Towards a complexification of Donaldson–Witten TQFT

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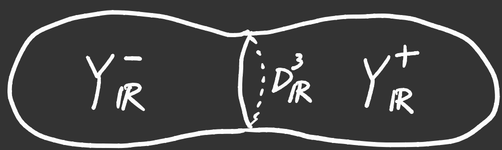
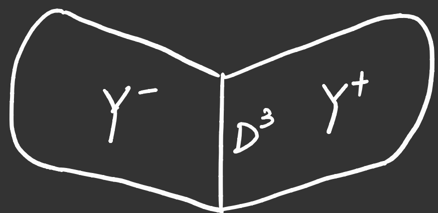
$\mathbb{R}$-ori 4-mfds	Calabi-Yau 4-folds
Donaldson invs $\# \{ \text{instantons} \}$	Donaldson-Thomas invs $\# \left\{ \begin{array}{l} SU(4) \text{ instantons} \\ \text{or coherent sheaves} \end{array} \right\}$
 	 $D^3 \in \lvert K_{Y^\pm}^{-1} \rvert$ CY3
Gluing formula (Donaldson-Witten TQFT)	?

Vect space on 3d:

Closed 3 mfd $D^3_{\mathbb{R}}$

$$CS : \mathcal{A}(D^3_{\mathbb{R}}) \longrightarrow \mathbb{R}$$

(Chern - Simons)

Instanton Floer homology

$$\mathcal{H}_{D^3_{\mathbb{R}}} = \text{"}H(\mathcal{A}/\mathcal{G}, CS)\text{"}$$

∞ - dim

"Morse homology"

CY 3-fold D

$$CS_{\mathbb{C}} : \mathcal{A}(D) \longrightarrow \mathbb{C}$$

(holomorphic Chern-Simons)

holo cousin of instanton Floer homology

$$\mathcal{H}_D = \text{"}H(\mathcal{A}/\mathcal{G}_{\mathbb{C}}, \varphi_{CS_{\mathbb{C}}}\mathbb{Q})\text{"}$$

∞ - dim

"Vanishing cycle homology"

More precisely: $\mathcal{M}_D$: moduli of (stable) bundles/sheaves on D (CY_3)

To make sense the homology $\mathcal{H}_{\mathcal{M}_D}$, the key str is:

<u>Thm</u>: (Pantev–Toën–Vaquié–Vezzosi + Brav–Bussi–Joyce)

$\mathcal{M}_D$ has (-1)-shifted symplectic structure

&

Locally is of form $\mathrm{crit}(\phi : \underset{\text{sm}}{\underset{=}{U}} \to \mathbb{C})$

$$\mathcal{H}_{\mathcal{M}_D} := H(\mathcal{M}_D, \bigcup_i \varphi_{\phi_i}\mathbb{Q})$$

Brav–Bussi–Dupont–Joyce–Szendrői

Kiem–Li

(ori data: Kontsevich–Soibelman)

Today: say $\mathcal{M}_D = \mathrm{crit}(\phi : W \to \mathbb{C})$

global critical locus

Then: $\mathcal{H}_{\mathcal{M}_D} = H_{\underline{F_0}}(W, \varphi_\phi \mathbb{Q})$ vanishing cycle homology

flavor sym

"critical homology"

Examples of $M_D = \mathrm{crit}\,(W \xrightarrow{\phi} \mathbb{C})$:

1). (GLSM)

$$W = \mathcal{O}_{\mathbb{P}^4}(-5) = \frac{(\mathbb{C}^5 \setminus \{0\}) \times \mathbb{C}}{\mathbb{C}^*} \xrightarrow{\phi} \mathbb{C}$$

$$\omega.t = (1, \dots, 1, -5)$$

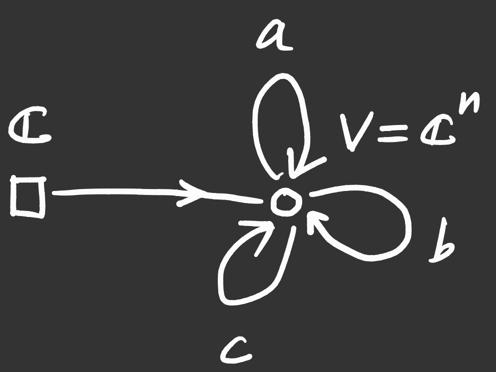
$$\phi(x_1, \dots x_5, p) = p \cdot (x_1^5 + \dots + x_5^5)$$

$$\mathrm{crit}(\phi) = \text{Quintic 3-fold } Q \,, \qquad H(W, \varphi_\phi) \cong H^{BM}(Q)$$

$$\Big\{ \sum_{i=1}^{5} x_i^5 = 0 \Big\} \hookrightarrow \mathbb{CP}^4 \qquad\qquad \text{dim reduction}$$

