Trace map on chiral Weyl algebras

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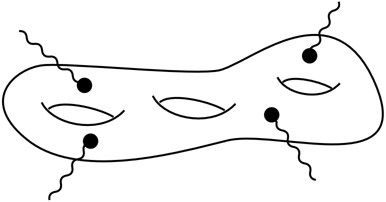
Overview

- Chiral homology: derived conformal blocks
- Chiral Weyl algebras
- Trace map on chiral Weyl algebras

Chiral homology: derived conformal blocks

The spaces of conformal blocks of two-dimensional conformal field theories have many interesting properties and connections to many different areas of mathematics and physics.

There are spaces of derived conformal blocks (chiral homologies) and the usual conformal blocks are their degree 0 part.



$$\vdots$$

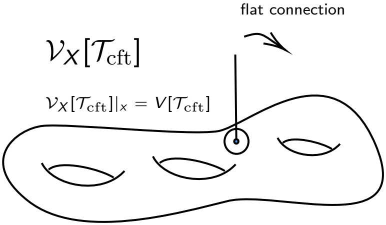
$$\mathrm{H}_1^{\mathrm{ch}}$$

$$\mathrm{H}_0^{\mathrm{c}h} = \text{conformal blocks}$$

Chiral homology: derived conformal blocks

Here we briefly recall the notion of chiral homology (by Beilinson and Drinfeld). Suppose that the space of local operators in our two-dimensional conformal field theory $\mathcal{T}_{\mathrm{cft}}$ can be described by a (quasi-conformal) vertex algebra $V[\mathcal{T}_{\mathrm{cft}}]$.

One can construct a vertex algebra bundle $\mathcal{V}_X[\mathcal{T}_{\mathrm{cft}}] \to X$ over a smooth Riemann surface X from $V[\mathcal{T}_{\mathrm{cft}}]$. This bundle is a left $\mathcal{D}_X$-module,i.e, a holomorphic ($\infty-$dim) vector bundle with a holomorphic connection.



Chiral homology: derived conformal blocks

From the left $\mathcal{D}_X$-module $\mathcal{V}_X[\mathcal{T}_{\mathrm{cft}}]$, we can get a right $\mathcal{D}_X$-module

$$\mathcal{A}_X[\mathcal{T}_{\mathrm{cft}}] := \mathcal{V}_X[\mathcal{T}_{\mathrm{cft}}] \otimes \omega_X.$$

The vertex algebra bundle structure on $\mathcal{V}_X[\mathcal{T}_{\mathrm{cft}}]$ gets translated to the chiral algebra (in the sense of Beilinson and Drinfeld) structure on $\mathcal{A}_X[\mathcal{T}_{\mathrm{cft}}]$.

Beilinson and Drinfeld construct a chain complex $\mathrm{C}^{\mathrm{ch}}_\bullet(X, \mathcal{A}_X[\mathcal{T}_{\mathrm{cft}}])$ which is a 2d chiral version of the usual Hochschild chain complex. The chiral homology is defined to be the homology of this complex

$$\mathrm{H}^{\mathrm{ch}}(X, \mathcal{A}_X[\mathcal{T}_{\mathrm{cft}}]) := H(\mathrm{C}^{\mathrm{ch}}_\bullet(X, \mathcal{A}_X[\mathcal{T}_{\mathrm{cft}}])).$$

Chiral homology: derived conformal blocks

By [Rozenblyum,2021], there is no higher chiral homology in the usual rational WZW model. In fact, there is a general open question:

Let V be a rational VOA. For any curve X, we have the corresponding chiral algebra $\mathcal{A}_X$. Is it true that all the higher chiral homologies of $\mathcal{A}_X$ vanish?

Beyond rational theories, one encounters

- $\dim H_0^{ch} = +\infty$ (chiral bosons).
- $H_{>0}^{ch} \neq 0$ (symplectic bosons).

Consider a holomorphic vector bundle E which is equipped with a symplectic pairing $\langle -, - \rangle : E \otimes E \to \omega_X$. Then the cohomology $H^\bullet(X, E)$ has a (-1)-shifted symplectic pairing

$$\int_X \langle -, - \rangle : H^\bullet(X, E) \otimes H^\bullet(X, E) \to \mathbb{C}.$$

We denote the BV algebra $\mathcal{O}(H^\bullet(X, E))$ by O_E.

The chiral quantum field theory (symplectic bosons) with Lagrangian

$$\int_X \langle \bar{\partial}\phi, \phi \rangle, \phi \in \mathcal{E} = \Omega^{0,\bullet}(X, E).$$

The algebraic structure of quantum observables in this theory is captured by the **chiral Weyl algebra** $\mathcal{A}_E$ associated to E.

It is expected that

$$\int \mathcal{D}\mathcal{E} \cdot e^{\frac{1}{\hbar}\int_X \langle \bar{\partial}_E \phi, \phi \rangle} \mathcal{O}_1 \cdots \mathcal{O}_n \sim \textbf{Chiral homology of } \mathcal{A}_E.$$

If we take $E = F \otimes \omega_X^{\frac{1}{2}}$ for a symplectic holomorphic vector bundle $F, w(-,-) : F \otimes F \to \mathcal{O}_X$. Then the chiral homology of $\mathcal{A}_E$ forms a $\mathcal{D}$-module on the moduli space of bundles which quantizes the **Gaiotto Lagrangian** ([Gaiotto], [Hitchin], [Ginzburg and Rozenblyum]) inside the Hitchin moduli space.

Theorem ([G])

The above path integral can be explicitly constructed as a map

$$\mathbf{Tr}_{\mathcal{A}_E} : C^{\mathrm{ch}}(X, \mathcal{A}_E) \to O_E$$

and satisfying

$$(d + \Delta_{\mathrm{BV}})\mathbf{Tr}_{\mathcal{A}_E} = 0.$$

Furthermore, the chain map $\mathbf{Tr}_{\mathcal{A}_E}$ is a quasi-isomorphism.

- The chiral homology (=BV cohomology of O_E) is concentrated in degree $\bullet = \dim H^0(X, E)$.
- The same method applies to chiral bosons and symplectic Fermions (have infinite-dimensional chiral homology groups).
- The variation of the analytic torsion $T(E)$ can be expressed as $\mathbf{Tr}_{\mathcal{A}_E}(J)$ for a current J.
- It is possible to generalize this to nonlinear symplectic bosons (chiral differential operators) and extend the Witten genera to higher genus curves.

Thank you!

String–Math

June 10–14, 2024

Resurgent large genus asymptotics of intersection numbers

j/w B. Eynard, E. Garcia-Failde, P. Gregori, D. Lewański

arXiv: AG/2309.03143

Alessandro Giacchetto

ETH Zürich

A case study: $m!$

Enumerative problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $$c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$$

Pro: exact
Con: recursive

Asymptotics: $$c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \mathrm{O}(m^{-1})\right)$$

Con: asymptotically exact
Pro: closed-form

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Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \frac{1}{12}m^{-1} + \frac{1}{288}m^{-2} + O\left(m^{-3}\right)\right)$

Con: asymptotically exact
Pro: closed-form

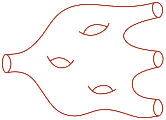
ψ-class intersection numbers

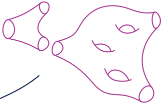
$$\langle\!\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\!\rangle = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \, (2d_i+1)!! \qquad d_1+\cdots+d_n = 3g-3+n$$

- Compute the perturbative expansion of topological 2d gravity
- Feynman diagrams of the Airy matrix model
- Volumes of moduli spaces of metric ribbon graphs
- Building block for all tautological intersection numbers

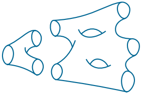
Recursive solution: Virasoro constraints

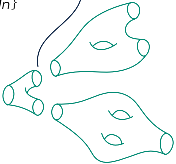
Witten conjecture/Kontsevich theorem, early '90s:





$$\langle\!\langle \tau_{d_1}\cdots\tau_{d_n}\rangle\!\rangle = \sum_{m=2}^{n}(2d_m+1)\,\langle\!\langle \tau_{d_1+d_m-1}\tau_{d_2}\cdots\widehat{\tau_{d_m}}\cdots\tau_{d_n}\rangle\!\rangle + \frac{1}{2}\sum_{a+b=d_1-2}\left(\langle\!\langle \tau_a\tau_b\tau_{d_2}\cdots\tau_{d_n}\rangle\!\rangle + \sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=\{d_2,\ldots,d_n\}}}\langle\!\langle \tau_a\tau_{I_1}\rangle\!\rangle\langle\!\langle \tau_b\tau_{I_2}\rangle\!\rangle\right)$$





Virasoro constraints/topological recursion.

