Developments on relations between GLSMs and Equivalences of Categories

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Outline

- Review of Ggauged Linear Sigma Models (GLSM) and their B-branes
- B-brane central charges
- Window categories via examples
- Monodromies: old and new
- Some future directions

Gauged Linear Sigma Models (GLSM)

These are a class of $\mathcal{N} = (2,2)$ supersymmetric 2d gauge theories that we will simply characterize by the 4-tuple $(G, \rho_{\mathrm{m}}, R, W)$, where

- G: Compact Lie group. G, $\mathfrak{g} := \mathrm{Lie}(G)$
- Matter: $\rho_{\mathrm{matter}} : G \to GL(V)$, $V \cong \mathbb{C}^N$
- Superpotential: G-invariant holomorphic polynomial $W \in (\mathrm{Sym}(V^\vee))^G$ such that there exists weights $R_i \in (0,2)$, $i = 1, \ldots, N$, that makes W quasi-homogeneous: $W(\lambda^{R_i}\phi_i) = \lambda^2 W(\phi_i)$.
- (Vector) R-charges: The weights R_i characterizes the action of the vector R-charge $U(1)_R$.

We will be concerned mainly with the coupling constants $t \in \mathfrak{z}_{\mathbf{C}}^\vee = \mathrm{Lie}(Z(G))_{\mathbf{C}}^\vee$

- $t_l := \zeta_l - i\theta_l \in \mathbb{C}$, $l = 1, \ldots, \mathrm{rk}(\mathfrak{z}^\vee)$

Denote the space of these constants by $\mathcal{M}_K$ (Stringy Kähler space)

and its B-branes...

B-branes are a class of boundary conditions (plus a boundary action) preseving half of the SUSY at the boundary requires us to specify a triplet (algebraic data) $\mathcal{B} = (\mathbf{T}, \rho_M, R_M)$.

- A $\mathbb{Z}_2$-graded, free $Sym(V^\vee)$ module denoted by $M = M_0 \oplus M_1$.

- A **matrix factorization** $\mathbf{T} \in End_{Sym(V^*)}(M)$ of $W \in Sym(V^*)$, i.e., a $\mathbb{Z}_2$-odd endomorphism such that $\mathbf{T}^2 = iW \cdot \mathbf{1}_M$

- A representation, $\rho_M : G \to GL(M)$ and a set of weights R_M compatible with ρ_{m} and R_i's respectively:

$$\lambda^{R_M}\mathbf{T}(\lambda^{R_i}\phi_i)\lambda^{-R_M} = \lambda\mathbf{T}(\phi_i)$$

$$\rho_M(g)^{-1}\mathbf{T}(\rho_{\mathsf{m}}(g)\cdot\phi)\rho_M(g) = \mathbf{T}(\phi).$$

for all $\lambda \in \mathbb{C}^\times$ and $g \in G$,

More B-branes

There is also **symplectic data**.

- G-invariant middle-dimensional subvariety of $\mathfrak{g}_{\mathbb{C}}$, or equivalently its intersection $L \subset \mathfrak{t}_{\mathbb{C}}$ with the Cartan algebra, which we refer to as the **contour**.

- Denote $\sigma \in \mathfrak{g}_{\mathbb{C}}$. An **admissible contour** is a G-invariant, middle dimensional L that is a continuous deformation of the real contour $L_{\mathbb{R}} := \{\Im(\sigma) = 0\}$ inside $\mathfrak{t}_{\mathbb{C}} \setminus \mathcal{H}$ where $\mathcal{H}$ is a hyperplane arrangement. such that the **imaginary part** of the boundary effective twisted superpotential

$$\widetilde{W}_{\text{eff},\rho}(\sigma) := \left(\sum_{\alpha>0} \pm i\pi\, \alpha \cdot \sigma\right) - \left(\sum_j (Q^j \cdot \sigma)\Big(\log(iQ^j \cdot \sigma)\Big)\right) - t \cdot \sigma + 2\pi i\, \rho \cdot \sigma$$

  approaches $+\infty$ in all asymptotic directions of L.

