Unitary Categorical Symmetry

Mathew Bullimore

Durham University

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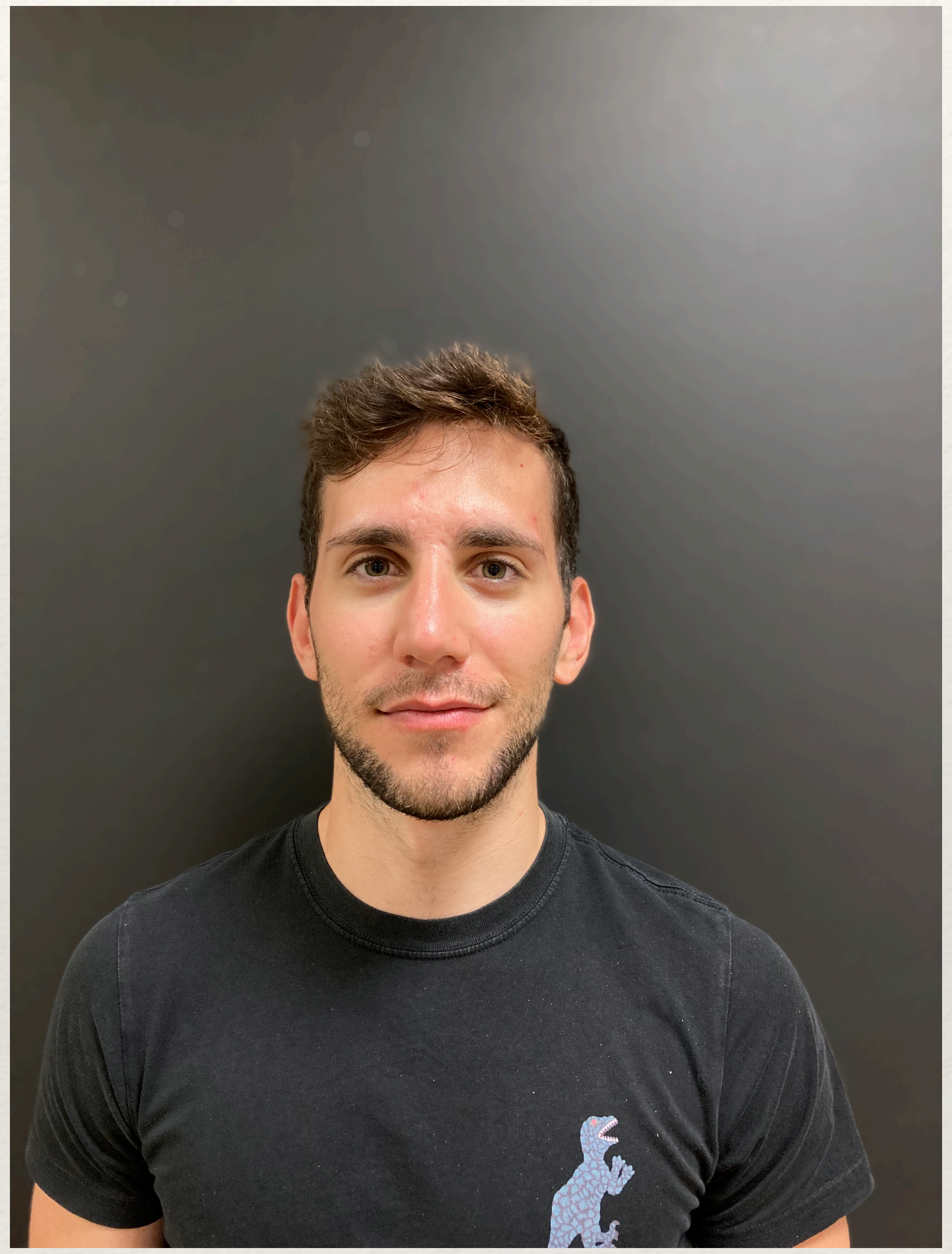
The Team