Large genus asymptotics

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{\left(\frac{2}{3}\right)^{2g-2+n}} \Big(1 + O(g^{-1})\Big)$$

Proved by Aggarwal (2020), Guo–Yang, (2021)

(combinatorial analysis of Virasoro constraints/determinantal formula)

Questions

- Universal strategy, adaptable to different problems?
- 'Geometric' meaning?
- Subleading corrections?

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Large genus asymptotics: our result

Answers (EGGGL)

- Universal strategy: resurgence + determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \qquad \xrightarrow{\text{quantisation}} \qquad \left(\hbar^2 \frac{d^2}{dx^2} - x\right) \psi(x,\hbar) = 0$$

- Subleading corrections: algorithm + properties

Uniformly in $d_1, \dots, d_n$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \alpha_k + O(g^{-k-1})\right)$$

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$$\left. + \frac{A^k}{(2g-3+n)^{\underline{k}}} \, \alpha_k + \mathrm{O}(g^{-k-1}) \right)$$

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$S = 1$
Stokes constant

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$A = 2/3$

$\psi \sim \frac{1}{\sqrt{2}x^{1/4}} e^{\pm \frac{A}{\hbar} x^{-3/2}}$

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Computable; polynomial in n and multiplicities of d_i

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$$\alpha_1 = -\frac{17-15n+3n^2}{12} - \frac{(3-n)(n-p_0)}{2} - \frac{(n-p_0)^2}{4}$$

where $p_0 = \#\{ d_i = 0 \}$

Darboux method

- $\widetilde{\varphi}(\hbar) = \sum_m a_m \hbar^m \xrightarrow{\text{Borel}} \widehat{\varphi}(s) = \sum_m \frac{a_m}{m!} s^m$
- Suppose $\widehat{\varphi}$ has log singularities $A_1, \ldots, A_n$:

$$\widehat{\varphi}(s) \sim -\frac{S_i}{2\pi} \widehat{\psi}_i(s - A) \log(s - A_i)$$

  S_i are the Stokes constants, $\widehat{\psi}_i(s) = \sum_m \frac{b_{i,m}}{m!} s^m$ are holomorphic

- Large m asymptotics:

$$a_m = \frac{S_1}{2\pi} \frac{\Gamma(m)}{A_1^m} \Big(b_{1,0} + \frac{A_1}{m-1} b_{1,1} + \frac{A_1^2}{(m-1)(m-2)} b_{1,2} + \cdots \Big)$$
$$+ \cdots$$
$$+ \frac{S_n}{2\pi} \frac{\Gamma(m)}{A_n^m} \Big(b_{n,0} + \frac{A_n}{m-1} b_{n,1} + \frac{A_n^2}{(m-1)(m-2)} b_{n,2} + \cdots \Big)$$

Darboux method: summary

Upshot:

Borel plane singularities $\Longrightarrow$ large order asymptotics

- Fact 1: Borel plane sings are well-understood for exponential integrals
- Fact 2: Borel plane sings behave well under sums/products

Example: $\mathrm{Ai}(x,\hbar)\cdot\mathrm{Bi}(x,\hbar)$

(the expansion coeff's of Ai and Bi are explicit, but the ones of $\mathrm{Ai}\cdot\mathrm{Bi}$ are not)

Determinantal formula

Take the generating series

$$W_n(x_1,\dots,x_n;\hbar) = \sum_{g\geqslant 0} \hbar^{2g-2+n} \sum_{d_1,\dots,d_n} \# \frac{\langle\langle \tau_{d_1}\cdots\tau_{d_n}\rangle\rangle}{x_1^{d_1}\cdots x_n^{d_n}}$$

Det. formula (Bergère–Eynard, Bertola–Dubrovin–Yang):

$$W_n(x_1,\dots,x_n;\hbar) = \begin{array}{c}\text{sum over permutations of } S_n\\ \text{involving } \mathrm{Ai} \text{ and } \mathrm{Bi}\end{array}$$

Example: $n=2$

$$W_2 = \frac{\mathrm{Ai}_1\mathrm{Bi}_1\mathrm{Ai}_2'\mathrm{Bi}_2' + \frac{1}{2}\mathrm{Ai}_1\mathrm{Bi}_1'\mathrm{Ai}_2\mathrm{Bi}_2' + \frac{1}{2}\mathrm{Ai}_1\mathrm{Bi}_1'\mathrm{Bi}_2\mathrm{Ai}_2'}{(x_1-x_2)^2} + (x_1 \leftrightarrow x_2)$$

where $\mathrm{Ai}_i = \mathrm{Ai}(x_i,\hbar)$, $\mathrm{Bi}_i = \mathrm{Bi}(x_i,\hbar)$.

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where $\mathrm{Ai}_i = \mathrm{Ai}(x_i,\hbar)$, $\mathrm{Bi}_i = \mathrm{Bi}(x_i,\hbar)$.

Singularity structure of $\widehat{W}_n$

Singularity strct of $\widehat{\mathrm{Ai}}$, $\widehat{\mathrm{Bi}}$ $\Longrightarrow$ Singularity strct of $\widehat{W}_n$

- $2n$ log singularities of $\widehat{W}_n$, located at

$$+\frac{4}{3}x_i^{3/2} \qquad \text{and} \qquad -\frac{4}{3}x_i^{3/2}, \qquad i = 1, \ldots, n$$

- Stokes constants: $S = 1$
- Holom. funct multiplying the log:
  - (A) at $+\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\mathrm{Ai}}_i$ with $\widehat{\mathrm{Bi}}_i$
  - (B) at $-\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\mathrm{Bi}}_i$ with $\widehat{\mathrm{Ai}}_i$

Bessel

Norbury's intersection numbers (super WP/JT, BGW tau function):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle^{\Theta} = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^{n} \psi_i^{d_i} (2d_i+1)!!$$

$$= S \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 2$

  Stokes constants of the Bessel ODE
- $A = 2$

  leading exp behaviour of K_0
- α_k polynomials in n and multiplicities of d_i

  are computable from the asymptotic expansion coeffs of K_0

r-Airy

Witten's r-spin intersection numbers (FJRW theory, top. gravity coupled to a WZW theory):

$$\langle\langle \tau_{a_1,d_1}\cdots\tau_{a_n,d_n}\rangle\rangle^{r\text{-spin}} = \int_{\overline{\mathcal{M}}_{g,n}} c_W(a_1,\ldots,a_n)\prod_{i=1}^{n}\psi_i^{d_i}(rd_i+a_i)!_{(r)}$$

$$= \frac{2^n}{2\pi}\frac{\Gamma(2g-2+n)}{r^{g-1-|d|}}\Bigg[\frac{S_{r,1}}{|A_{r,1}|^{2g-2+n}}\left(\alpha_0^{(r,1)}+\frac{|A_{r,1}|}{2g-3+n}\alpha_1^{(r,1)}+\cdots\right)$$

$$+\cdots$$

$$+\frac{S_{r,\lfloor\frac{r-1}{2}\rfloor}}{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^{2g-2+n}}\left(\alpha_0^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\frac{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^K}{2g-3+n}\alpha_1^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\cdots\right)$$

$$+\frac{\delta_r^{\text{even}}}{2}\frac{S_{r,\frac{r}{2}}}{|A_{r,\frac{r}{2}}|^{2g-2+n}}\left(\alpha_0^{(r,\frac{r}{2})}+\frac{|A_{r,\frac{r}{2}}|^K}{2g-3+n}\alpha_1^{(r,\frac{r}{2})}+\cdots\right)\Bigg]$$

where $S_{r,\alpha}$, $A_{r,\alpha}$, $\alpha_k^{(r,\alpha)}$ are obtained from the r-Airy ODE.