[Davison]

2). (quiver with potential)



$$W = \mathrm{Hom}(V,V)^{\times 3} \times V /\!\!/_{GL(V)} \xrightarrow{\phi} \mathbb{C}$$

$$(a,b,c,v) \longmapsto \mathrm{tr}\, a[b,c]$$

$$F_0 \leq F = (\mathbb{C}^*)^3 \circlearrowright \qquad (t_1,t_2,t_3)\cdot(a,b,c) = (t_1\cdot a,\ t_2\cdot b,\ t_3\cdot c)$$

$$F_0 = \{t_1\cdot t_2\cdot t_3 = 1\}$$

$\mathrm{Crit}\,\phi = \mathrm{Hilb}^n(\mathbb{C}^3)$, $\quad (\mathrm{Crit}\,\phi)^{F_0} =$ 3D Young diagrams w/ n-boxes

$H_{F_0}(W, \varphi_\phi)$ coho DT theory

(relevant to geo rep theory)

Relative 4d theory:

(Y, D) sm log CY_4 pair ($D \overset{i}{\hookrightarrow} Y$ anti-canonical divisor)

WANT to restrict sheaves F on Y to D

i^*F & $\mathbb{L}i^*F$

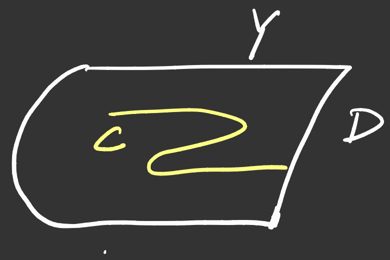
classical restriction derived restriction

(geo better) (homological alge better)

Consider those F such that $i^*F = \mathbb{L}i^*F$ $\circledast$

(relative condition)

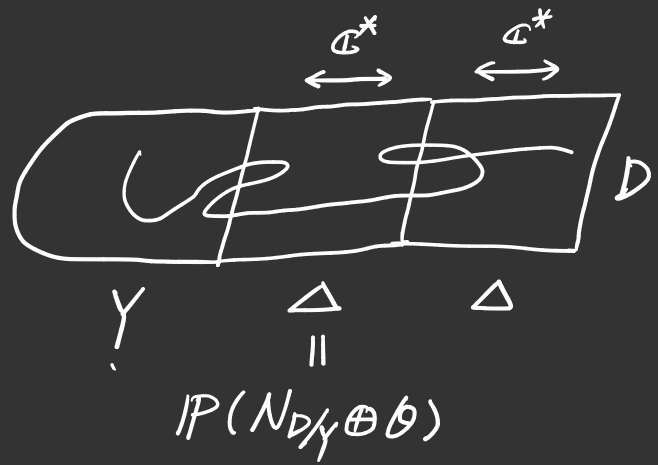
Say $F = I_C$ ideal sheaf of $C \subseteq X$



then ⊛ requires C intersect D in a nice way.

$\Rightarrow$ Moduli of such F has no compactness.

solution: consider such F on expanded pairs:



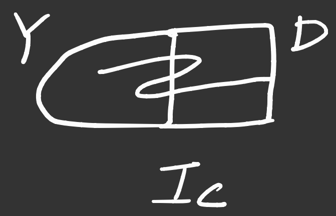
Fix topo data (curve classes etc)

$\Downarrow$

only need finitely many bubbles

Li-Wu: $\exists$ a moduli of sheaves on expanded pairs w/ relative conditions on D's. And a restriction map

$$\mathcal{M}_{Y,D} \xrightarrow{r} \mathcal{M}_D$$



$$\longmapsto \quad I_C|_D$$

- $\mathcal{M}_{Y,D}$ is proper if Y is so.

Given $\mathcal{M}_{Y,D} \xrightarrow{r} \mathcal{M}_D = \mathrm{Crit}(W \xrightarrow{\phi} \mathbb{C})$

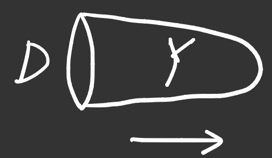
$\underline{\underline{Q}}$: $\mathcal{H}_{\mathcal{M}_D} = H(W, \varphi_\phi \mathbb{Q}) \xrightarrow{?} H^{BM}(\mathcal{M}_{Y,D})$ (related to Joyce conj)