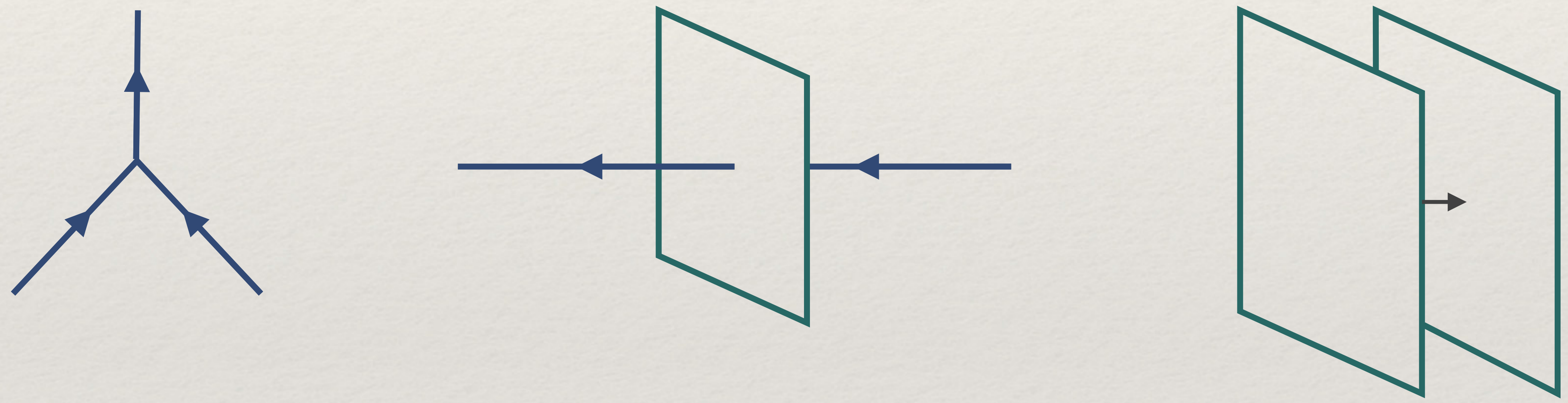
Thomas Bartsch



Andrea Grigoletto

Introduction

- This talk is about categorical symmetry in QFT.
- The philosophy is that symmetries are topological defects. [Gaiotto-Kapustin-Seiberg-Willett '14]



  - What is their mathematical structure?
  - How do they act on and organise observables?
  - How is unitarity / reflection positivity incorporated?

Plan

- Part 1: Examples.
- Part 2: General Expectations.
- Part 3: Solitonic Symmetries.
- Part 4: Further Applications.

Part 1: Examples

One dimension

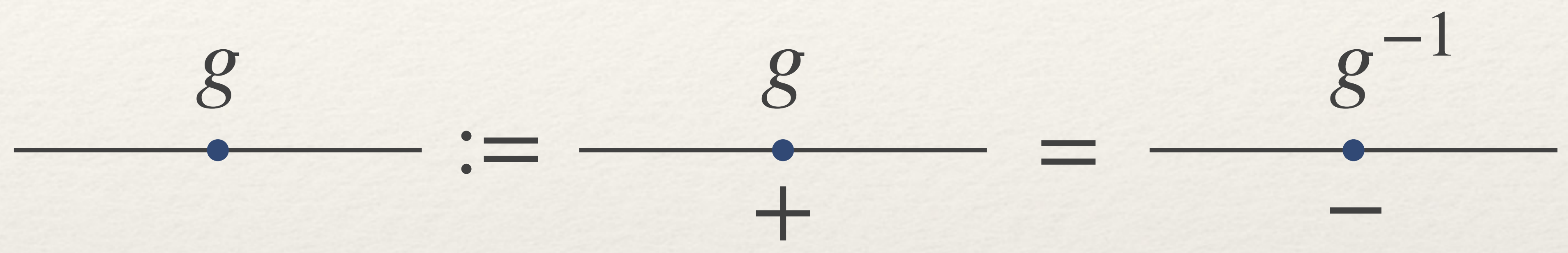
- Quantum mechanics with a finite symmetry group G.
- There are unitary operators U_g commuting with the Hamiltonian.
- They compose according to

$$U_g U_h = \alpha(g,h)\, U_{gh} \ .$$

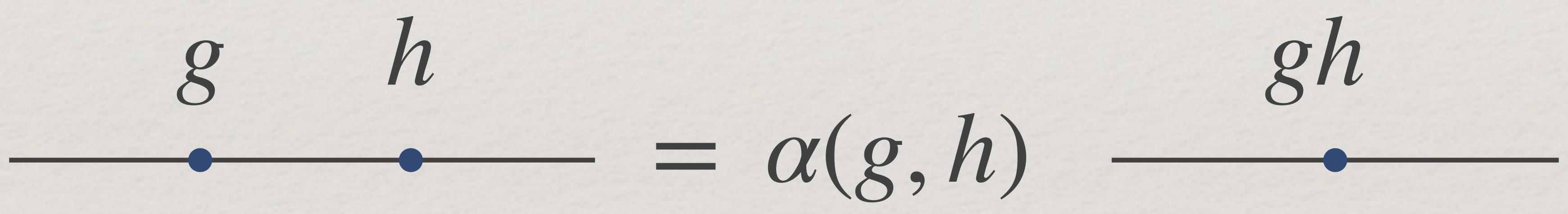
- There is a 't Hooft anomaly captured by a normalised 2-cocycle $\alpha : G \times G \to U(1)$.
- The Hilbert space transforms in a projective unitary representation of G.

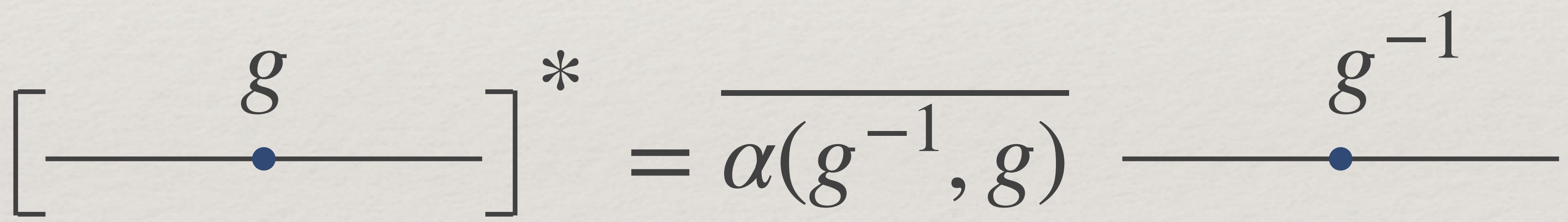
One dimension

- Now formulate in the language of categorical symmetries.
- Topological defects associated to oriented points:



- They generate the twisted group algebra ${}^{\alpha}\mathbb{C}[G]$.



