{X}_t,\mathbb{Z}), \quad \Omega_t \in H^{2,0}(\mathcal{X}_t)$$

- Is there a geometric interpretation of $\gamma(t) = \sum_{n=0}^{\infty} B_n t^n$?

ONE WAY OF ANALYZING THE B_n SEQUENCE

- The function $\gamma(t)$ can be shown to be a solution of the homogeneous fourth order equation $\mathscr{D}_4(\gamma(t)) = 0$ at $t = \infty$, where

$$\mathscr{D}_4 = \theta \cdot \mathscr{L}_3$$

$$= \theta^4 - t(\theta+1)(2\theta+1)\left(17\theta^2 + 17\theta + 5\right) + t^2(\theta+1)^3(\theta+2)$$

- The operator $\mathscr{D}_4$ is MUM at $t = 0$ with canonical basis $\omega_0 = \omega, \omega_1, \omega_2, \omega_3$; is it CY?
- No! $\mathscr{D}_4$ is not self-adjoint!

NOT SELF-ADJOINT? NO PROBLEM! (SOME CAVEATS APPLY)

- Pretend like we didn't know $\mathscr{D}_4$ is not self-adjoint, and attempt to compute holomorphic prepotential, virtual Yukawa coupling, and virtual instanton numbers (W. Yang, 2021)

Mirror Map: $\tau(t) = \dfrac{\omega_1(t)}{\omega_0(t)}$, $q = \exp(2\pi i \tau)$

$$\mathfrak{F} = -\frac{1}{12}Y_{111}\left(-\frac{\omega_3}{\omega_0} + \frac{\omega_1\omega_2}{\omega_0^2}\right) - \frac{1}{2}Y_{011}\frac{\omega_1^2}{\omega_0^2} - \frac{1}{2}Y_{001}\frac{\omega_1}{\omega_0} - \frac{1}{6}Y_{000},$$

Holomorphic Prepotential: $\mathfrak{F} := \dfrac{\omega_3}{\omega_0} = \tau^3 + \text{holo}.$

Virtual Yukawa Coupling: $\mathscr{Y} := \dfrac{d^3\mathfrak{F}}{d\tau^3} = 6 + \displaystyle\sum_{k=1}^{\infty} k^3 N_k \frac{q^k}{1-q^k}$

INTEGRALITY AND MODULARITY

- Surprisingly, one finds experimentally that the virtual instanton numbers are 6-periodic integers

$$N_1 = -42, \quad N_2 = -39, \quad N_3 = -44, \quad N_4 = -39,$$
$$N_5 = -42, \quad N_6 = -34, \quad N_{k+6} = N_k$$

and also that $\mathscr{Y}$ is a modular form of weight 4 for $\Gamma_0(6)^+$

- Yang found a similar story for the Dwork pencil of K3 surfaces, with 2-periodic virtual instanton numbers $N_1 = -480$, $N_2 = -240$, $N_{k+2} = N_k$

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 - 4tX_0X_1X_2X_3 = 0$$

THE BIGGER PICTURE

- Common organization of these examples are the modular pencils of K3 surfaces for $\Gamma_0(N)^+$, $N = 2,3,4,5,6,7,8,9,11$, up to rational pullback, appearing in particular in the work of Golyshev (2005) on mirror symmetry for the 17 deformation classes of rank 1 Fano threefolds and Golyshev & Zagier (2016) in their proof of the Gamma conjecture for these Fanos
- Results from Golyshev & Zagier that the virtual Yukawa coupling $\mathscr{Y} = A_N \sum_{M|N} M^2 h_M G_4(M\tau)$ is a rational combination of Eisenstein series, with $G_4(\tau) = \frac{1}{240} + \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}$ the weight 4 Hecke normalized Eisenstein series, and prepotential $\mathfrak{F}$ is an Eichler integral
- Hence virtual instanton numbers are always N-periodic integers