Thank you for the attention!

Weil–Petersson volumes?

Weil–Petersson volumes satisfy the determinantal formula.

Problem

Understand the WP quantum curve:

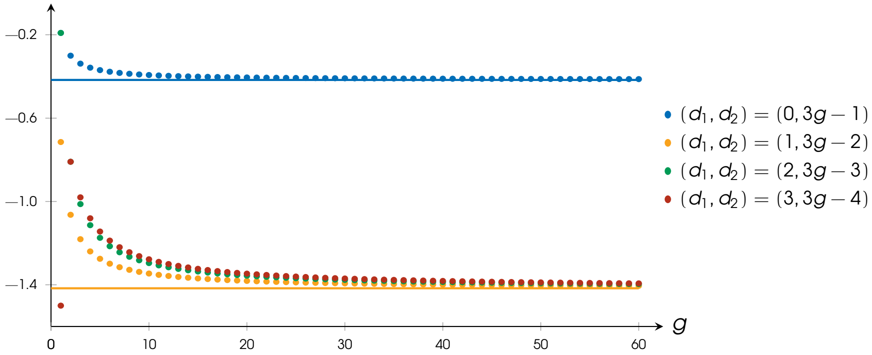
$$y^2 - \frac{\sin^2(2\pi\sqrt{x})}{4\pi^2} = 0 \qquad \xrightarrow{\textit{quantisation}} \qquad ??$$

(aka wave/Baker–Akhiezer function)

Visualising the large genus asymptotics

$$\frac{2g-3+n}{2/3}\left(\frac{\langle\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\rangle}{\frac{2^n}{4\pi}\frac{\Gamma(2g-2+n)}{(2/3)^{2g-2+n}}}-1\right)=\alpha_1(n,p_0)+O(g^{-1})$$

For $n=2$:



Modular Resurgent Structures and Spectral Traces of local $\mathbb{P}^2$

Veronica Fantini
IHÉS

Based on arXiv:2404.11550 and arXiv:2404.10695 joint with C. Rella

String Math 2024
ICTP, Trieste

Resurgence

Going beyond perturbation theory and characterizing analytic functions through their asymptotics

- Resurgence provides an effective tool to study perturbative expansions by computing sub-leading order contributions [Écalle]

$$\phi_0(\hbar) = \sum_{n=0}^{\infty} a_n \hbar^n \in \mathbb{C}[\![\hbar]\!]_1 \rightsquigarrow e^{-\frac{\omega}{\hbar}} \phi_\omega(\hbar) \text{ with } \phi_\omega(\hbar) \in \mathbb{C}[\![\hbar]\!]_1 \,, \omega \in \Omega \subseteq \mathbb{C}$$

- Among its different applications in mathematics and physics, it has been largely applied in **topological strings** [Alexandrov, Alim, Couso-Santamaría, Edelstein, Grassi, Gu, Iwaki, Kashani-Poor, Klemm, Mariño, Pasquetti, Pioline, Rella, Schiappa, Schwick, Teschner, Vonk, ...]

- Also particularly interesting is studying resurgence of the **asymptotic expansions of analytic functions**: indeed the resurgent structures reveal certain properties of the original analytic function

$$\Phi_0(\hbar) \textbf{ analytic} \rightsquigarrow \phi_0(\hbar) \in \mathbb{C}[\![\hbar]\!]_1 \rightsquigarrow e^{-\frac{\omega}{\hbar}} \phi_\omega(\hbar) \text{ with } \phi_\omega(\hbar) \in \mathbb{C}[\![\hbar]\!]_1 \,, \omega \in \Omega \subseteq \mathbb{C}$$

- By studying the resurgent structure of the **first fermionic spectral trace of local** $\mathbb{P}^2$ we show that the generating functions of the Stokes constants are holomorphic quantum modular forms [**VF** -Rella]

modular resurgent structures $\rightsquigarrow$ *holomorphic quantum modular forms*

First fermionic spectral trace of local $\mathbb{P}^2$

Non perturbative completion of the topological string free energies

- The **Topological String/Spectral Theory correspondence** (TS/ST) identifies as non-perturbative completion of topological strings on toric CY 3-folds X the spectral determinant Ξ of certain quantum mechanical operators O_X built from the quantization of the mirror curve Σ_κ where $\kappa \in \mathscr{M}_{\mathrm{cpx}}$ [Grassi—Hatsuda—Mariño]

$X = \mathcal{O}(-3) \to \mathbb{P}^2$

$\Sigma_\kappa = \{x, y \in \mathbb{C} \mid e^x + e^y + e^{-x-y} + \kappa = 0\}$ with $\kappa \in \mathbb{H}/\Gamma_1(3)$

$\mathsf{O}_{\mathbb{P}^2}(\mathsf{x}, \mathsf{y}) := e^{\mathsf{x}} + e^{\mathsf{y}} + e^{-\mathsf{x}-\mathsf{y}}$, where $[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar$

- The spectral determinat $\Xi(\kappa, \hbar) = \det(1 + \kappa\, \mathsf{O}_{\mathbb{P}^2}^{-1})$ is an entire function of κ and it is analytic in $\hbar$

- The Nth **fermonic spectral trace** $Z(N, \hbar)$ is defined as the expansion at the orbifold point: $\Xi(\kappa, \hbar) = 1 + \sum_{N=1}^{\infty} Z(N, \hbar)\, \kappa^N$

$$Z(1,\hbar) = \frac{1}{\sqrt{3\mathsf{b}}} e^{-\frac{\pi \mathrm{i}}{36}\mathsf{b}^2 + \frac{\pi \mathrm{i}}{12}\mathsf{b}^{-2} + \frac{\pi \mathrm{i}}{4}} \frac{(q^{2/3}; q)_\infty^2}{(q^{1/3}; q)_\infty} \frac{(e^{2\pi \mathrm{i}/3}; \tilde{q})_\infty}{(e^{-2\pi \mathrm{i}/3}; \tilde{q})_\infty^2}$$

holomorphic/anti-holomorphic block

$\hbar \propto \tau^{-1}$

with $q = e^{2\pi \mathrm{i} \mathsf{b}^2} = e^{3\mathrm{i}\hbar}$ and $\tilde{q} = e^{-2\pi \mathrm{i}/\mathsf{b}^2} = e^{2\pi \mathrm{i}\tau}$, and where $(a; q)_\infty$ is the q-Pochhammer symbol

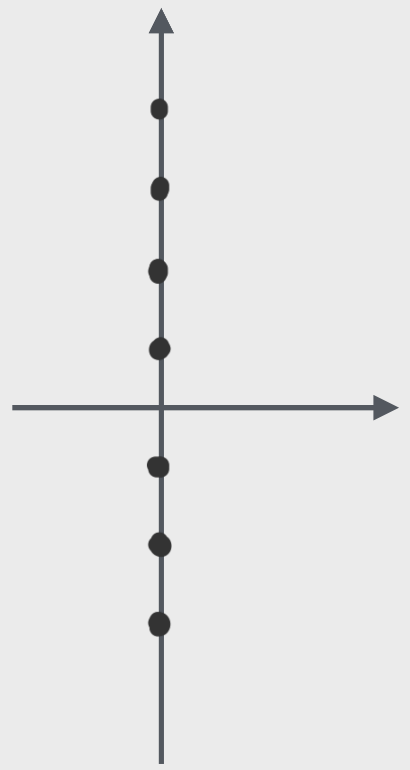
Resurgent Structure of the asymptotics of $\log Z(1,\hbar)$

Similar resurgent structures appear in both the strong/weak coupling regimes [Rella]

Resurgence Structure of *weak* asymptotics $\phi(\hbar) \in \mathbb{C}[\![\hbar]\!]_1$

- **Singularities** are simple poles at $\rho_m = \mathcal{A}_0 m\,,\ m \in \mathbb{Z}_{\neq 0}$

$$\mathcal{A}_0 = \frac{4\pi^2}{3}\mathrm{i}$$



- **Stokes constants** $\{S_m\,, m \in \mathbb{Z}_{\neq 0}\}$
- **L-function** $L_0(s) := \dfrac{1}{3\sqrt{3}\mathrm{i}} \displaystyle\sum_{m>0} \frac{S_m}{m^s} = \zeta(s) L(s+1, \chi_{3,2})$