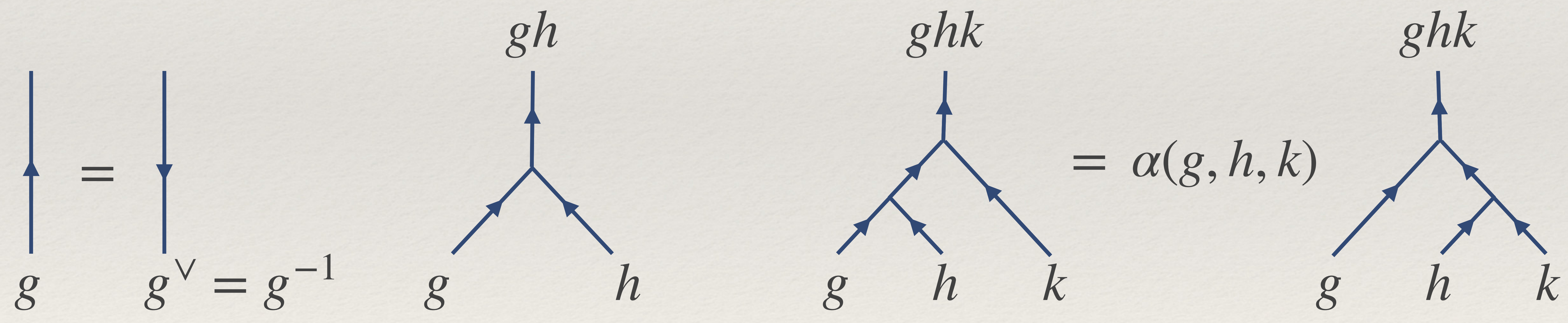


- This is a finite-dimensional C*-algebra.
- I will call it a finite-dimensional O_1- † algebra.

["Dagger n-categories", Ferrer et. al. '24]

Two dimensions

- Two-dimensional unitary theory with finite symmetry group G.
- 't Hooft anomaly captured by a normalised 3-cocycle $\alpha : G \times G \times G \to U(1)$.
- There are topological defects $g \in G$ associated to oriented lines.



Two dimensions

- Consider action on the Hilbert space on S^1.

- This generates the group algebra $\mathbb{C}[G]$.
- It is a finite-dimensional C*-algebra / O_1- † algebra.
- However, it does not see the ’t Hooft anomaly.

Two dimensions

- Consider action on the twisted sector Hilbert spaces on S^1.

- This generates the twisted Drinfeld double ${}^\tau\mathbb{C}[G/\!\!/G]$.
- It is a finite-dimensional C*-algebra/O_1- † algebra.
- It sees the 't Hooft anomaly via transgression.

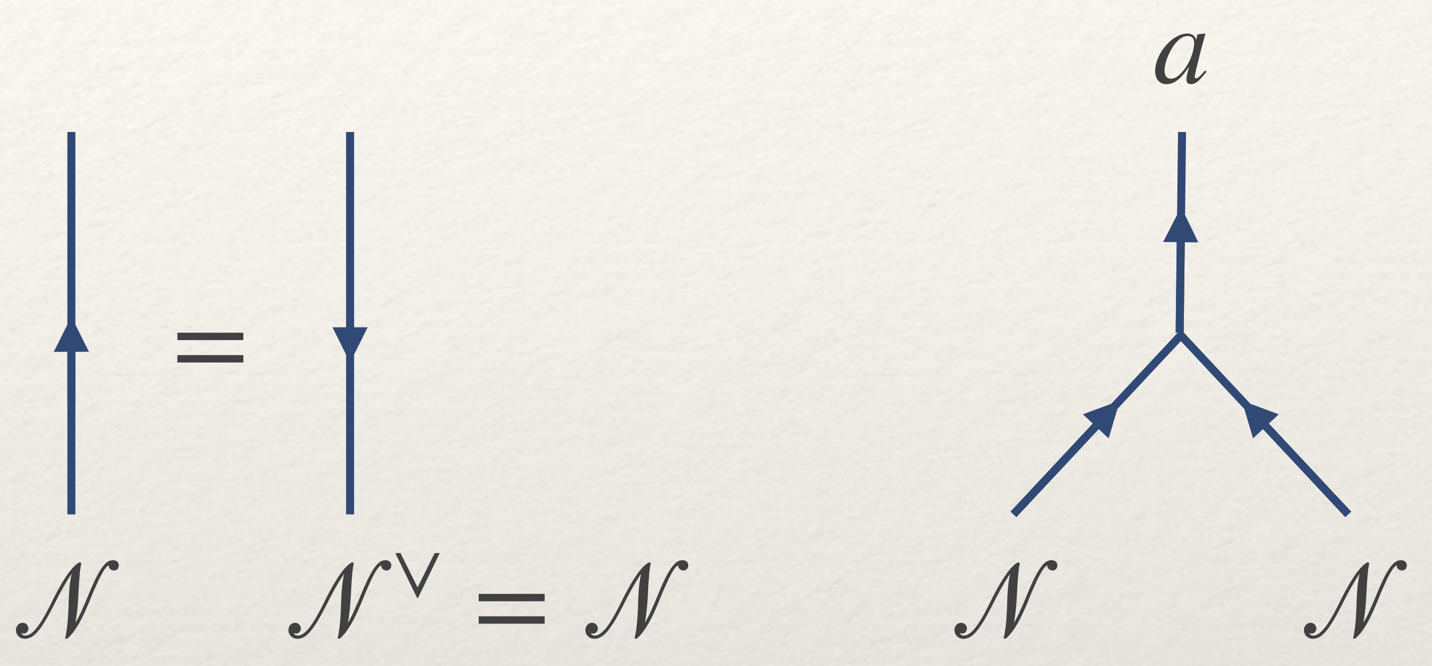
$$\tau_x(g,h) = \frac{\alpha(g,h,x)\,\alpha({}^{gh}x,g,h)}{\alpha(g,{}^{h}x,h)}$$