The full B-brane is then given by $(\mathcal{B}, L_t)$. We will denote te category spanned by $\mathcal{B}$'s by

$$MF_G(W)$$

$Z_{D^2}(\mathcal{B}, L_t; t)$ partition function of GLSM

We can compute the partition function on a disk/hemisphere D^2, expressed as an integral over Lie$(T_G)_{\mathbb{C}}$ and it depends on the boundary conditions $\mathcal{B}$, on t and on an integration contour L_t. The exact partition function $Z_{D^2}(\mathcal{B}, L_t; t)$ for a GLSM on D^2 takes a Mellin-Barnes integral form:

$$Z_{D^2}(\mathcal{B}, L_t; t) = \int_{L_t} d^{\mathsf{rk}(G)}\sigma Z_{\mathsf{class}} Z_{\mathsf{gauge}} Z_{\mathsf{matter}} f_{\mathcal{B}}(\sigma),$$

where

$$Z_{\mathsf{gauge}} := \prod_{\alpha>0} \alpha \cdot \sigma \sinh(\pi\alpha \cdot \sigma)$$

$$Z_{\mathsf{matter}} := \prod_{j=1}^{\dim(V)} \Gamma\left(iQ_j \cdot \sigma + \frac{R_j}{2}\right)$$

$$Z_{\mathsf{class}} := e^{it\cdot\sigma}$$

$$f_{\mathcal{B}}(\sigma) := \mathsf{Tr}_M\left(e^{i\pi\mathbf{r}_*} e^{2\pi\rho(\sigma)}\right),$$

Contour: The contour L_t must be a middle dimensional continuos deformation of $L_{\mathbb{R}} := \{\mathrm{Im}(\sigma) = 0\}$ such that $Z_{D^2}(\mathcal{B}, L_t; t)$ is absolutely convergent.

Collection of Facts

Some facts about $Z_{D^2}(\mathcal{B}, L_t; t)$:

1. The conditions on L_t can be interpreted as absolute convergence of $Z_{D^2}(\mathcal{B}, L_t; t)$.

2. $Z_{D^2}(\mathcal{B}, L_t; t)$ satisfies a differential equation with only regular singuarities when the GLSM is nonanomalous i.e. $\rho_{\mathrm{m}} : G \to SL(V)$.

Some facts about $MF_G(W)$:

1. In general, $MF_G(W) \not\cong D(Y_\zeta, W_\zeta)$ where

$$Y_\zeta := \frac{\mu^{-1}(\zeta)}{G}, \qquad W_\zeta := W\Big|_{Y_\zeta}$$

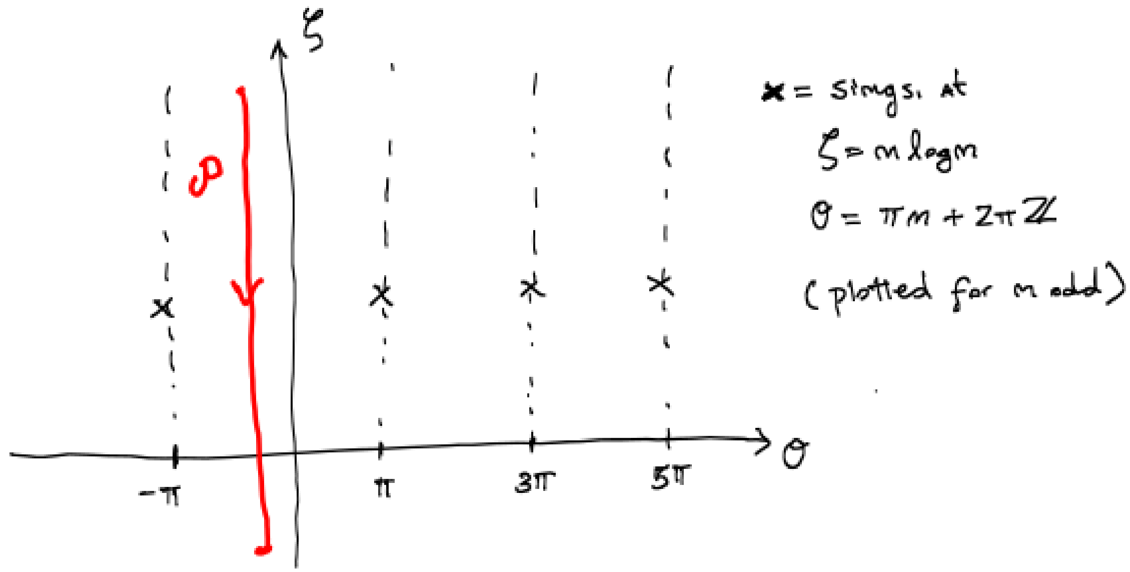
   Y_ζ will be referred as the classical Higgs branch.