K-theory version: $K_0(W, \phi) \xrightarrow{?} K_0(\mathcal{M}_{Y,D})$

$$\begin{array}{c} K_0(W,\phi) \\ \| \\ K_0(MF(W,\phi)) \end{array}$$

Matrix factorization category $\quad \mathcal{E}_0 \rightleftarrows \mathcal{E}_1$, $\; d^2 = \phi$

By pushforward to pt, obtain $\mathcal{H}_{\mathcal{M}_D} \longrightarrow \mathbb{Q}$



- Approximation by alge cycles:

$Z(\phi)$: zero locus of $\phi : W \to \mathbb{C}$ w/ $i : Z(\phi) \hookrightarrow W$

Milnor triangle $\psi_\phi \to \varphi_\phi \to i^*$

$$\Downarrow$$

canonical map: $H^{BM}(Z(\phi)) \longrightarrow H(W, \varphi_\phi)$

(Borel-Moore homology)

cycle map: $A(Z(\phi)) \longrightarrow H^{BM}(Z(\phi))$

(Chow gp of alge cycles)

composition:

$A(Z(\phi)) \longrightarrow H(W, \varphi_\phi)$

"alge cycles on $H(W, \varphi_\phi)$."

K-theory version:

$$D^b(Z(\phi))/\mathrm{Perf}(Z(\phi)) \simeq MF(W,\phi)$$

Orlov

Ballard-Favero-Katzarkov

$$\Downarrow K_0(-).$$

$$K_0(Z(\phi)) \twoheadrightarrow K_0(W,\phi)$$

Thm: (C.-Zhao-Zhou 24)

$$\exists \quad \begin{aligned} r^!: \quad & A_*(Z(\phi)) \longrightarrow A_*(M_{Y,D}) \\ r^!: \quad & K_0(Z(\phi)) \longrightarrow K_0(M_{Y,D}) \end{aligned}$$

satisfying natural functorial property

Some key features:

1. Need framework of shifted symplectic geometry

$$\begin{tikzcd} \mathcal{M}_{Y,D} \arrow[r, "r"] \arrow[r, swap, "Lag"] \arrow[dr, bend right, swap, "(-2)\text{-symp}"] & \mathcal{M}_D = \mathrm{crit}(W \xrightarrow{\phi} \mathbb{C}) \arrow[d, "Lag\ fib"] & \rightsquigarrow (-1)\text{-symp} \\ & W & \end{tikzcd}$$

2. (-2)-shifted symp $\Rightarrow$ special obstruction theory

(obs bdl has quadratic form)

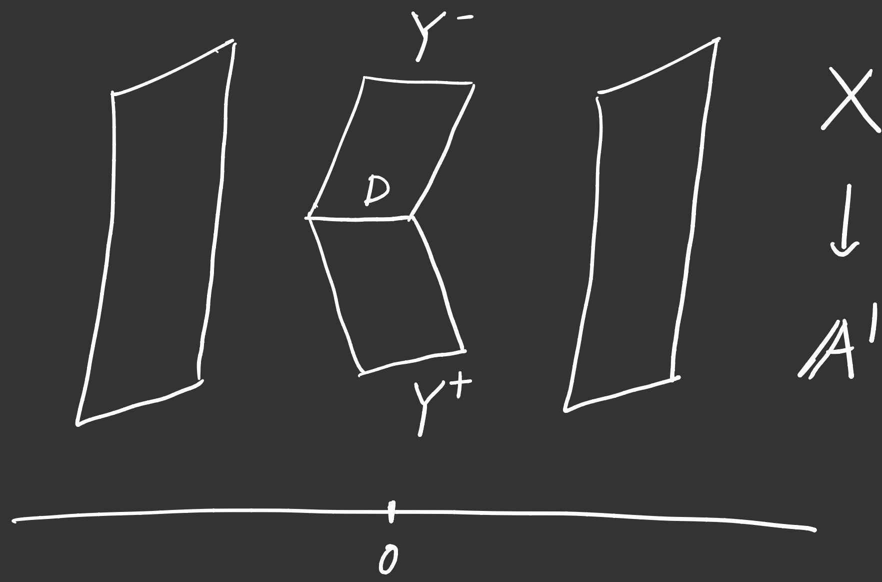
$SO(n,\mathbb{C})$-characteristic class (Edidin-Graham)

Localized to isotropic section (Oh-Thomas)

family version (H. Park)

Gluing formula:

Given a degeneration of CY_4



Thm (CZZ): There is a gluing formula relating invs of generic fibers X_t $(t \neq 0)$ and $(Y_\pm, D)$.