[Dijkgraaf-Pasquier-Roche '90][Willerton '05]

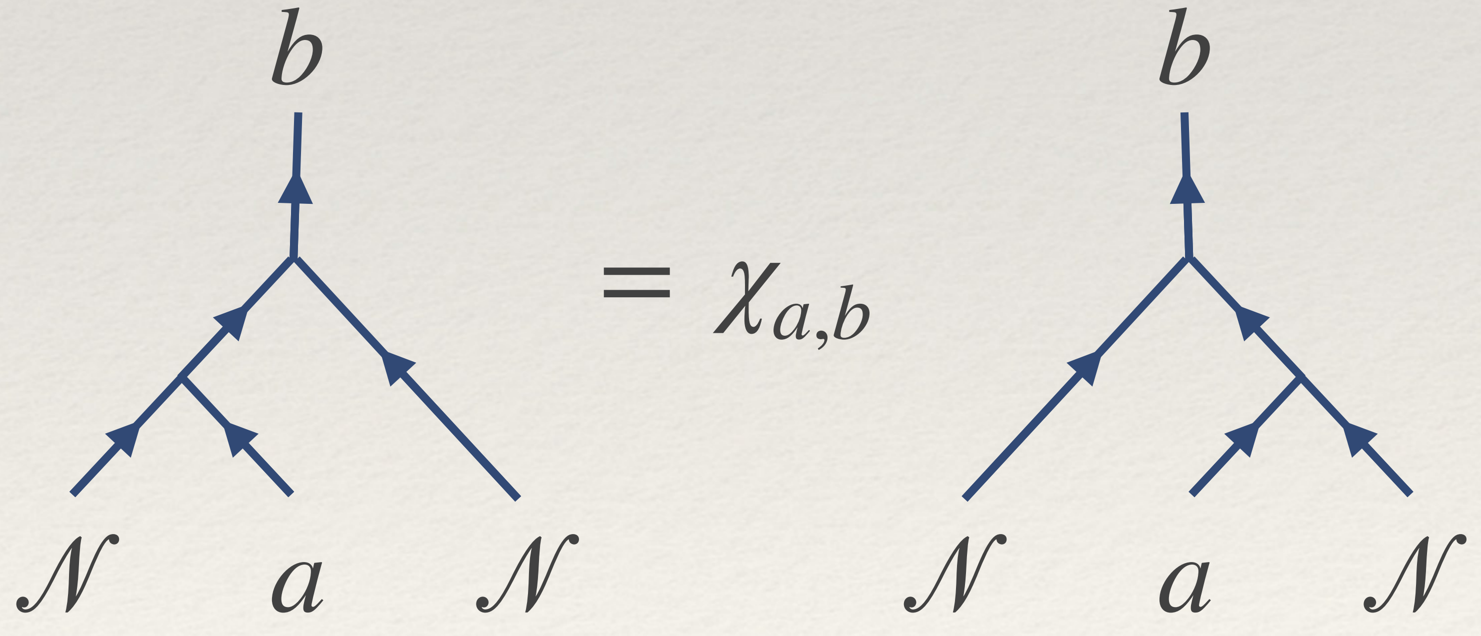
Two dimensions

- Tambara-Yamagami symmetry:
  - Anomaly free abelian group A.
  - Additional non-invertible line $\mathcal{N}$.

$$\mathcal{N}^\vee = \mathcal{N} \qquad \mathcal{N}^2 = \oplus_{a \in A}\, a$$



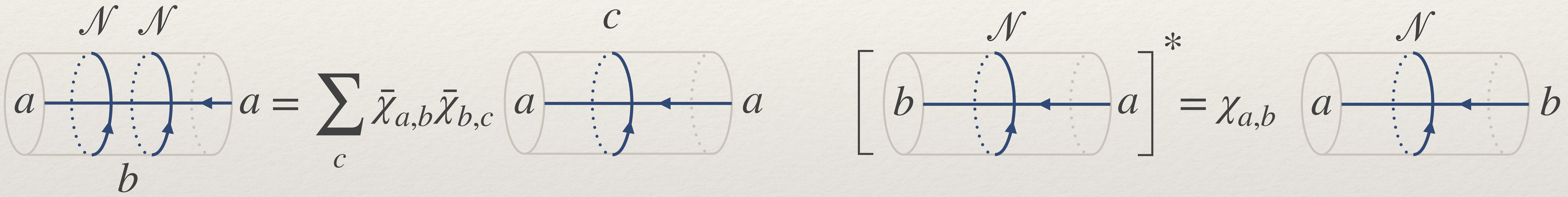
- Additional data fixing associators involving $\mathcal{N}$:
  - Symmetric bi-character $\chi : A \times A \to U(1)$.
  - A choice of square root $s = \pm 1/\sqrt{|A|}$.



[Tambara-Yamagami '98]

Two dimensions

- Consider action on the twisted sector Hilbert spaces on S^1.