2. There exist infinite embedings $D(Y_\zeta, W_\zeta) \cong \mathcal{W}_s \hookrightarrow MF_G(W)$

Example: CY hypersurface in $\mathbb{P}^{n-1}$

$$Z_{D^2}(\mathcal{B}, L_t; t) = \int_{L_t} d\sigma \Gamma(i\sigma)^n \Gamma(-ni\sigma + 1) e^{it\sigma} f_{\mathcal{B}}(\sigma),$$

We can then set the following question: for a fixed θ_* (recall $t = \zeta - i\theta$), if we consider the path $\mathcal{P}$ from $\zeta \gg 1$ to $\zeta \ll -1$, does it exist an integration contour L_t, such that L_t is a continuous deformation of $L_{\mathbb{R}}$ and $Z_{D^2}(\mathcal{B}, L_t; t)$ is convergent at every $t \in \mathcal{P}$?



- Yes, only if the weights q_a of ρ_M belong to an interval that depends only on $k := \lfloor \frac{\theta_*}{2\pi} \rfloor$. We denote

$$\mathcal{W}_k := \left\{ \mathcal{B} \in MF_{U(1)}(W) : \text{ weights of } \rho_M \text{ belong to } (-n/2 - k, n/2 - k) \right\}$$

- There exits functors $\mathcal{F}_k$ and $\mathcal{G}_k$ implementing the equivalences

$$\mathcal{F}_k : D^b Coh(X) \cong \mathcal{W}_k \qquad \mathcal{G}_k : \mathcal{W}_k \cong MF_{\mathbb{Z}_n}(f_n)$$

Example: Pfaffian-Grassmnnian

Consider the following example with $G = U(2)$. Denote

$$(p_a, x_a) \in V \cong (\mathbb{C}^7) \oplus (\mathbb{C}^2)^{\oplus 7}, \qquad a = 1\ldots,7$$

$$\rho_{\mathrm{matter}}(g) \circ (p_a, x_a^\alpha) = (\det(g)p_a, g^\alpha{}_\beta x_a^\beta) \qquad g \in U(2)$$

$$W = \sum_{a,b,c=1}^{7} p_a A^{a,bc} x_a^\alpha x_b^\beta \varepsilon_{\alpha\beta}$$

$$Z_{D^2}(\mathcal{B}, L_t; t) = \int_{L_t \subset \mathbb{C}^2} d^2\sigma(\sigma_1 - \sigma_2)\sinh(\pi(\sigma_1 - \sigma_2))\Gamma(i\sigma_1)^7\Gamma(i\sigma_2)^7$$
$$\times \Gamma(-i\sigma_1 - i\sigma_2 + 1)^7 e^{it(\sigma_1+\sigma_2)} f_{\mathcal{B}}(\sigma_1, \sigma_2)$$

Example: Pfaffian-Grassmnnian

The function $Z_{D^2}(\mathcal{B}, L_t; t)$ is annihilated by the differential operator

$$\begin{aligned}\mathcal{L} = 9\Theta^4 &- z\left(519\Theta^4 + 1020\Theta^3 + 816\Theta^2 + 306\Theta + 45\right)\\ &-z^2\left(2258\Theta^4 + 10064\Theta^3 + 15194\Theta^2 + 9546\Theta + 2166\right)\\ &+z^3\left(1686\Theta^4 + 5256\Theta^3 + 4706\Theta^2 + 1350\Theta + 12\right)\\ &-z^4\left(295\Theta^4 + 608\Theta^3 + 478\Theta^2 + 174\Theta + 26\right)\\ &+z^5(\Theta + 1)^4,\end{aligned}$$

$$z := -e^{-t}$$

It has five singular points:

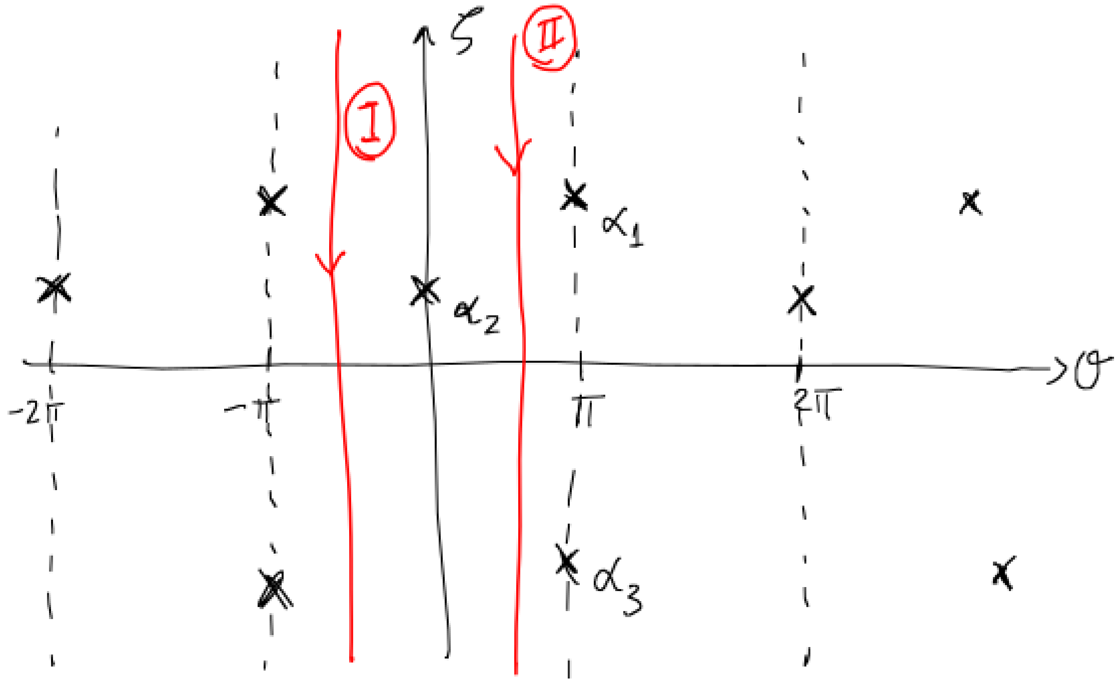
$$z \in \{0, \infty, \alpha_1, \alpha_2, \alpha_3\} \qquad \alpha_a := (1 + e^{\frac{2\pi i a}{7}})^{-7}$$

Example: Pfaffian-Grassmnnian

Notation: $\mathbb{C}$ - trivial rep. $S \cong \mathbb{C}^2$ - fundamental

$$\mathbb{C}(m) := \mathbb{C} \otimes (\det S)^m$$

$$S^\ell(m) := \mathrm{Sym}^\ell S \otimes (\det S)^m$$



(I)