- Applications:

Eg 1: $\mathbb{C}^4$ & $\mathrm{Hilb}^n(\mathbb{C}^4) \curvearrowleft T = \{(t_1, \cdots, t_4) \in (\mathbb{C}^*)^4 \mid t_1 \cdot t_2 \cdot t_3 \cdot t_4 = 1\}$

Torus fixed pts $\mathrm{Hilb}^n(\mathbb{C}^4)^T =$ solid partitions (4D Young diagrams) of size n

No known closed formula of their counts

MacMahon gave a wrong guess at 1915

$$[\mathrm{Hilb}^n(\mathbb{C}^4)]^{vir}_T \in A^T_n(\mathrm{Hilb}^n(\mathbb{C}^4))$$

L: T-equi line bdl. $L^{[n]}$: tautological bdl on $\mathrm{Hilb}^n(\mathbb{C}^4)$

$$L^{[n]}|_Z = H^0(\mathbb{C}^4, L|_Z) \cong \mathbb{C}^n$$

Thm: (CZZ). We have

$$1+\sum_{n=1}^{\infty} q^n \cdot \int_{[\mathrm{Hilb}^n(\mathbb{C}^4)]_T^{vir}} e_T(L^{[n]}) = M(-q)^{\int_{\mathbb{C}^4} c_3^T(\mathbb{C}^4)\cdot c_1^T(L)},$$

where $M(q)=\prod_{n\geq 1}\frac{1}{(1-q^n)^n}$ is MacMahon function.

Pf: By using gluing formula & similar pole analysis of Maulik-Nekrasov-Okounkov-Pandharipande on 3-folds

Rk: 1. Nekrasov (2017) conj a K-theoretic version of above formula

Kool-Rennemo (in progress) prove it by factorization seq of Okounkov

2. We also compute zero dim relative invs for all log CY local curves

i.e. $\mathrm{Tot}_C(L_1 \oplus L_2 \oplus L_3)$, $\mathbb{C}^3 \times \{p_1, \cdots, p_r\}$

$L_1 \otimes L_2 \otimes L_3 \cong \omega_C(p_1 + \cdots + p_r)$

genus of C arbitrary

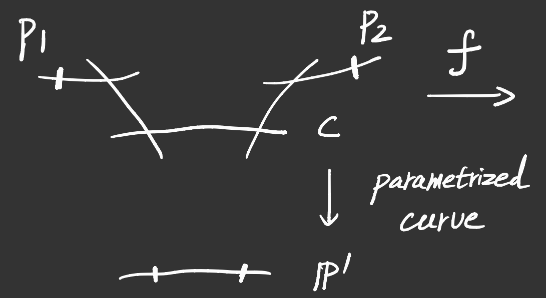
Eg 2: $Y = \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2) \oplus \mathcal{O}_{\mathbb{P}^1}(l_3) \xrightarrow{\pi} \mathbb{P}^1$

$$D = \coprod_{i=1}^{n} \pi^{-1}(p_i) \cong \coprod_{i=1}^{n} \mathbb{C}^3 \qquad \log CY: \sum_{i=1}^{3} l_i = n-2$$

$r: M_{Y,D} \to M_D$ has components

$$QM_{0,n}(\mathrm{Hilb}^d(\mathbb{C}^3), d)_{\mathbb{P}^1} \xrightarrow{r} \mathrm{Hilb}^d(D) = \underbrace{\mathrm{Hilb}^d(\mathbb{C}^3) \times \cdots \times \mathrm{Hilb}^d(\mathbb{C}^3)}_{n\text{-copy}}$$

$(n=2)$



$\xrightarrow{f} \mathrm{Hilb}^d(\mathbb{C}^3) \quad \overset{r}{\longmapsto} \quad \big(f(p_1), f(p_2)\big)$

Above relative 4d theory defines Gromov-Witten type invs of

$$\mathrm{Hilb}^d(\mathbb{C}^3)$$

$$\Rightarrow$$

singular space if $d \geq 4$

Thm (C.-Zhao 23): We can replace $\mathrm{Hilb}^d(\mathbb{C}^3)$ by any quiver with potential

(or crit locus in GIT)

& define their GW type invs.

Rk: 1. This gives mathematical approach to "partition functions" of

- 3d $N=2$ SUSY gauge theory ($\Sigma \times S^1$)
- gauged linear sigma model (GLSM)

(after Witten, FJR, CFGKS, KL, TX, FK, $\cdots$)

2. Can be used to extract information of quantum gp actions on $\mathcal{H}_{M_D}$

(along w/ ideas & works of Okounkov school on Nakajima quiver vars)

<u>Related question/direction:</u>

Stable envelopes as Maulik-Okounkov, Aganagic-Okounkov

for quivers with potentials

& connections to above invs.