MIRROR SYMMETRY

- Golyshev (2005) shows that the quantum operator $\mathscr{D}_4 = \theta \cdot \mathscr{L}_3$ is precisely the Borel-Laplace transform of the A-side connection for small quantum cohomology on the associated mirror rank 1 Fano
- The modular K3 pencils are mirror to anticanonical K3s in the Fano
- Following Stienstra (2005) on instanton expansions for noncritical string on local CY threefolds, one defines Q by $\frac{d}{d\tau} \log Q = \frac{1}{2\pi i} \iint_\Sigma \Omega_{t(\tau)}$ and computes the Lambert expansion

$$Q \frac{d}{dQ} \log q = 1 + \sum_{k=1}^{\infty} k^2 \widetilde{N}_k \frac{Q^k}{1 - Q^k}$$

GROMOV-WITTEN INVARIANTS OF LOCAL CY FOURFOLDS

- Computing "dual" instanton numbers yields the following for $N = 2$ (here $\mathsf{N}_k = -\widetilde{N}_k/4$)

$$
\begin{gathered}
\mathsf{N}_1 = -20 \\
\mathsf{N}_2 = -820 \\
\mathsf{N}_3 = -68060 \\
\mathsf{N}_4 = -7486440 \\
\mathsf{N}_5 = -965038900 \\
\mathsf{N}_6 = -137569841980 \\
\mathsf{N}_7 = -21025364147340 \\
\mathsf{N}_8 = -3381701440136400 \\
\mathsf{N}_9 = -565563880222390140 \\
\mathsf{N}_{10} = -97547208266548098900 \\
\mathsf{N}_{11} = -17249904137787210605980 \\
\mathsf{N}_{12} = -3113965536138337597215480 \\
\vdots
\end{gathered}
$$

- These are precisely the $g = 0$ Gromov-Witten invariants of local $\mathbb{P}^3$ as computed by Klemm & Pandharipande (2007) via the Aspinwall-Morrison formula
- Conjecture (S., Malmendier, 2024): such behavior persists for each associated local CY fourfold K_F
- This is consistent with Iritani's (2023) recent work on the mirror symmetric Gamma conjecture

Quantum K Rings of Partial Flag Varieties

Irit Huq-Kuruvilla (Virginia Tech Dept of Mathematics)

Based on:
Relations in Twisted Quantum K Rings, arxiv: 2406.00916,
A Presentation for the Quantum K-Ring of Partial Flag Varieties, to appear

June 12, 2024

Quantum K-Theory

$QK^*(X)$ is a Q-deformation of the ring $K^*(X)$, given by deforming the product using the $n = 3, g = 0$ K-theoretic Gromov-Witten invariants of X.
Q are Novikov's variables, indexed by curve classes in $H_2(X)$.
It also involves a deformation of the K-theoretic Poincare pairing, which we denote $(,)_Q$.

The quantum product, which we denote $*_Q$ is defined by via structure constants in the following way: If Y_1, Y_2, Y_3 are subvarieties of X, then:

$$(\mathcal{O}_{Y_1} *_Q \mathcal{O}_{Y_2}, \mathcal{O}_{Y_3})_Q =$$

$$\sum_d Q^d \chi(\overline{M}_{0,3,d}(X); \mathcal{O}^{vir} \otimes \prod_i ev_i^* \mathcal{O}_{Y_i})$$

Example

If $X = \mathbb{P}^n$ then $K^*(X)$ is generated by $\mathcal{O}(-1)$ and has the relation:

$$(1 - \mathcal{O}(-1))^{n+1} = 0$$

$QK^*(X)$ has the same generator (over $\mathbb{C}[[Q]]$), but the relation is now:

$$(1 - \mathcal{O}(-1))^{n+1} = Q$$

In general, these rings are harder to compute than in quantum cohomology, and comparatively few examples are known where the rings are completely described.

Flag Varieties

The (type A) flag variety $X = Fl(v_1, \dot{;} v_k; N)$ is the moduli space of flags of vector subspaces:

$$V_1 \subset V_2 \subset V_3 \cdots \subset V_k \subset \mathbb{C}^N$$

Satisfying $dim(V_i) = v_i$
It is expressible as $Sl(N)/P$, for P a parabolic subgroup.

V_i determines the tautological bundle $\mathcal{S}_i$.