$\mathbb{C}(1) \cdots \mathbb{C}(6)\ \mathbb{C}(7)$

$S(1) \cdots S(6)\ S(7)$

$S^2S\ \ S^2S(1) \cdots S^2S(6)$

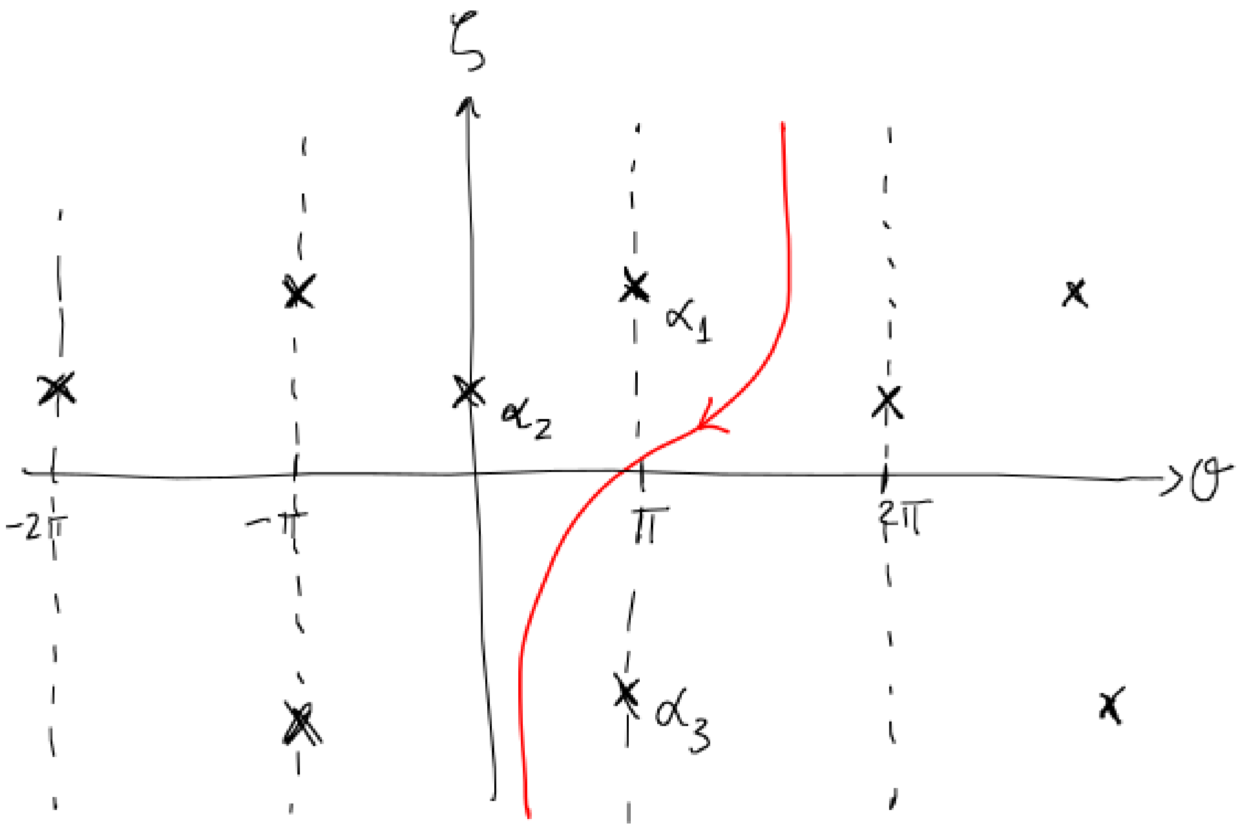
(II)

$\mathbb{C}(1) \cdots \mathbb{C}(6)\ \mathbb{C}(7)$

$S\ \ S(1) \cdots S(6)$

$S^2S\ \ S^2S(1) \cdots S^2S(6)$

Example: Pfaffian-Grassmnnian



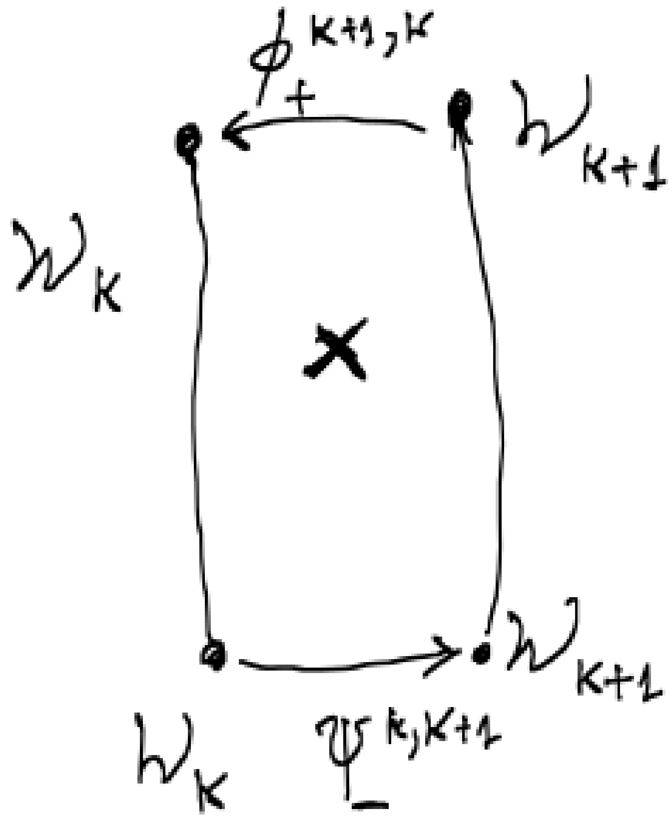
Conj:

$\mathbb{C}$ $\mathbb{C}(1)$ ... $\mathbb{C}(6)$

S $S(1)$ .... $S(6)$

$S^2 S$ $S^2 S(1)$ $S^2 S(6)$

Monodromies, Categorically

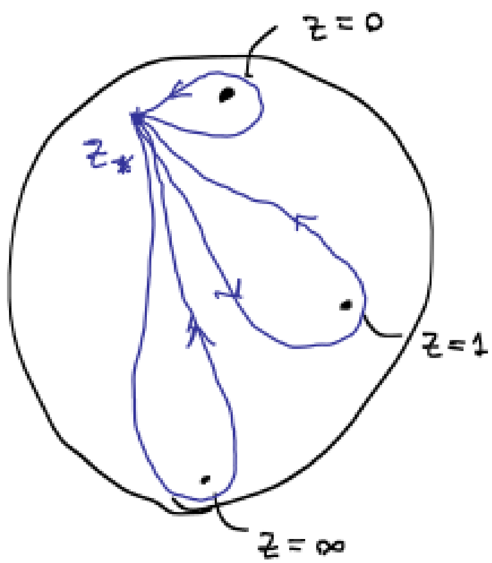


Example: Monodromies, Categorically

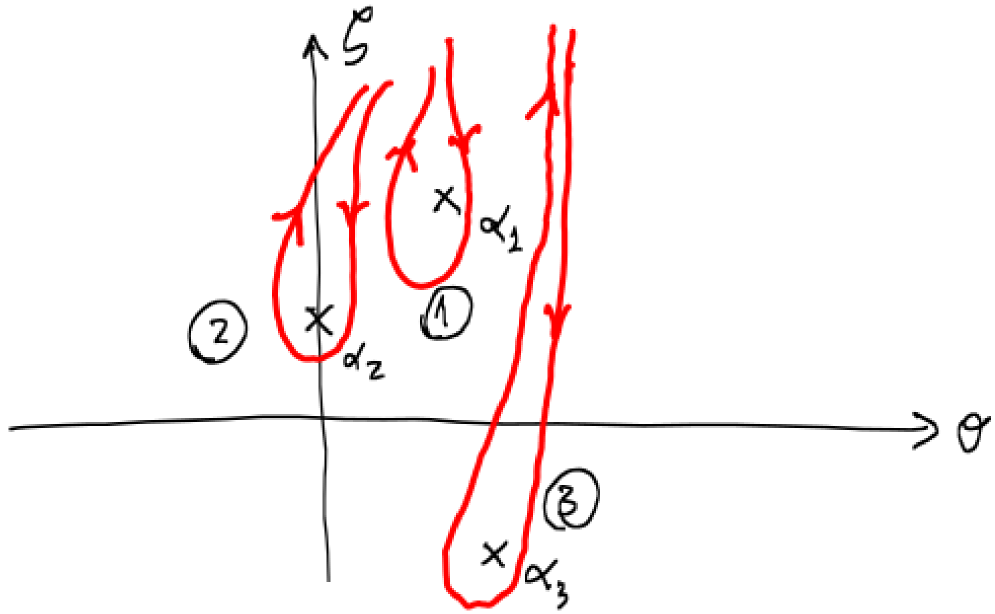
$$M_0 \longrightarrow - \otimes \mathcal{O}_X(5)$$

$$M_1 \longrightarrow T_{\mathcal{O}_X}$$

$$M_\infty \longrightarrow T_{\mathcal{O}_X}^{-1} \circ (- \otimes \mathcal{O}_X(-5))$$



Example: Monodromies, Categorically



$$M_{\alpha_1} \longrightarrow T_{\mathcal{O}_X}$$

$$M_{\alpha_2} \longrightarrow T_{S_X}$$

$$M_{\alpha_3} \longrightarrow T_{\mathsf{Sym}^2 S_X(1)}$$

Two-parameter families

Next we consider GLSMs corresponding to a resolution of the determinantal variety

$$Z(A,k) = \{\phi \in B \mid \operatorname{rank} A(\phi) \le k\},$$

where B is smooth projective and A is a section of the bundle $\mathrm{Hom}(\mathcal{E},\mathcal{F})$. This resolution is given an incidence correspondence. Define $B_{\mathcal{E},k}$:

$$G(k,\mathcal{E}) \longrightarrow B_{\mathcal{E},k} \xrightarrow{\ \pi\ } B$$

then

$$X_A := \tilde{Z}(A,k) = \{p \in B_{\mathcal{E},n-k} \mid A(\pi(p)) \circ p = 0\},$$