- This generate the Tambara-Yamagami "tube algebra".
- This is again a finite-dimensional C*-algebra / O_1- † algebra.
- There are $\frac{1}{2}|A|(|A|+7)$ irreducible C*-representations.

[Bartsch-MB-Grigoletto '23]

Two dimensions

- What mathematical structure do we need to get a O_1- $\dagger$ algebra on S^1?
- Start from a unitary fusion category:
  - $\vee$: duals of objects.
  - $\dagger$: dagger on morphisms.
  - Compatibility of ($\vee$, $\dagger$).
- This ensures (almost) consistent graphical calculus on oriented 2-manifolds.
- I will call this an O_2- $\dagger$ fusion category.

["Dagger n-categories", Ferrer et. al. '24]

Two dimensions

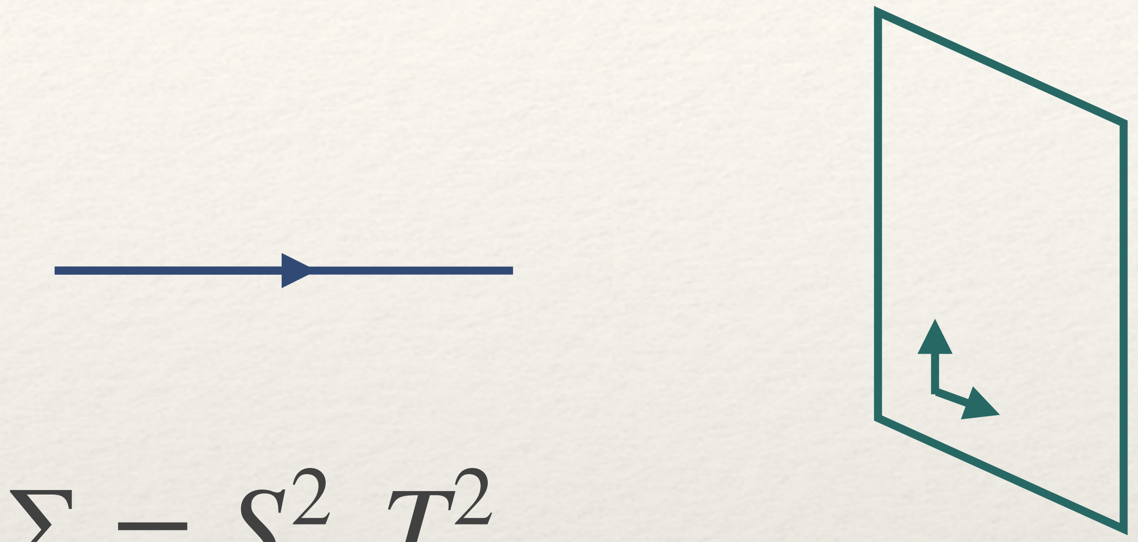
- Consider action on twisted sector Hilbert spaces on S^1.
- This generates the tube algebra of C .

$$\mathrm{Tube}_{S^1}(\mathsf{C}) = \int_{S^1} \mathsf{C}$$

  - C is a unitary fusion category $\longrightarrow$ $\mathrm{Tube}_{S^1}(\mathsf{C})$ is a C^*-algebra.
  - i.e. O_2- $\dagger$ fusion structure on C $\longrightarrow$ O_1- $\dagger$ algebra structure on $\mathrm{Tube}_{S^1}(\mathsf{C})$.
  - Categorical unitarity implies physical unitarity.
- What about higher dimensions?