Let $\Lambda_y(V) := \sum_i y^i \bigwedge^i V$
The classical K ring of X is determined by the following application of the Whitney sum formula:

$$\Lambda_y(\mathcal{S}_i)\Lambda_y(\frac{\mathcal{S}_{i+1}}{\mathcal{S}_i}) = \Lambda_y(\mathcal{S}_{i+1})$$

The Whitney Conjectures

$QK(X)$ described for full flags by Naito-Sagaki-Maeno, Grassmanians by Gu-Sharpe-Mihalcea-Sharpe-Zou, incidence varieties by Xu.
Not known in general, but conjectured:

Conjecture (Gu-Mihalcea-Sharpe-Xu-Zhang-Zou)

The quantum K-theory of X is determined by the relation:

$$\Lambda_y(\mathcal{S}_i) * \Lambda_y(\frac{\mathcal{S}_{i+1}}{\mathcal{S}_i}) =$$

$$\Lambda_y(\mathcal{S}_{i+1}) - y^{v_{i+1}-v_i}\frac{Q_i}{1-Q_i}det(\frac{\mathcal{S}_{i+1}}{\mathcal{S}_i}) * (\Lambda_y(\mathcal{S}_i) - \Lambda_y(\mathcal{S}_{i-1}))$$

Where Q_i corresponds to the H_2 element $-c_1(det(\mathcal{S}_\rangle))$

GMSXZZ give a physics argument for why these conjectures should be true.

Physical Inspiration

Physically, $QK(X)$ is is the OPE ring of a certain 3D GLSM with gauge group $\prod_i U(v_i)$.

Quantum "Schubert Calculus" corresponds to the Wilson line operators.

Apply Fourier transform to $\Sigma \times S^1$ to reduce the problem to 2D which gives rise to a twisted superpotential.

After a certain choice of Chern-Simons terms, the superpotential becomes:

Superpotential

$$
\begin{aligned}
\mathcal{W} \;\; = \;\; & \frac{1}{2}\sum_{i=1}^{s}(v_i - 1)\sum_{j=1}^{v_i}(\ln X_{ij})^2 \\
& - \sum_{i=1}^{s}\sum_{1\le j<k\le v_i}(\ln X_{ij})(\ln X_{ik}) \\
& + \sum_{i=1}^{s}\left(\ln\left((-1)^{v_i-1}q_i\right)\right)\sum_{j=1}^{v_i}(\ln X_{ij}) \\
& + \sum_{i=1}^{s}\sum_{j=1}^{v_i}\sum_{k=1}^{v_{i+1}}\mathrm{Li}_2\left(X_{ij}/X_{i+1,k}\right)
\end{aligned}
$$

Li_2 is the dilogarithm.

Bethe Ansatz

Critical locus of $\mathcal{W}$ is given by the Bethe Ansatz equations:

$$(-1)^{v_i-1} Q_i X_{ij}^{v_i} \prod_{k=1}^{v_{i-1}} (1 - \frac{X_{i-1,k}}{X_{ij}}) = \prod_{k=1}^{v_i} X_{ik} \prod_{\ell=1}^{v_{i+1}} (1 - \frac{X_{i+1,\ell}}{X_{ij}})$$

If we interpret X_{ij} as Chern roots of $\mathcal{S}_i$, then a formal* symmetrization of these equations gives us the Whitney relations. This leaves us with a few questions:

What do these equations really mean?
What kind of ring do they live in?
Why does the symmetrization procedure make sense?

Abelian/non-Abelian Correspondence

For any GIT quotient, $A//G$, we can consider the *abelianization*, the quotient $A//T$ for T the maximal torus of G. By a theorem of Harada-Landweber, the K-rings of these spaces are related in the following way:

Theorem (Harada-Landweber)

There is a surjective map $\phi : K(A//T)^W \to K(A//G)$ (and more stuff!)

In our case,

$$X = A/G = (\bigoplus_i Hom(\mathbb{C}^{v_i}, \mathbb{C}^{v_{i+1}}))//\prod_i Gl(v_i)$$

The corresponding abelian quotient $A//T$ is a tower of projective bundles. ϕ identifies various $\mathcal{O}(-1)$s with Chern roots of $\mathcal{S}_i$.

Abelian/non-Abelian Correspondence for QK Rings

The corresponding claim for quantum K-rings is literally false, but we can modify it by considering a twisted quantum K-ring on the abelian side.