We will focus on the case $B = \mathbb{P}^n$ and $\mathcal{E},\mathcal{F}$ being direct sum of line bundles. Then X_A takes a much simpler form

$$X_A = \{(\phi, x) \in \mathbb{P}^d \times Gr(n-k,n) \mid A(\phi)^{ij} x_j = 0\} \tag{1}$$

Two-parameter families

Consider the GLSM with the following matter content:

	Φ_a	P_i	X_i
$U(1)$	1	-1	0
$U(2)$	0	$\mathbf{2}$	$\overline{\mathbf{2}}$
$U(1)_R$	$2(1-\epsilon-\delta)$	2ϵ	2δ

for $a = 1, \cdots, 8$, $i = 1, \cdots, 4$.

$$W = \sum_{i,j=1}^{n} \mathrm{Tr}(P_i A(\Phi)^{ij} X_j),$$

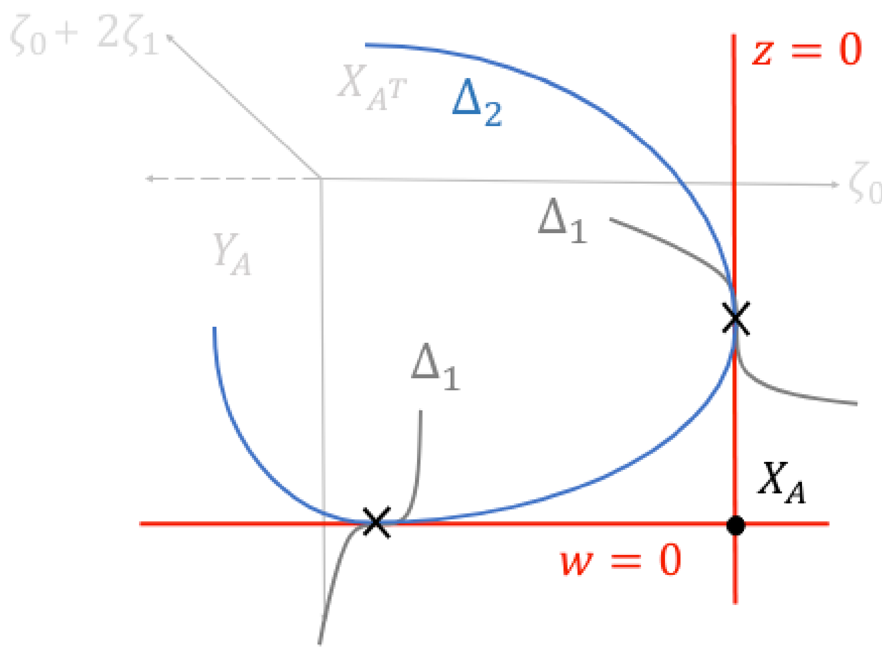
The Higgs branch geometries are given by

$$X_A: \quad \{(\phi, x) \in \mathbb{P}^7 \times Gr(2,4) \mid A(\phi)_i \cdot x^\alpha = 0\}$$

$$X_{A^T}: \quad \{(\phi, p) \in \mathbb{P}^7 \times Gr(2,4) \mid p_\alpha \cdot A(\phi)^j = 0\}$$

$$Y_A: \quad \{(p, x) \in \mathbb{P}(S^{\oplus 4}) \to Gr(2,4) \mid p \cdot A^a \cdot x = 0\}$$