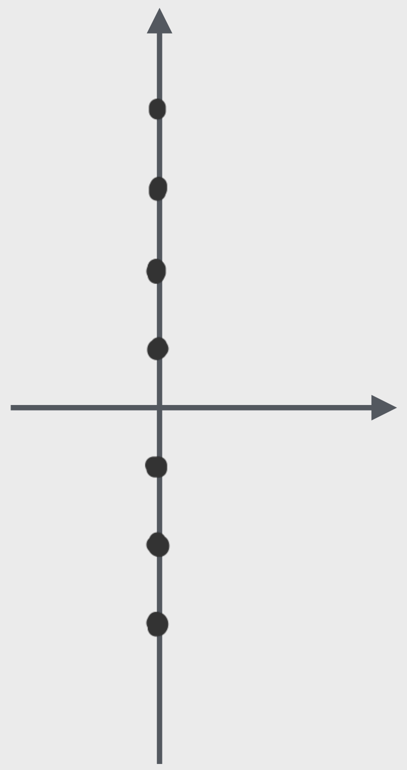
$$\Lambda_0(s) := -\,i\frac{3^{\frac{s}{2}-2}}{\pi^{s+1}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+1\right)L_0(s)$$

$$\boxed{\Lambda_0(s) = \Lambda_\infty(-s)}$$

Resurgence Structure of *strong* asymptotics $\psi(\tau) \in \mathbb{C}[\![\tau]\!]_1$

- **Singularities** are simple poles at $\eta_m = \mathcal{A}_\infty m\,,\ m \in \mathbb{Z}_{\neq 0}$

$$\mathcal{A}_\infty = \frac{2\pi}{3}\mathrm{i}$$



- **Stokes constants** $\{R_m\,, m \in \mathbb{Z}_{\neq 0}\}$
- **L-function** $L_\infty(s) := \dfrac{1}{3} \displaystyle\sum_{m>0} \frac{R_m}{m^s} = \zeta(s+1) L(s, \chi_{3,2})$

$$\Lambda_\infty(s) := \frac{3^{\frac{s}{2}-1}}{\pi^{s+1}}\Gamma\left(\frac{s+1}{2}\right)^2 L_\infty(s)$$

Quantum Modularity

Modular properties of the generating functions of the Stokes constants

Let $q = e^{2\pi i y}$, the **generating functions** of the Stokes constants in both regimes are defined respectively as follows

$$f_0(y) := \sum_{m>0} S_m\, q^m = 3 \log \frac{(e^{2\pi \mathrm{i}/3}\, q; q)_\infty}{(e^{-2\pi \mathrm{i}/3}\, q; q)_\infty} \qquad f_\infty(y) := \sum_{m>0} R_m\, q^m = 3 \log \frac{(q^2; q^3)_\infty}{(q^1; q^3)_\infty}$$

Theorem [**VF**—Rella] The functions $f_0\,, f_\infty : \mathbb{H} \to \mathbb{C}$ are holomorphic quantum modular functions for the group $\Gamma_1(3)$

Let $f_0^\star\,, f_\infty^\star$ be the **Fricke involution** of respectively $f_0\,, f_\infty$

Theorem [**VF**—Rella] The functions $f_0^\star\,, f_\infty^\star : \mathbb{H} \to \mathbb{C}$ are holomorphic quantum modular functions for the group $\Gamma_1(3)$

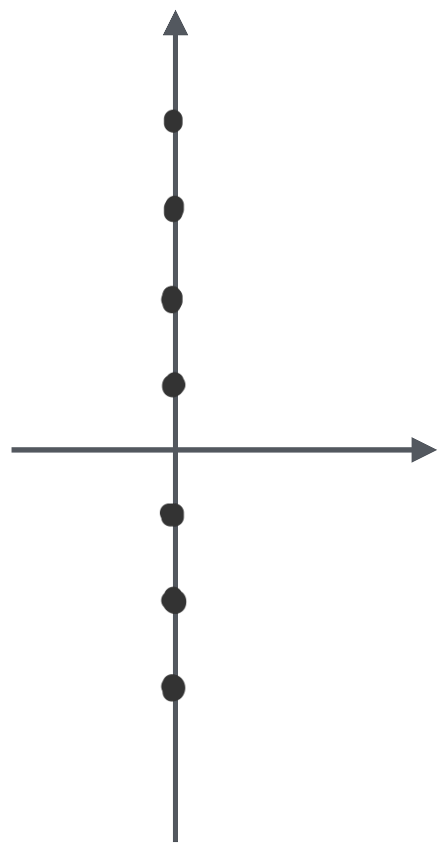
Recall that for local $\mathbb{P}^2$, the moduli space $\mathscr{M}_{\mathrm{cpx}} \cong \mathbb{H}/\Gamma_1(3)$ and the free energies are quasi-modular functions for $\Gamma_1(3)$ [Aganagic—Bouchard—Klemm, Coates—Iritani]

Modular Resurgent Structures

Resurgent series coming from holomorphic quantum modular forms

Definition [VF—Rella] An asymptotic series $\tilde{f} \in \mathbb{C}[\![y]\!]_1$ has a **modular resurgent structure** if the following conditions holds.

- The Borel transform of $\tilde{f}$ has a tower of **singularities** at $\zeta_n = \mathscr{A}n\,,\ \ n \in \mathbb{Z}_{\neq 0}$, for some constant $\mathscr{A} \in \mathbb{C}$
- For every $n \in \mathbb{Z}_{\neq 0}$ the resurgent series at the singularity ζ_n is the **Stokes constant** $A_n \in \mathbb{C}$
- The Stokes constants A_n are the coefficients of an **L-function** $L(s) = \sum_{n\neq 0} \frac{A_n}{n^s}$ analytic for $\Re(s) > \alpha$ and which admits a meromorphic continuation



Conjecture [VF—Rella] Let $f\colon \mathbb{H} \to \mathbb{C}$ be a q-series with $q = e^{2\pi i y}$. If its asymptotic series $\tilde{f} \in \mathbb{C}[\![y]\!]_1$ as $y \to 0$ and $\Im(y) > 0$ has a modular resurgent structure, then $f(y)$ is a holomorphic quantum modular form for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.

modular resurgent structures $\rightsquigarrow$ *holomorphic quantum modular forms*

Highlights

Overview of results from the study of local $\mathbb{P}^2$ and on modular resurgent structures

- Weak/Strong asymptotics of $\log Z(1,\hbar)$ have modular resurgent structures: f_0 , f_∞ are holomorphic quantum modular functions
- L-functions play a crucial role in defining the followings
    - **Strong-weak resurgent symmetry** is the result of a fully-flagged net of relations involving perturbative/non-perturbative contributions in both the strong and weak coupling regimes
    - **New paradigm of resurgence**: new series *resurge* as prescribed by the **functional equation** that governs the analytic continuation of the L-function whose coefficients are the Stokes constants
- Modular resurgent structures also appears in the study of **Maass cusps forms**
- The **effectiveness of the median resummation** allows to reconstruct the generating function of the Stokes constants from their asymptotic expansion
    - In the example of local $\mathbb{P}^2$, we proved it for f_0 and conjectured it for f_∞
    - We conjecture that the median resummation of modular resurgent series given by the asymptotic of a q-series reconstructs the q-series itself