This means we calculate Gromov-Witten invariants using $\mathcal{O}^{vir} \otimes \pi_{n+1*} ev_{n+1}^* Eu_\lambda(\bigoplus_{r \in \text{roots}} L_r)$

These invariants also have a ring structure, and many standard results on ring relations pass over to this case.

Conjecture

Let ϕ_Q be the map extending ϕ by acting on Novikov variables and taking the limit $\lambda \to 1$:

$$\phi_Q : QK^{tw}(A//T)^W \to QK(A//G)$$

Whitney Relations

Theorem (HK)

The Abelian/non-Abelian correspondence for quantum rings holds for flag varieties.

Theorem (HK)

The Bethe Ansatz equations come from relations in $QK^{tw}(A//T)$, if we interpret each X_{ij} as its corresponding tautological bundle on $A//T$.

Thus symmetric combinations of the Bethe equations descend* to relations in $QK(X)$

Corollary

The Whitney relations give a presentation of $QK(X)$

Maulik-Okounkov Lie algebras and BPS Lie algebras—together at last.

Tommaso Maria Botta (ETH Zurich)

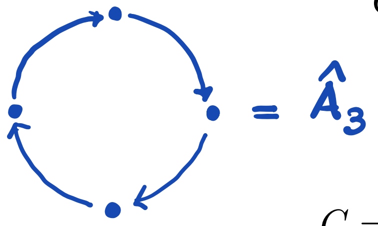
Based on: ArXiv2312.14008. Joint with **Ben Davison.**

String Math 2024

ICTP Trieste, 13.06.2024

Two tales from the 20th century

- Fix a quiver $Q = (Q_0, Q_1)$ without 1-loops (just for now!).
- Its generalized Cartan matrix is given by $C = 2 - Q - Q^T$.
- We get a Kac-Moody Lie algebra (over $\mathbb{Q}$): $\mathfrak{g}_Q^{\mathsf{KM}} = \mathfrak{g}_Q^{\mathsf{KM},+} \oplus \mathfrak{h} \oplus \mathfrak{g}_Q^{\mathsf{KM},-}$
- Nakajima quiver varities: $X(\mathsf{w},\mathsf{v}) = \mu_{\mathsf{w},\mathsf{v}}^{-1}(0)^{\zeta-\mathsf{ss}}/G_{\mathsf{v}} \subset [T^* \operatorname{Rep}_{\mathsf{w},\mathsf{v}} / G_{\mathsf{v}}]$



$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

- **Nakajima** (90s): Set $\mathbb{N}_{\mathsf{w}}^{T_{\mathsf{w}}} := \bigoplus_{\mathsf{v}\in\mathbb{N}^{Q_0}} H_{T_{\mathsf{w}}}^*(X(\mathsf{w},\mathsf{v}),\mathbb{Q})$ There is a canonical morphism

$$U(\mathfrak{g}_Q^{\mathsf{KM}}) \to \operatorname{End}(\mathbb{N}_{\mathsf{f}}^{T_{\mathsf{f}}})$$

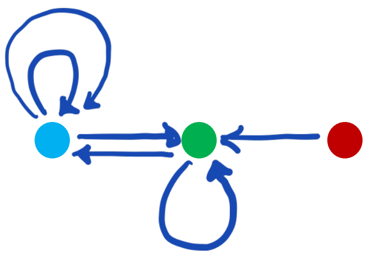
- **Kac Polynomials** ($80s$): We count (set-theoretically) iso-classes of representations of points in $\operatorname{Rep}_{\mathsf{d}}$ over $\mathbb{F}_q$:

$$\mathsf{a}_{Q,\mathsf{v}}(q) := \# \left\{ \begin{matrix} \text{isomorphism classes of absolutely indecomposable} \\ \mathsf{v}\text{-dimensional } Q\text{-modules over } \mathbb{F}_q \end{matrix} \right\} \in \mathbb{N}[q].$$

- **Kac Conjecture** (Hausel's theorem): Let $\mathfrak{g}_{Q,\mathsf{v}}^{\mathsf{KM}}$ be a root space of $\mathfrak{g}_Q^{\mathsf{KM}}$. Then $\mathsf{a}_{Q,\mathsf{v}}(q=0) = \dim(\mathfrak{g}_{Q,\mathsf{v}}^{\mathsf{KM},+})$.