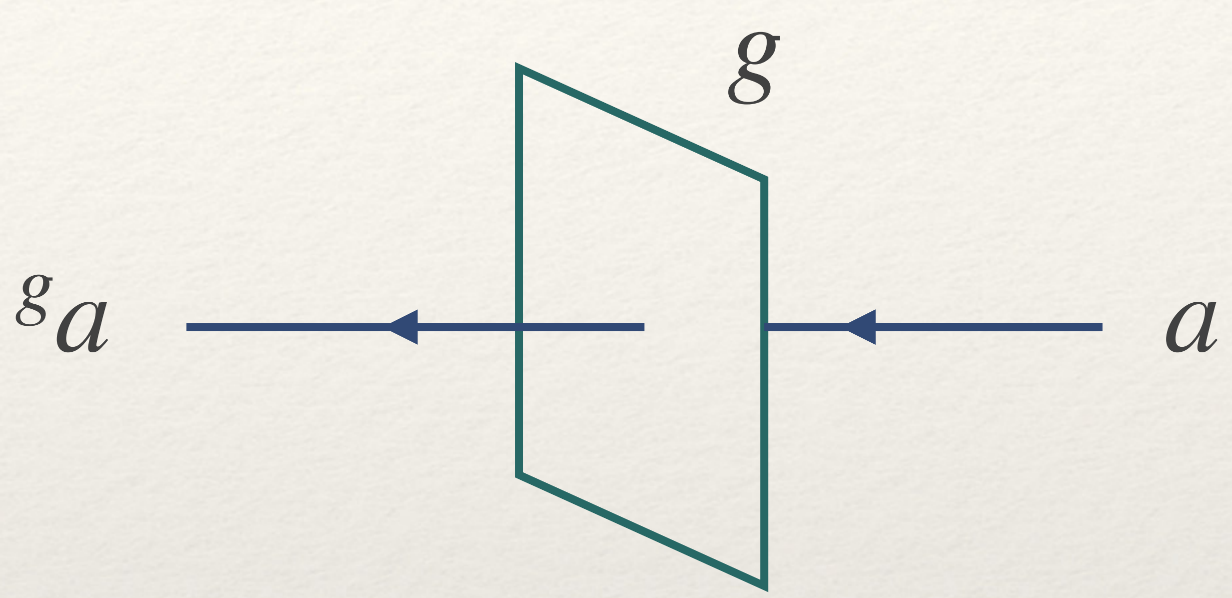
Three dimensions

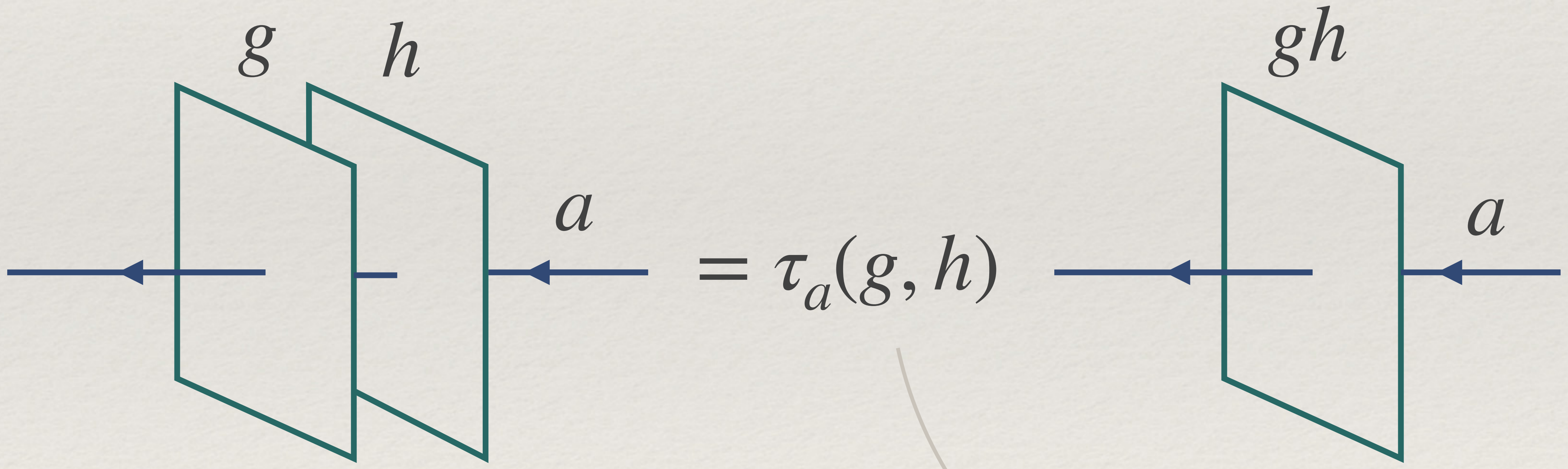
- There are new features in three dimensions.
  - Topological lines and surfaces.
  - Hilbert space on closed oriented surfaces $\Sigma = S^2, T^2, \ldots$
  - Various types of twisted sectors.
- I will focus on an example!



Three dimensions

- Consider a split 2-group symmetry:
  - 0-form symmetry G.
  - 1-form symmetry A.
  - An action $G \to \operatorname{Aut} A$.
  - 't Hooft anomaly $\lambda : G \times G \to A^\vee$.



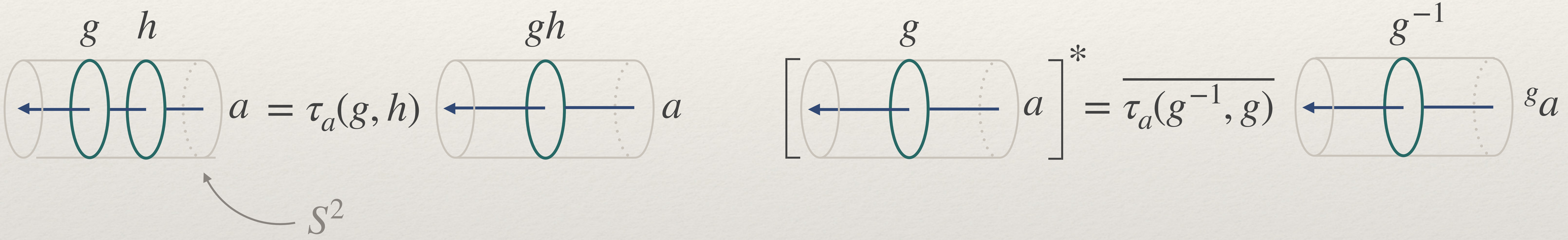


$\tau_a(g, h) := \langle \lambda(g, h), {}^{gh}a \rangle$

- What is the O_1- † algebra associated to a closed oriented surface Σ ?

Three dimensions

- Consider action on twisted sectors on $\Sigma = S^2$.



- This generates the twisted groupoid algebra ${}^{\tau}\mathbb{C}[G//A]$.
- It is again a finite-dimensional O_1- † algebra.
- It sees the mixed 't Hooft anomaly via the twist.