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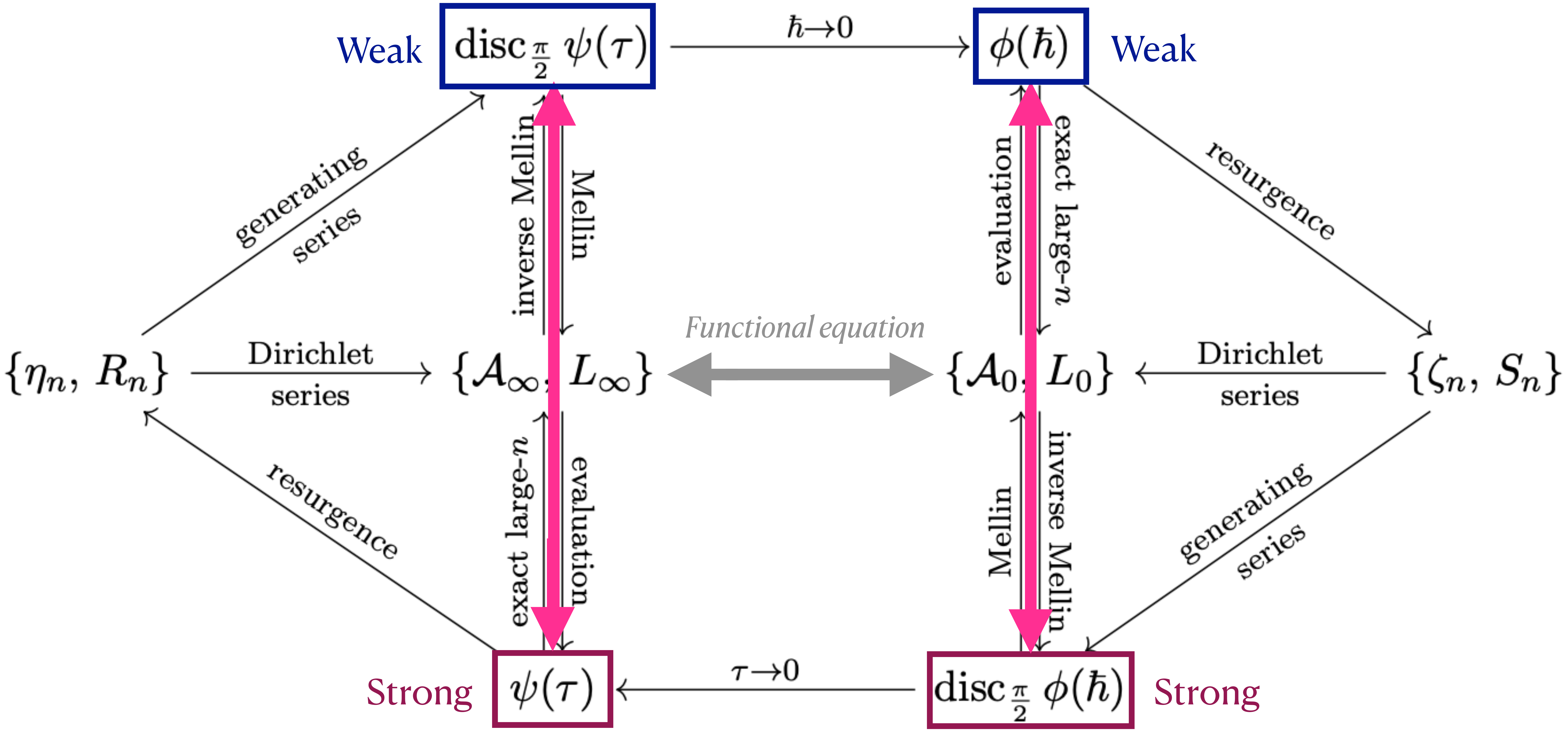
Thank you for your attention

Strong-weak resurgent symmetry

Perturbative/non perturbative contributions in the strong/weak regimes satisfy a rich net of relations



New paradigm of resurgence

New functions *resurge* from the functional equation

$$
\begin{tikzcd}
L_{\infty,\pm}(s) \arrow[r, "\text{inverse Mellin}"] \arrow[rrrd, leftrightarrow, "\text{functional equation}"] & f_\infty(y) \arrow[r, "y\to 0"] & \tilde{f}_{\infty,\pm}(y) \arrow[r, "\text{resurgence}"] \arrow[d] & \{S_m\} \arrow[d, "L\text{-function}"] \\
\{R_m\} \arrow[u, "L\text{-function}"] & \tilde{f}_{0,\pm}(y) \arrow[l, "\text{resurgence}"'] \arrow[u] & f_0(y) \arrow[l, "y\to 0"'] & L_{0,\pm}(s) \arrow[l, "\text{inverse Mellin}"']
\end{tikzcd}
$$

$\tilde{f}$ perturbative / f non perturbative

Correlators on ABJ(M) line-defects from string worldsheet on $AdS_4 \times \mathbb{CP}^3$:

The role of the Kalb-Ramond field

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String-Math, June 2024

IFLP CONICET UNLP

The ABJ(M) model and its gravitational dual

- Context: Precise example of AdS/CFT correspondence.
- ABJ(M): a 3-dimensional SCFT, precisely it is a $\mathcal{N} = 6$ Super Chern-Simons gauge theory with matter, which gauge group is $U_k(N+\ell) \times U_{-k}(N)$.

  Aharony- Bergman-Jafferis-Maldacena, 2008 & Aharony- Bergman-Jafferis, 2008
- Large N limit: M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$.
- In a t´Hooft limit ($N >> k^5 \to \infty$ and $\frac{N}{k} := \lambda$, finite) is accessible a Type IIA String Theory description and the geometry collapses to $AdS_4 \times \mathbb{CP}^3$
- For $\ell \neq 0$ we need include a non-vanishing Kalb-Ramond field arround a non-trivial cycle $\mathbb{CP}^1 \subset \mathbb{CP}^3$

$$\frac{1}{2\pi}\int_{\mathbb{CP}^1} B^{(2)} = \frac{\ell}{k}\,, \text{ where } B^{(2)} = \frac{\mathcal{B}}{2}dA\,.$$

- This flat configuration couples with the end points of open strings.

Wilson Lines as 1d Defect CFT

- A Wilson Line is an non-local operator defined along an open curve:

$$\mathcal{W}_{\mathcal{R}}[\mathcal{C}] = \frac{1}{\dim(\mathcal{R})} \mathrm{Tr}_{\mathcal{R}} \left(\mathcal{P} e^{i \int_{\mathcal{C}} \mathcal{L}(\tau) d\tau} \right)$$

- A straight Wilson Line on a d-dimensional CFT partially breaks the conformal group: $SO(2,1) \times SO(d-1) \subset SO(2,d)$
- For SCFT, there are some operators with an unbroken supersymmetry subgroup. Ex.: $\frac{1}{6}$-BPS WL in ABJ(M)

Drukker-Plefka-Young, 2009

$$\mathcal{L}(x(t)) = \begin{pmatrix} A_\mu \dot{x}^\mu(x) - \frac{2\pi i}{k} \mathcal{M}_J^I \bar{C}^J C_I & 0 \\ 0 & \hat{A}_\mu \dot{x}^\mu(x) - \frac{2\pi i}{k} \hat{\mathcal{M}}_I^J C^I \bar{C}_J \end{pmatrix}$$

preserves a $SO(2,1) \times SO(2) \times SU(3) \times (4\,\text{supercharges})$
this is $SU(1,1|1) \times SU(2) \times SU(2) \subset OSP(6|4)$

Correlators on a line defect

- This 1-dimensional defect supported on the 3-dimensional background defines itself a CFT_1. (Drukker, Kawamoto 2006) We can calculate:

$$\left\langle \mathcal{O}_1\left(t_1\right)\cdots\mathcal{O}_n\left(t_n\right)\right\rangle_{\mathcal{C}} = \frac{\left\langle \operatorname{tr}\left(\mathcal{P}\mathcal{O}_1\left(t_1\right)\cdots\mathcal{O}_n\left(t_n\right)e^{i\int_{\mathcal{C}} dt\mathcal{L}(t)}\right)\right\rangle}{\langle \mathcal{W}[\mathcal{C}]\rangle}$$

- This n-point function are constrained by the conformal structure, for example:

$$\left\langle \mathcal{O}_\Delta\left(t_1\right)\mathcal{O}_\Delta\left(t_2\right)\mathcal{O}_\Delta\left(t_3\right)\mathcal{O}_\Delta\left(t_4\right)\right\rangle_{\mathcal{C}} = \frac{1}{\left(t_{12}t_{34}\right)^{2\Delta}}G(u;\lambda),$$

where $u = \frac{t_{12}t_{34}}{t_{13}t_{24}}$

Gravitational dual to line defect

- Claim: This setup provides a correspondence AdS_2/CFT_1
- Using Holographic dictionary for Wilson Lines: Maldacena,1998

$$\langle W(\mathcal{C})\rangle = \mathcal{Z}_{\text{open string}}\Big|_{\mathcal{C}} \simeq e^{-S_E[X_{cl},\ldots]}\Bigg|_{X(z=0):\,\mathcal{C}}$$

Recall: AdS_4: $ds^2_{AdS_4} = \frac{dz^2+dt^2+dx_i dx^i}{z^2}$

- For the straight line the dual worldsheet geometry is AdS_2
- The edge of the open string take Dirichlet b.c on $z=0$ and for the case of $\frac{1}{6}-BPS$ WL take Neumann b.c on a $\mathbb{CP}^1 \subset \mathbb{CP}^3$
- d.o.f: In the bosonic sector there are
  - 6 massless scalars ($\mathbb{CP}^3$ directions)
  - 2 massive complex scalars with $m^2=2$ associated with AdS_4 directions, (transverse to AdS_2 fluctuations).