Generalized Kac-Moody Lie algebras (Borcherds Lie algebra)

1. **What if we want to recover the whole Kac polynomial?**
2. **What if we want to allow 1-loops in Q?**



If so, then the Lie algebra better be $\mathbb{Z}$-graded, and we need to take care of three kind of vertices $i \in Q_0$:

- Real, i.e. without loops: $C_{ii} = 2$.
- Isotropic, i.e. with exactly one loop: $C_{ii} = 0$.
- Hyperbolic, i.e. with more than one loop: $C_{ii} < 0$.

$$Q = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} -2 & -2 & 0 \\ -2 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

This is the realm of GKM Lie algebras, a.k.a Borcherds Lie algebras.

The Lie algebra is still generated by triples of the form $\{e_i, h_i, f_i\}$ (with multiplicities) and relations, but

- If i is isotropic, these are not modelled on $\mathfrak{sl}(2)$, but rather on the 3-dim Heisenberg Lie algebra.
- The Serre relations $[e_i, e_j]^{1-C_{ij}} = 0$ apply iff i is real or if $C_{ij} = 0$.

GKM Lie algebras from Geometry

Maulik-Okounkov theory

- Arises in connection to GW theory of $X(\mathsf{w}, \mathsf{v})$.
- Produces a Lie algebra and a Yangian

$$\mathfrak{g}_Q^{\mathsf{MO}} \subset Y_T\left(\mathfrak{g}_Q^{\mathsf{MO}}\right) \subset \prod_{\mathsf{w}\in\mathbb{N}} \mathrm{End}\left(\mathbb{N}_{\mathsf{w}}^{T_{\mathsf{w}}}\right)$$

- These are reconstructed from the braidings (R-matrices) of $Y_T\left(\mathfrak{g}_Q^{\mathsf{MO}}\right)$, which are geometrically defined:

$$R = \mathrm{Stab}_-^\vee \circ \mathrm{Stab}_+ \in \mathrm{End}\left(\mathbb{N}_{\mathsf{w}'} \otimes \mathbb{N}_{\mathsf{w}''}\right)_{\mathsf{loc}}$$

- There is a decomposition

$$\mathfrak{g}_Q^{\mathsf{MO}} = \mathfrak{g}_Q^{\mathsf{MO},+} \oplus \mathfrak{h}_Q^{\mathsf{MO}} \oplus \mathfrak{g}_Q^{\mathsf{MO},-}$$

- $\mathfrak{g}_Q^{\mathsf{MO}}$ controls the quantum multiplication

$$c_1(\lambda)\star_{\mathsf{quantum}} = c_1(\lambda) \cup -\hbar \sum_{\mathsf{v}>0} \frac{q^{\mathsf{v}}}{1-q^{\mathsf{v}}} \alpha_{\mathsf{v}}\alpha_{-\mathsf{v}} + \ldots$$

for $\alpha_{\mathsf{v}} \in \mathfrak{g}_{Q,\mathsf{v}}^{\mathsf{MO}},\ \alpha_{-\mathsf{v}} \in \mathfrak{g}_{Q,-\mathsf{v}}^{\mathsf{MO}}$

Cohomological Hall Algebras (CoHAs)

- Categorifies DT theory of Jacobi algebras.
- Produces a (bi-)algebra (the CoHA), defined as

$$H\mathcal{A}_{\Pi_Q}^T := \bigoplus_{\mathsf{v}\in\mathbb{N}^{Q_0}} H_T^{\mathsf{BM}}\left([\mu_{\mathsf{v}}^{-1}(0)/G_{\mathsf{v}}], \mathbb{Q}\right)$$

$$H^{\mathsf{BM}}\left([\mu_{\mathsf{v}}^{-1}(0)/G_{\mathsf{v}}], \mathbb{Q}\right) = H^*(T\,\mathrm{Rep}(\mathsf{v}), \varphi_{\mathrm{Tr}W}\mathbb{Q})$$

(Kontsevich-Soibelman, Schiffmann-Vasserot, Davison)