$\tau_a(g, h) = \langle \lambda(g, h), {}^{gh}a \rangle$

[Bartsch-MB-Grigoletto '23]

Three dimensions

- What mathematical structure do we need to get a O_1- $\dagger$ algebra on any Σ ?
- Unitary fusion 2-category:
  - $\vee$: duals of objects.
  - $\dagger_1$, $\dagger_2$: dagger for 1-morphisms, 2-morphisms.
  - Compatibility conditions on ($\vee$, $\dagger_1$, $\dagger_2$).
- This ensures (almost) consistent graphical calculus on oriented 3-manifolds.
- I will call this an O_3- $\dagger$ fusion 2-category.

→ ["Dagger n-categories", Ferrer et. al. '24]

Three dimensions

- Consider a 2-dimensional oriented surface Σ.
- There is an associated tube algebra.

$$\mathrm{Tube}_\Sigma(\mathsf{C}) = \int_\Sigma \mathsf{C}$$

  - C is a unitary fusion 2-category $\longrightarrow$ $\mathrm{Tube}_\Sigma(\mathsf{C})$ is a C^*-algebra.
  - O_3- † fusion structure on C $\longrightarrow$ O_1- † algebra structure on $\mathrm{Tube}_\Sigma(\mathsf{C})$.
  - Categorical unitarity implies physical unitarity.
- What is the general structure in higher dimensions?

Part II: General Expectations

Symmetry Type

- Introduce notion of symmetry type.
- This incorporate principles of relativity and unitarity.
- This is a tangential structure $\rho : H_d \to O_d$.
  - Relativity requires that $SO_n \subset \mathrm{Im}(\rho)$.
  - Unitarity encoded in extended symmetry group

$$1 \to H_d \to \hat{H}_d \to \mathbb{Z}_2 \to 1 \,.$$

H_d	$\hat{H}_d$	
SO_d	O_d	bosons
$Spin_d$	Pin_d^+	fermions
O_d	$\mathbb{Z}_2 \times O_d$	bosons, T
Pin_d^+	$\widehat{Pin}_d^+$	fermions, $T^2 = (-1)^F$
Pin_d^-	$\widehat{Pin}_d^-$	fermions, $T^2 = 1$

[Freed-Hopkins '16]

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Examples!

H_d	$\hat{H}_d$	
SO_d	O_d	bosons
$Spin_d$	Pin_d^+	fermions
O_d	$\mathbb{Z}_2 \times O_d$	bosons, T
Pin_d^+	$\widehat{Pin}_d^+$	fermions, $T^2 = (-1)^F$
Pin_d^-	$\widehat{Pin}_d^-$	fermions, $T^2 = 1$

[Freed-Hopkins '16]

Symmetry Categories

- Finite symmetries are captured by a collection $\{\mathsf{C}_{n,d}\}$, $n = 0, \ldots, d$.
  - $\mathsf{C}_{n,d} \sim n$-dimensional topological defects in d dimensions.
  - $\mathsf{C}_{n,d} \sim n$-category whose structure depends on $\rho : H_d \to O_d$.
  - A universal feature is equivalences

$$\mathsf{C}_{n,d} \longrightarrow \Omega\, \mathsf{C}_{n+1,d} \, .$$

"n-dimensional defects are junctions between trivial $(n+1)$-dimensional defects"

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"n-dimensional defects are junctions between trivial $(n+1)$-dimensional defects"

Symmetry Categories

- Choose a symmetry type $\rho : H_d \to O_d$.
- The expectation is that $\mathsf{C}_{n,d}$ is a "$\hat{H}_d$ -dagger E_{d-n}-fusion n-category".
  - n-category captures n-dimensional topological defects.
  - E_{d-n}-fusion corresponds to fusion in transverse $\mathbb{R}^{d-n}$.
  - $\hat{H}_d$- $\dagger$ structure includes compatible daggers ($\dagger_0$, $\dagger_1$, …, $\dagger_n$).
- The idea is to have a (almost) consistent graphical calculus on H_d - manifolds.

[Baez-Dolan '95]["Dagger n-categories", Ferrer et. al. '24]

Examples

- We already encountered examples with symmetry type $H = SO$.
  - O_1-dagger E_1-algebra = finite-dimensional C*-algebra.
  - O_2-dagger E_1-fusion category = unitary fusion category.
  - O_3-dagger E_1-fusion 2-category = unitary fusion 2-category.
- What about other symmetry types?
  - $Spin \to$ super or $\mathbb{Z}_2$-graded analogues.
  - $O, Pin^{\pm} \to$ supplement with real structure.

Integration

- The action of symmetries on a QFT comes from "integration" over H_k -manifolds M_k.

$$\mathsf{C}_{n,d} \longrightarrow \int_{M_k} \mathsf{C}_{n,d}$$

$\hat{H}_d$ -dagger	$\hat{H}_{d-k}$ -dagger
E_{d-n}-fusion	E_{d-n}-fusion
n-category	$(n-k)$-category

- Example: $H = SO$, $n = k = d - 1$, finite-dimensional C*-algebra $\text{Tube}_{M_{d-1}}(\mathsf{C}) = \int_{M_{d-1}} \mathsf{C}$.
- This acts on the Hilbert space on M_{d-1}.