(Correa, V. Giraldo-Rivera, G. Silva, 2020)

Scalars fields in AdS_2 and the problem of b.c

- AdS/CFT dictionary: $Z_{CFT_d}[J] = Z_{\text{strings on } AdS_{d+1}}[\phi]\Big|_{\text{b.c:} J(x)}$

  (Witten, 1998)

  - $\mathcal{O}$ with scale dimension $\Delta \longleftrightarrow \phi(x)$ with mass m such that

$$\Delta = \frac{d}{2} \pm \frac{d}{2}\sqrt{1 + \frac{4m^2}{d^2}}\,, \text{ in the range: } -\frac{d^2}{4} \leq m^2 < -\frac{d^2}{4} + 1$$

- Near boundary: $\phi(x) = \alpha(x) z^{\Delta_-} + \beta(x) z^{\Delta_+}$
  - For massless scalar in AdS_2: $\phi(x) = \alpha(x) + \beta(x) z$
  - Dirichlet b.c: $\alpha(x) = J(x)$ fixed
  - Neumann b.c: $\beta(x) = J(x)$ fixed (Witten, 2001 T. Hartman, L. Rastelli, 2006)
  - Mixed b.c (compatible with susy and conformal symmetry)

$$\chi\dot{\alpha} + \beta = J(x)$$ (Correa, V. Giraldo-Rivera, G. Silva, 2020)

Claim: We find that the Kalb-Ramond field coupled to the edge of open string at a $\mathbb{CP}^1 \subset \mathbb{CP}^3$ induced a kind of mixed b.c over the worldsheet fluctuations

The action for semi-classical fluctuations

- We will focus on the coordinates along a particular $\mathbb{CP}^1 \subset \mathbb{CP}^3$

$$S_{\mathbb{CP}^1} = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{\det\left(g_{\mu\nu} + \partial_\mu Y^A \partial_\nu Y^A\right)}$$

- Parameterizing the transverse fluctuations around fixed direction of the $\mathbb{CP}^1$ (defined by a vector n^A) as $Y^A = n^A + \frac{\sqrt{2\pi}}{\lambda^{1/4}} y^A - \frac{\pi}{\sqrt{\lambda}} y^2 n^A + \mathcal{O}(\frac{1}{\lambda})$, with $n^A y^A = 0$
- The truncated action up to a boundary term its read as

$$S_{\mathbb{CP}^1} = \int d^2\sigma \sqrt{g} \Big[\tfrac{1}{2} \partial_\mu y^A \partial^\mu y^A + \frac{\pi}{\sqrt{\lambda}} y^A y^B \partial_\mu y^A \partial^\mu y^B + \\ + \frac{\pi}{4\sqrt{\lambda}} (\partial_\mu y^A \partial^\mu y^A)^2 - \frac{\pi}{2\sqrt{\lambda}} \partial_\mu y^A \partial^\mu y^B \partial_\nu y^A \partial^\nu y^B + \mathcal{O}(\tfrac{1}{\lambda}) \Big] \, .$$

Obs.: This quartic terms gives the interaction vertex and play a crucial role.

Mixed boundary conditions from Kalb-Ramond coupling

- The imposition of Neumann boundary conditions requiring a boundary term added to the quadratic action:

$$\tilde{S}^{(2)}_{\mathbb{CP}^1} = S^{(2)}_{\mathbb{CP}^1} + \int_{-\infty}^{\infty} dt\ \partial_z y^A y^A \big|_{z=0}$$

$$\Rightarrow\ \delta\tilde{S}^{(2)}_{\mathbb{CP}^1}\big|_{\text{on-shell}} = \int_{-\infty}^{\infty} dt\ \delta(\partial_z y^A) y^A \big|_{z=0}$$

- The KR field is flat $\rightarrow$ coupling to the open string adds just a boundary term.
  We are interesting in Neumann b.c over a $\mathbb{CP}^1$
- At quadratic order the

$$S^{(2)}_{\mathbb{CP}^1} \ = \ \int d^2\sigma \left(\sqrt{g}\tfrac{1}{2}\partial_\mu y^A \partial^\mu y^A - \mathcal{B}\epsilon^{ABC} n^A \partial_z y^B \partial_t y^C\right)\ , \ (1)$$

- In addition de KR field leads to an extra quartic terms wich contributing as new interaction vertex $-\mathcal{B}\epsilon^{ABC}\frac{1}{2}n^A y^2 \partial_z y^B \partial_t y^C$

The on-shell variation, together with the additional bdry. term becomes

$$\delta\tilde{S}^{(2)}_{\mathbb{CP}^1} = \int_{-\infty}^{\infty} dt\ \delta\left(\partial_z y^A - \mathcal{B}\epsilon^{ABC} n^B \partial_t y^C\right) y^A\Big|_{z=0} .$$

- At classic the problem it is reduced to

$$\boxed{\Box y^A = 0\,, \qquad \left(\partial_z y^A - \mathcal{B}\,\epsilon^{ABC} n^B \partial_t y^C\right)\Big|_{z=0} = J^A(t)\,,}$$

- First step: find the Green function for this problem

$$G^{AB} = -\langle y^A y^B\rangle_{0,n} = (\delta^{AB} - n^A n^B)\mathsf{G}_\mathsf{s}(\sigma,\sigma') + \epsilon^{ABC} n^C \mathsf{G}_\mathsf{a}(\sigma,\sigma')\,,$$

and take the mean value over the $\mathbb{CP}^1$.

4-point function and the conformal structure

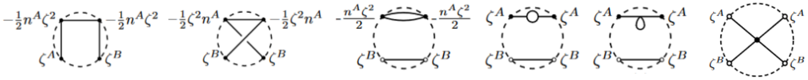
- We compute explicitly the 4-point correlators in terms of the Green's function for our b.c. At non-trivial leading order $(1/\lambda)$ we find:

$$\langle Y^A(t_1)\,Y^A(t_2)\,Y^B(t_3)\,Y^B(t_4)\rangle \propto \frac{G_{\mathsf{S}}(\lambda,u)}{(t_2-t_1)^{2\Delta}(t_4-t_3)^{2\Delta}}$$

with $\quad \mathsf{G}_{\mathsf{S}}^{(2)}(u) = \frac{4}{(1-\mathcal{B}^2)^2}\left[\log^2(1-u) + \pi^2\mathcal{B}^2\Theta(u-1)\right]\,.$

- Is the conformal structure preserved in the next-to-leading order?

We need evaluate several diagrams, including bulk vertex and loops corrections



4-point functions and conformal structure

- To avoid this complication at order $1/(\sqrt{\lambda})^3$, we can compute $\partial_{t_1}\partial_{t_2}\partial_{t_3}\partial_{t_4}\left\langle Y^A(t_1)Y^A(t_2)Y^B(t_3)Y^B(t_4)\right\rangle$.
- Finally, we can compute explicitly

$$t_{12}^2 t_{34}^2 \partial_{t_1}\partial_{t_2}\partial_{t_3}\partial_{t_4} G_{\mathsf{S}}^{(3)} = P(u) + P\left(\tfrac{u}{u-1}\right) ,$$

where

$$\begin{aligned} P(u) = &- \frac{8u^2}{(1-\mathcal{B}^2)^3}\left[4+\log(u^2)\right] \\ &+ \frac{8}{(1-\mathcal{B}^2)^2}\left[2+u+2u^2+\left(\tfrac{2}{u}-1+\tfrac{u^3}{(1-u)^3}\right)\log(u^2)\right] \\ &+ \frac{8}{(1-\mathcal{B}^2)}\left[-d_2+32\pi f_1\left(1+u^2\right)\right] . \end{aligned}$$

In accordance with the conformal symmetry of the line, we observe that the anomalous terms cancel and we can determine the precise function of the cross-ratio. Furthermore, crossing symmetry is manifestly.