- The multiplication and its action on $\mathbb{N}_{\mathsf{w}}^{T_{\mathsf{w}}}$ are given by Hecke correspondences.
- There is a **GKM Lie algebra** (Davison-Hennecart-Schlegel Mejia)

$$\mathfrak{g}_Q^{\mathsf{BPS},+} \oplus \mathfrak{h}_Q^{\mathsf{BPS}} \oplus \mathfrak{g}_Q^{\mathsf{BPS},-}$$

and a PBW-type isomorphism

$$\mathrm{gr}\left(H\mathcal{A}_{\Pi_Q}\right) \cong \mathrm{Sym}\left(\mathfrak{g}_Q^{\mathsf{BPS},+} \otimes H(B\mathbb{C}^\times)\right)$$

- Kac polynomials are recovered by the graded dimensions of root spaces of $\mathfrak{g}_Q^{\mathsf{BPS},+}$ (Davison+Mozgovoy)

Main results

Theorem [B-Davison]

For every quiver Q, there is an isomorphism $\mathfrak{g}_Q^{\mathsf{MO}} \cong \mathfrak{g}_Q^{\mathsf{BPS}} \otimes_{\mathbb{Q}} H^*(BT)$ (modulo center) intertwining the natural actions on cohomology of quiver varieties. In particular, the MO Lie algebra is GKM (and defined over $\mathbb{Q}$).

Corollary [B-Davison, Schiffmann-Vasserot]

There is an isomorphism $Y_T(\mathfrak{g}_Q^{\mathsf{MO}})^+ \cong H\mathcal{A}^T_{\Pi Q}$ intertwining the natural actions on cohomology of Nakajima quiver varieties.

Corollary (**Okounkov's Conjecture**) [B-Davison, Schiffmann-Vasserot]

For any dimension vector $\mathsf{v} \in \mathbb{N}^{Q_0}$, there is an equality: $\sum_{k\in\mathbb{Z}} \dim((\mathfrak{g}^{\mathsf{MO}}_{Q,\mathsf{v}})^k)q^{k/2} = \mathsf{a}_{Q,\mathsf{v}}(q^{-1})$.

Theorem [B-Davison]

There is a canonical injective morphism $\Psi : H_T(X(\mathsf{w},\mathsf{v}),\mathbb{Q}) \to H_T^{\mathsf{BM}}([\mu^{-1}_{\mathsf{w},\mathsf{v}}(0)/G_{\mathsf{v}}],\mathbb{Q}) = H\mathcal{A}^T_{\Pi_{Q_{\mathsf{w}}},(\mathsf{v},1)}$ whose image is $\mathfrak{g}^{\mathsf{BPS}}_{Q_{\mathsf{w}},(\mathsf{v},1)} \otimes H^*(BT) \subset H\mathcal{A}_{\Pi_{Q_{\mathsf{w}}},(\mathsf{v},1)}$.

Remark: The map Ψ is called non-abelian stable envelope, and we borrow if from Aganagic and Okounkov's work.

The End

Thank you!

Twisted Tools for (Untwisted) Quantum Field Theory

String Math 2024

Justin Kulp
with Davide Gaiotto, and Jingxiang Wu
and earlier works with Kasia Budzik,
Brian Williams, and Matthew Yu.

Simons Center for Geometry and Physics

13/June/2024

arXiv:2403.13049, arXiv:2306.01039, arXiv:2207.14321

Outline and Punchlines

- QFTs have "higher" **multilinear k-ary operations** ("brackets")

$$\{-,-,\ldots,-\} \tag{1}$$

  - Control: **deformations**, **OPEs**, and **anomalies**
  - Factorization algebras, operads, and ∞-algebras
- Familiar to high energy physicists and mathematicians who have studied twisted SQFTs (descent relations) or SFT
  - Can go very far in Holomorphic/Topological theories

Three Takeaways

1. QFTs have higher brackets defined by η-vector
2. Brackets are computable and encode anomalies and OPEs
3. Non-renormalization theorem for HT theories