Part III : Solitonic Symmetries

Universal Example

- Symmetry type $\rho : H_d \to O_d$ but no internal symmetry.
- n-dimensional topological defects ~ n-dimensional TQFTs embedded in d-dimensions.
- This is the realm of the tangle hypothesis.
- Symmetry categories are variations on n-Hilbert spaces

$$\mathsf{C}_{n,d} = \mathsf{Hilb}^{H}_{n,d} \,.$$

[Baez-Dolan '95][Baez '96]

Universal Example

- Let's look at low dimensional examples.
- Consider symmetry type $H = SO$. ← I omit the superscript for $H = SO$.
- Point-like defects:
  - $\mathsf{Hilb}_{0,0} = \mathbb{C}$ as a C*-vector space.
  - $\mathsf{Hilb}_{0,1} = \mathbb{C}$ as a C*-algebra.
  - $\mathsf{Hilb}_{0,2} = \mathbb{C}$ as a commutative C*-algebra.

Universal Example

- Let's look at low dimensional examples.
- Consider symmetry type $H = SO$. ← I omit the superscript for $H = SO$.
- Line defects:
  - $\mathsf{Hilb}_{1,1} = \mathsf{Hilb}$, as a unitary category.
  - $\mathsf{Hilb}_{1,2} = \mathsf{Hilb}$, as a unitary fusion category.
  - $\mathsf{Hilb}_{1,3} = \mathsf{Hilb}$, as a unitary braided fusion category.
  - $\mathsf{Hilb}_{1,4} = \mathsf{Hilb}$, as a unitary symmetric fusion category.

Universal Example

- What about surfaces?
- First, isomorphism classes of invertible objects in $\mathsf{Hilb}^H_{n,d}$:
  - $n < d \longrightarrow \quad \mathrm{Hom}(\Omega^H_n, U(1))$
  - $n = d \longrightarrow \begin{cases} \mathrm{Hom}(\Omega^H_d, U(1)) & \text{if } d = \text{odd} \\ \mathrm{Hom}(\Omega^H_d, U(1)) \times \mathbb{R}_{>0} & \text{if } d = \text{even} \end{cases}$
- This feeds into: [Freed-Hopkins '16]
  - $\mathsf{Hilb}_{2,2} \sim$ commutative †-Frobenius algebras.
  - $\mathsf{Hilb}_{2,3} \sim$ normalised commutative †-Frobenius algebras.

Universal Example

- What about surfaces?
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- This feeds into: [Freed-Hopkins '16]
  - $\mathsf{Hilb}_{2,2} \sim$ commutative †-Frobenius algebras.
  - $\mathsf{Hilb}_{2,3} \sim$ normalised commutative †-Frobenius algebras. ⟶ Fix $r_j = 1$.

Solitonic Symmetries

- Consider a d-dimensional sigma model with target X . ← I assume finite $\pi_{\leq n}X$.
- Solitonic symmetries arise from topology of X .
- n-dimensional defects ~ n-dimensional TQFTs embedded in d dimensions with coupling to X

$$\mathsf{C}_{n,d} := [\, \pi_{\leq n}X\, , \mathsf{Hilb}^{H}_{n,d}\,]$$

  - Here $\pi_{\leq n}X$ is the homotopy n-groupoid of X.
  - A type of unitary higher local systems on X .
  - Invertible part reproduces $(d-n-1)$-form symmetry $\mathrm{Hom}(\Omega^{H}_{n}(X), U(1))$.

[Chen-Tanizaki '23][Pace '23][Pace-Zhu-Beaudry-Wen '23]

Example

- Consider a 4-dimensional sigma model with $H = Spin$.

$$B^3\mathbb{Z}_2 \longrightarrow \pi_{\leq 3}X \longrightarrow B^2\mathbb{Z}_2$$

- There is a simple object $\mathcal{N}$ such that:
  - $\mathcal{N}$ is obtained by coupling to minimal TQFT $\mathcal{A}_{2,1}$.
  - $\mathcal{N}^2 = \mathcal{Z}_2 =$ decoupled $\mathbb{Z}_2$ Dijkgraaf-Witten theory.
  - $\mathcal{N}^\vee = \mathcal{N}$.

[Hsin-Lam-Seiberg '18]

- This is historically the first example of a non-invertible symmetry in 4 dimensions.

[Kaidi-Ohmori-Zheng '21][Choi-Cordova-Hsin-Lam-Shao '21]

Example

- Consider a 4-dimensional sigma model with $H = Spin$ to $X = \mathbb{CP}^1$.

$$B^3\mathbb{Z} \longrightarrow \pi_{\leq 3}X \longrightarrow B^2\mathbb{Z}$$

← Violates finiteness assumption!

- There are simple objects $\mathcal{N}_{N,p}$ such that:
  - $\mathcal{N}_{N,p}$ is obtained by coupling to minimal TQFT $\mathcal{A}_{N,p}$.

    [Hsin-Lam-Seiberg '18]
  - $\mathcal{N}_{N,p}$ have non-invertible fusion rules.
  - $\mathcal{N}^\vee_{N,p} = \mathcal{N}_{N,2N-p}$.
- This reproduces a $\mathbb{Q}/\mathbb{Z}$ type classification.

[Cordova-Ohmori '22][Choi-Lam-Shao '22][Chen-Tanizaki '23]

Part IV : Applications

Applications

- Applications to gapped systems and 't Hooft anomalies for categorical symmetries.
- Incorporate symmetry type and unitarity.
  - Gapped systems as functors $F : \mathsf{C}_{d,d} \longrightarrow \mathsf{Hilb}^{H}_{d,d}$.
  - 't Hooft anomalies as obstructions to functors $F : \mathsf{C}_{d-1,d} \longrightarrow \mathsf{Hilb}^{H}_{d-1,d}$.
- Many known results in low dimensions provide strong consistency checks.
- There remains much to be done!

Questions?