Conclusions and Outlook

For more details see: Correa, D.H., Ferro, M.G. Giraldo-Rivera, V.I. “Mixed boundary conditions in AdS2/CFT1 from the coupling with a Kalb-Ramond field.” J. High Energ. Phys. 2024, 141 (2024) [arXiv:2312.13258]

- The fluctuations on the open string dual to the $1/6$ BPS bosonic Wilson line in the ABJ(M) model satisfy a boundary conditions mixing longitudinal and transverse derivatives, where the mixing parameter comes from the Kalb-Ramond field.
- As evidenced by the 4-point function obtained as a function of the cross-ratio, a $1/6$ BPS bosonic Wilson line with local operator insertions in the ABJ(M) model constitutes a CFT_1.

Outlook

Future directions:

- Try to use analytic bootstrap techniques to investigate its correlation functions. This works for the 1/2 BPS Wilson Line but our case is less symmetric.
- Inclusion of Green-Schwarz fermions in the world-sheet, to determine their boundary conditions and to compute fermionic 1-loop Witten diagrams.

Thanks!

Partition function of Argyres-Douglas theories on the blowup

Ideal Majtara, SISSA

String-Math 2024, ICTP Trieste, June 10-14 2024;

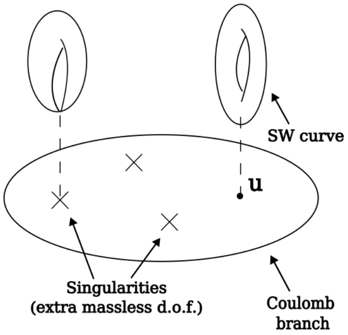
work in collaboration with
G. Bonelli, P. Gavrylenko, A. Tanzini

SISSA

Argyres-Douglas theories

$\mathcal{N} = 2$ 4d susy gauge theories
low-energy theory is determined by the SW curve.

Argyres-Douglas (AD) theories are strongly coupled isolated 4d SCFTs, which appear as singularities of the Coulomb branch.



mutually non-local light d.o.f. $\Rightarrow$ no known lagrangian description!

Usual localization techniques are difficult to apply...

Interesting informations on AD theories can be extracted using the theory of integrable systems.

Example: H_0 theory (AD point of $SU(2)$ $N_f = 1$,$SU(3)$ $N_f = 0$).

Painlevé - gauge theory correspondence

The SW curve is associated to an integrable system:

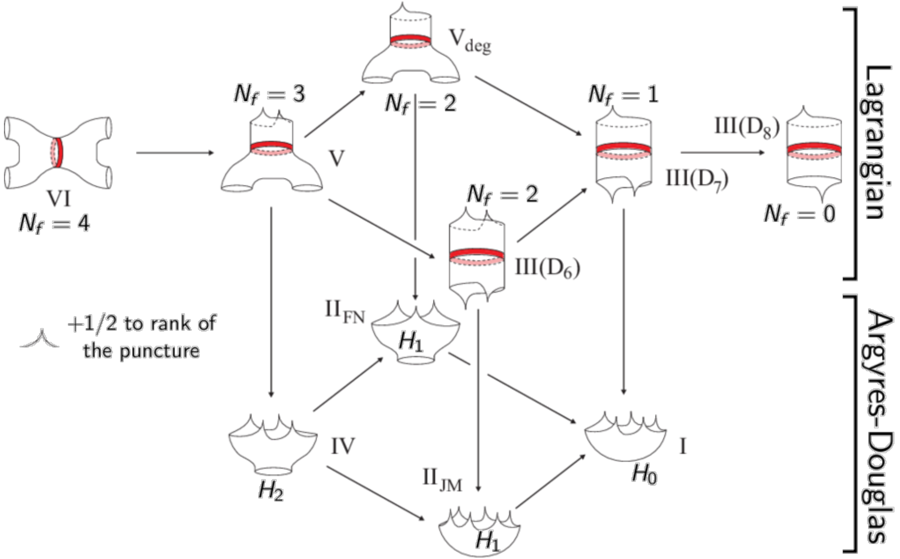
SW curve = Spectral curve

If we introduce a self-dual ($\epsilon_1 = -\epsilon_2 = \epsilon$) Omega background this system acquires a time dependence $\Rightarrow$ Painlevé equations ($SU(2)$).

for $SU(2)$ gauge theories, this gives the so called Painlevé-gauge theory correspondence:

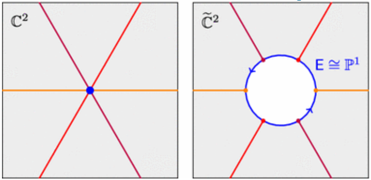
- Painlevé time $\leftrightarrow$ gauge coupling scale $t = \Lambda e^{\epsilon s}$;
- Painlevé Hamiltonian $H(q, p, t) \leftrightarrow$ Coulomb parameter u;
- Painlevé free parameters $\leftrightarrow$ masses of the matter fields;
- $\epsilon \to 0$ (autonomous) limit $\leftrightarrow$ SW theory

The map can be generalized to higher rank theories.


VI
$N_f = 4$
$N_f = 3$
V
$\mathrm{V}_{\mathrm{deg}}$
$N_f = 2$
$N_f = 2$
$\mathrm{III}(\mathrm{D}_6)$
$N_f = 1$
$\mathrm{III}(\mathrm{D}_7)$
$\mathrm{III}(\mathrm{D}_8)$
$N_f = 0$
Lagrangian
$\mathrm{II}_{\mathrm{FN}}$
H_1
+1/2 to rank of the puncture
IV
H_2
$\mathrm{II}_{\mathrm{JM}}$
H_1
I
H_0
Argyres-Douglas

Nekrasov partition function on the blowup

Consider the Nekrasov partition function on the blowup $\hat{\mathbb{C}}^2$ of $\mathbb{C}^2 \simeq \mathbb{R}^4$ with a surface observable $\mathcal{I}$ on $E \simeq \mathbb{C}P^1$ (exceptional divisor).



$$\hat{Z}_{Nek}(\epsilon_1, \epsilon_2, s) = \langle \Omega | e^{s\mathcal{I}(E)} | \Omega \rangle_{\hat{\mathbb{C}}^2}$$

Painlevé tau function τ $\left(H \sim \frac{\partial}{\partial t} \log \tau\right)$ $\longleftarrow$ Blowup factor $\mathcal{B} = \hat{Z}(\epsilon_1, \epsilon_2, s)/Z(\epsilon_1, \epsilon_2)$.

$\downarrow$ $\downarrow$

Expansion around a zero (HRZ-like expansion) $\longleftrightarrow$ Topological Operator/State correspondence

Autonomous limit

Taking the SW limit $\epsilon_1, \epsilon_2 \to 0$ we obtain the following result

$$\lim_{\epsilon_1,\epsilon_2\to 0} \mathcal{B}(\epsilon_1,\epsilon_2,s) \propto \underbrace{e^{Ts^2}}_{\text{Contact term}} \underbrace{\sigma(s;g_2,g_3)}_{\text{Weierstrass sigma}} ,$$

The result is universal. Informations about the theory are encoded in the elliptic invariants $g_2 = g_2(u,\Lambda), g_3 = g_3(u,\Lambda)$ which define the Weierstrass parametrization of the SW curve:

$$y^2 = 4x^3 - g_2 x - g_3 ,$$

e.g. for AD H_0: $g_2 = c,\ g_3 = u$.

The result is directly related to the integrable system:

SW blowup factor = autonomous tau function

σ is the tau function of the integrable system associated to SW.

Equivariant Fintushel-Stern blowup formula

From Donaldson-Witten theory topological correlators on a 4-manifold X correspond to Donaldson invariants.