The η-Function

Deformations and the Beta Function

- Given a QFT T, it can be deformed by turning on interactions

$$S_T + \sum_i g^i \int_{\mathbb{R}^d} \mathcal{I}_i \tag{2}$$

  - g^i are "coordinates" on "theory space"
  - Work perturbatively in couplings g^i (formal power series)
- Generic QFT (point) is not scale invariant
  - Scale xform on T is traded for a change of the couplings
- Encode infinitesimal scale transformations in **beta function**

$$\boldsymbol{\beta} = \sum_i \beta^i(g) \frac{\partial}{\partial g^i} \tag{3}$$

$$\beta^i(g) = \underbrace{(\Delta_i - d)}_{\text{Classical}} g^i + O(g^2) \tag{4}$$

The Eta Function

- Can compute analog of β **for any type of transformation**.
  Ex. Non-relativistic scale transformations $(t, x) \mapsto (\lambda^z t, \lambda x)$
- Consider $T \hookrightarrow (\widetilde{T}, Q_{\mathrm{BRST}})$ described in a BV-BRST formalism.
  - $\widetilde{T}$ is bigger ambient theory with ghosts, anti-ghosts, anti-fields, etc. and odd nilpotent symmetry Q_{BRST}
  - Observables in T are recovered from $\widetilde{T}$ by taking Q_{BRST} coho
- Deform T by deforming $\widetilde{T}$ without breaking BRST symmetry
  - Deformations $\mathcal{D}[\widetilde{T}]$ are a formal pointed dg-supermanifold.
- BRST symmetry will be encoded in **eta function**

$$\boldsymbol{\eta} = \sum_i \eta^i(g) \frac{\partial}{\partial g^i} \tag{5}$$

  - Linear term tells us if interaction $\mathcal{I}$ explicitly violates BRST symmetry; higher order terms do so "quantum mechnically"

L_∞-Algebras

- Since $Q^2 = 0$, the eta function $\boldsymbol{\eta}^2 = 0$.
  - Gives **quadratic constraints** on coefficient functions $\eta^i(g)$

$$\eta^i(g) = \sum_{n>0} \frac{1}{n!} \sum_{j_1\cdots j_n} \eta^i_{j_1\cdots j_n} g^{j_1} \cdots g^{j_n} \tag{6}$$

- Define the following multilinear operation $\mathrm{Int}^{\otimes n} \to \mathrm{Int}$

$$\{\mathcal{I}_{j_1}, \cdots, \mathcal{I}_{j_n}\} = \eta^i_{j_1\cdots j_n} \mathcal{I}_i \tag{7}$$

  - The BRST variation is a **Maurer-Cartan equation**:

$$\boldsymbol{\eta}\mathcal{I} = \{\mathcal{I}\} + \frac{1}{2!}\{\mathcal{I},\mathcal{I}\} + \frac{1}{3!}\{\mathcal{I},\mathcal{I},\mathcal{I}\} + \dots \tag{8}$$

$\boldsymbol{\eta}^2 = 0 \quad \Leftrightarrow$ Coefficients $\eta^i_{j_1\cdots j_n}$ and brackets $\{\cdot, \dots, \cdot\}$ define an **L_∞-algebra** on Int.

BRACKETS ARE COMPUTABLE

- Write $[Q, \mathcal{I}_i] = \sum_j \eta_i^j \mathcal{I}_j + d\mathcal{J}_i$. In a sharp cutoff regularization:

$$\{\mathcal{I}, \mathcal{I}\}(x_2) \overset{\text{Sharp Cutoff}}{=} \int_{|x_{12}|=\epsilon} \mathcal{I}(x_1)\mathcal{J}(x_2) + \mathcal{J}(x_1)\mathcal{I}(x_2) \tag{9}$$

Ex. 2d G gauge theory and S_{Matter} with G global symmetry

$$S_T = -\frac{1}{4}\int d^2x\, F_{\mu\nu}F^{\mu\nu} + S_{\text{Matter}} \tag{10}$$

Ex. Free fermions with vector current $J_a^\mu = \bar{\psi}\gamma^\mu t_a \psi$.

- Add ghosts $T \hookrightarrow (\widetilde{T}, Q)$ and study **interaction** $\mathcal{I} = A_\mu J^\mu$
- Recover well-known **1-loop anomaly for G-gauge theory**

$$\{A_\mu J^\mu, A_\nu J^\nu\} = \#\, cF_{12}\,. \tag{11}$$

Generalizations

- Systematically compute corrections to Q on local operators

$$Q\,\mathcal{O} = \{\mathcal{O}\} + \{\mathcal{I},\mathcal{O}\} + \frac{1}{2}\{\mathcal{I},\mathcal{I},\mathcal{O}\} + \cdots . \tag{12}$$

  - Useful for computing perturbative corrections to BPS operators in twisted SQFTs (like Konishi anomaly corrections)

- **Position dependent interactions**
  - Momentum-inflow p^i at each vertex $\otimes_i \mathrm{Int}_{p^{(i)}} \to \mathrm{Int}_{\sum_i p^{(i)}}$
  - Reflects "momentum-coloured operad" structure

$$\{\mathcal{I}_{i_1}\ {}_{p^{(1)}}\ \mathcal{I}_{i_2}\ {}_{p^{(2)}}\ \ldots\ {}_{p^{(n-1)}}\ \mathcal{I}_{i_n}\} \tag{13}$$

- Special case: holomorphic momentum inflow recovers **λ-brackets** and n-Lie or homotopy conformal algebras

- **Auxiliary, boundary, and defect systems**

  Ex. All OPE coefficients of chiral VOA from higher brackets

Holomorphic-Topological Theories

Holomorphic-Topological Theories

- "**Holomorphic-Topological**" means flat spacetime has structure of $\mathbb{C}^H \times \mathbb{R}^T$ with coords $(x^{\mathbb{C}}, \bar{x}^{\mathbb{C}}, x^{\mathbb{R}})$
  - Anti-holomorphic translations in $\mathbb{C}^H$ and translations in $\mathbb{R}^T$ are gauge symmetries (Q_{BRST}-exact)
  - Interested in theories with action

$$\int_{\mathbb{C}^H \times \mathbb{R}^T} \left[(\Phi, \mathrm{d}\,\Phi) + \mathcal{I}(\Phi)\right]\, d^H x^{\mathbb{C}} \tag{14}$$

  - Φ is a "superfield," and $dx^{\mathbb{R}}$ and $d\bar{x}^{\mathbb{C}}$ are "superspace coordinates" (superfields are form-valued).
- In such theories, we will be interested in brackets of the form

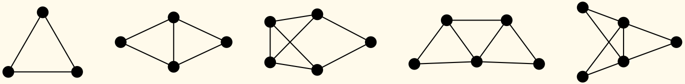
$$\{\mathcal{O}_1\,{}_{\lambda_1} \cdots {}_{\lambda_{n-1}}\,\mathcal{O}_n\} \tag{15}$$

Holomorphic-Topological Integrals

- The **Feynman integrals** that contribute will take the form:

$$I_\Gamma(\lambda; z) = \int_{\mathcal{M}^{|\Gamma_0|-1}} \left[\prod_{v \in \Gamma_0}^{v \neq v_*} \mathrm{dVol}_v e^{\lambda_v \cdot x_v^{\mathbb{C}}} \right] \mathrm{d} \left[\prod_{e \in \Gamma_1} P_\epsilon(x_{e(0)} - x_{e(1)} + z_e) \right]$$

- Non-vanishing Feynman diagrams are **n-Laman graphs**



  - $n = H + T$
  - Follows by counting form degrees of the integrand

- $I_\Gamma(\lambda; z)$ has a number of symmetries/identities: symmetries from the graph, and under shifts of z_e.

Quadratic Identities and Non-Renormalization

- Feynman integrals (more generally diagrams) satisfy infinite collections of (geometric) **quadratic identities**:

$$\sum_{\text{Laman } S} \sigma(\Gamma, S) I_{\Gamma[S]}(\lambda + \partial; z) \cdot I_{\Gamma(S)}(\lambda; z) = 0\,. \tag{16}$$

  - Identities imply (higher)-associativity of the accompanying brackets in a diagram-by-diagram way
  - Can **bootstrap** all Feynman integrals from these identities?

- A purely graphical proof demanding consistency between (H, T) and $(H+1, T-2)$ theories, implies:

Non-Renormalization Theorem

All loop graphs in $(H, T \geq 2)$-theories must vanish.