The blowup factor $\mathcal{B}_{SW}$ gives then the relation between Donaldson polynomial of X and the ones of its blowup $\hat{X} = X \# CP^2$:

Fintushel-Stern blowup formula

Let $\Phi_X(p, S)$ be the generating function of Donaldson polynomials

$$\Phi^{\hat{X}}(p, \Sigma + sE) = \mathcal{B}_{SW}(s, p)\Phi^X(p, \Sigma)$$

The same reasoning can be applied in the NS Omega background:

$\mathcal{B}_{SW}(s) \to \mathcal{B}_{NS}(s, \epsilon)$, Donaldson inv. $\to$ Equivariant Donaldson inv.

Equivariant Fintushel-Stern blowup formula

The blowup factor $\mathcal{B}_{NS}(s, \epsilon)$ in the blowup formula for equivariant Donaldson invariants is given by the Painlevé tau function.

This a direct consequence of Nakajima-Yoshioka blowup relations (Notice: these relations exchange self-dual with NS).

Blowup factor and local observables

The Weierstrass sigma σ admits the following expansion

$$\sigma(s; g_2, g_3) = \sum_{n,m=0}^{\infty} a_{nm} \left(\frac{g_2}{2}\right)^n (2g_3)^m \frac{s^{4n+6m+1}}{(4n+6m+1)!} \ .$$

The result has a physical meaning [Moore & Witten]:

Topological Operator/state correspondence

Blowup is a local change of topology	$\Rightarrow$	Can be expressed as a sum of topological local operators = polynomials in u (and Λ).

The same logic applies to the theory in the Omega background!

Blowup factor in the NS omega background

$$\mathcal{B}_{NS}(s, \epsilon_1) = \lim_{\epsilon_2 \to 0} \frac{\hat{Z}_{Nek}(s)}{Z_{Nek}} = \sum_{n=0}^{\infty} c_n(u_{NS}(\epsilon_1), \Lambda, \epsilon_1) \frac{s^{n+1}}{(n+1)!} \ .$$

Valid in all the moduli space $\Rightarrow$ Can be applied to AD!

HRZ-like expansions

This gives a natural ansatz for τ as an expansion in s.

$$\tau(s) = \sum_{n=0}^{+\infty} c_n \frac{s^{n+1}}{(n+1)!} \ .$$

Interpretation of this expansion from the Painlevé equation side?

HRZ-like expansions

$$\tau(0) = \mathcal{B}_{NS}(0) = 0 \Rightarrow \text{Expansion around a zero of } \tau!$$

Mathematically, this expansion was first studied by Hone, Ragnisco and Zullo (HRZ) which found a way to determine the coefficients c_n recursively [Hone, Ragnisco & Zullo].

Comparing with the gauge theory we obtain the following dictionary:

$$\left.\begin{array}{l} \blacktriangleright \text{ The position } t_0 \text{ of the zero of } \tau \text{ is the coupling } t_0 \propto \Lambda \\ \blacktriangleright \text{ The hamiltonian } H_0 \text{ is the Coulomb parameter, } H_0 \propto u. \end{array}\right\} \textit{Initial conditions}$$

- s measures the displacement from the zero t_0.

Example: AD theory H_0 (PI equation)

$$\frac{2n(n^2-1)(n-6)}{(n+1)!}c_n = -\sum_{l=1}^{n-1} P^4_{l+1,n-l+1} c_l c_{n-l} + g_2 \sum_{l=0}^{n-4} c_l c_{n-l-4} - 2\epsilon \sum_{l=0}^{n-5} c_l c_{n-l-5} \ ,$$

$$P^k_{nm} = \frac{k!}{n!m!}\sum_{l=0}^{k}(-1)^l \binom{n}{l}\binom{m}{k-l} \ , \quad c_0 = 1, c_1 = 0 \text{ (gauge)}, c_6 = -6g_3 \text{ (resonance)} \ .$$

$$c_N = \sum_{4m+6n+5l=N} a_{mnl} \left(\frac{g_2}{2}\right)^m (2g_3)^n \epsilon^l$$

Explicit values of the first coefficients ($\alpha = g_2/2, \ \beta = 2g_3$):

c[0]= 1

c[1]= 0

c[2]= 0

c[3]= 0

c[4]= $-\alpha$

c[5]= $6\,\epsilon$

c[6]= $-3\,\beta$

c[7]= 0

c[8]= $-9\,\alpha^2$

c[9]= $84\,\alpha\,\epsilon$

c[10]= $-18\,\alpha\,\beta - 294\,\epsilon^2$

c[11]= $216\,\beta\,\epsilon$

c[12]= $69\,\alpha^3 - 54\,\beta^2$

c[13]= $-1650\,\alpha^2\,\epsilon$

c[14]= $513\,\alpha^2\,\beta + 18\,774\,\alpha\,\epsilon^2$

c[15]= $-18\,720\,\alpha\,\beta\,\epsilon - 78\,624\,\epsilon^3$

c[16]= $321\,\alpha^4 + 4968\,\alpha\,\beta^2 + 144\,144\,\beta\,\epsilon^2$

c[17]= $-52\,488\,\alpha^3\,\epsilon - 89\,424\,\beta^2\,\epsilon$

c[18]= $33\,588\,\alpha^3\,\beta + 14\,904\,\beta^3 + 1\,112\,436\,\alpha^2\,\epsilon^2$

c[19]= $-1\,358\,640\,\alpha^2\,\beta\,\epsilon - 8\,670\,816\,\alpha\,\epsilon^3$

c[20]= $160\,839\,\alpha^5 + 257\,580\,\alpha^2\,\beta^2 + 15\,053\,040\,\alpha\,\beta\,\epsilon^2 + 27\,734\,616\,\epsilon^4$

Integrality

The coefficients of the HRZ-like expansion satisfy a recursion relation with rational coefficients.

However, some highly non-trivial cancellations arise and for all Painlevé equations the coefficients a_{mnl} seem to be actually integers!

- For σ this can be proved using the theory of Schur polynomials or elliptic curves [Ayano, Ônishi].
- For τ checked numerically to very high order ($n \sim 100$) but no proof.

Conjecture

The coefficients are all integers and are related to counting of BPS states.

The coefficients are universal for $\epsilon \to 0$ but differ when $\epsilon \neq 0 \Rightarrow$ They measure the coupling of the soliton (blowup) with gravity.

Modularity and non-perturbative Z_{Top}

The HRZ-like expansion is given by $g_2, g_3 \Rightarrow$ modular invariant.

From the IR theory this modularity arises because the blowup factor is a holomorphic function of u.

The modularity of τ is directly related to holomorphic anomaly equations (BCOV)

Topological string interpretation

τ is a non-perturbative completion of topological string	$\Rightarrow$	The theory is manifestly background independent (= holomorphic + modular).

$$\tau = Z_{Top}\left(a, \frac{\partial}{\partial x}, \Lambda e^{\epsilon s}\right) e^{-\frac{1}{2}E_2 x^2}\sigma(x, g_2, g_3)\Big|_{x=0},$$

$$\partial_{E_2}\tau = 0 \Leftrightarrow \partial_{E_2} Z_{Top} = \frac{1}{2}\frac{\partial^2}{\partial a^2} Z_{Top} \ .$$

Conclusions and future directions

- The HRZ-like expansion of τ corresponds to the topological OPE of $\mathcal{B}_{NS}$ and is valid around any point of moduli space $\Rightarrow$ We can apply to AD theories! Can we use similar techniques to compute other observables?
- In NS limit the blowup factor $\mathcal{B}_{NS}$ corresponds exactly to the Painlevé tau function! What is $\mathcal{B}$ for generic Omega background? Tau function of Quantum Painlevé?
- Natural equivariant generalization of Fintushel-Stern blowup formula. Can we derive this geometrically? Does integrality have a topological origin?
- The SW blowup factor $\mathcal{B}_{SW}(s) \sim \sigma(s)$ can be derived from the "u-plane integral" of Moore & Witten. Can we derive $\mathcal{B}_{NS}(s, \epsilon)$ from a "Quantum u-plane integral"?
- Modularity of τ implies holomorphic anomaly equations! Can we use the expansion of τ to fix the holomorphic ambiguity for compact CY3?

Thank you for your attention