Two-parameter families

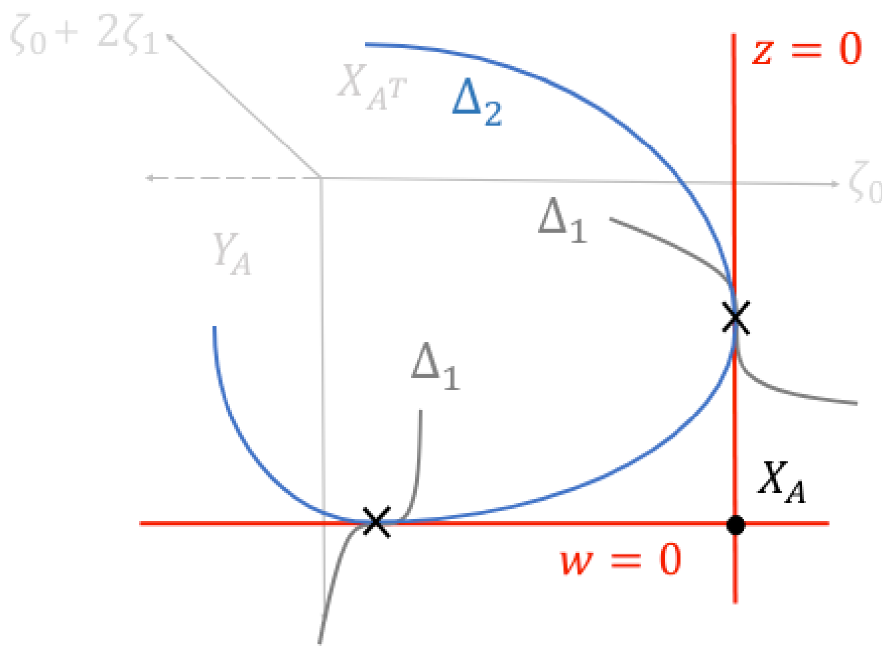


$$\Delta_1 : \quad (1+w)^4 - 2z(1-6w+w^2) + z^2 = 0$$

$$\Delta_2 : \quad -(1+w)^8 + 4z(1+34w+w^2)(1+w)^4 - 2z^2(3-372w+1298w^2$$
$$-372w^3+3w^4) + 4z^3(1+34w+w^2) - z^4 = 0$$

$$w := e^{t_1}, \qquad z := e^{-t_0}$$

Two-parameter families



Crossing between X_A and Y_A phases:

$$-\frac{n(n-k)}{2} < \frac{\theta_0}{2\pi} + q^0 < \frac{n(n-k)}{2},$$

Crossing between X_A and X_{A^T} phases:

$$-\frac{k+1}{2} < \frac{\theta_1}{2\pi} + q^\alpha < \frac{k+1}{2}, \qquad \text{For all } \alpha$$

In our case $(k, n) = (2, 4)$ and $\alpha = 1, 2$.

Two-parameter families

The monodromy around these two boundaries (with basepoint at the X_A phase) can be computed explictly, for instance, around ζ_0-boundary

$$T_X := - \otimes \mathcal{O}_{X_A}(-4, 0)$$

where $\mathcal{O}_{X_A}(0, -1)$ stands for $\mathcal{O}_{X_A}$ twisted by $\det^{-1} S_{X_A}$

Monodromy around ζ_1-boundary, T_Y, is complicated, but we can find an expression in terms of simpler spherical twists. Consider the intersection

$$\mathcal{L} := S^3_\varepsilon \cap (\Delta_1 \cup \Delta_2 \cup \{w = 0\})$$

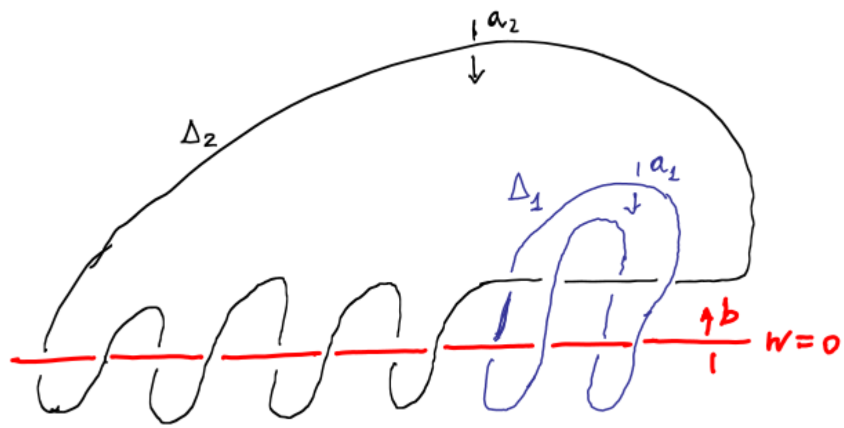
where S^3_ε is a 3-sphere centered at the intersection $\Delta_1 \cap \Delta_2 \cap \{w = 0\}$. Then

$$S^3_\varepsilon \setminus \mathcal{L}$$

becomes a link complement.

Two-parameter families

For the case at hand, $S^3_\varepsilon \setminus \mathcal{L}$ takes the form of a nested link:



then $\pi_1(S^3_\varepsilon \setminus \mathcal{L})$ is generated by a_1, a_2 and b, with the assignments:

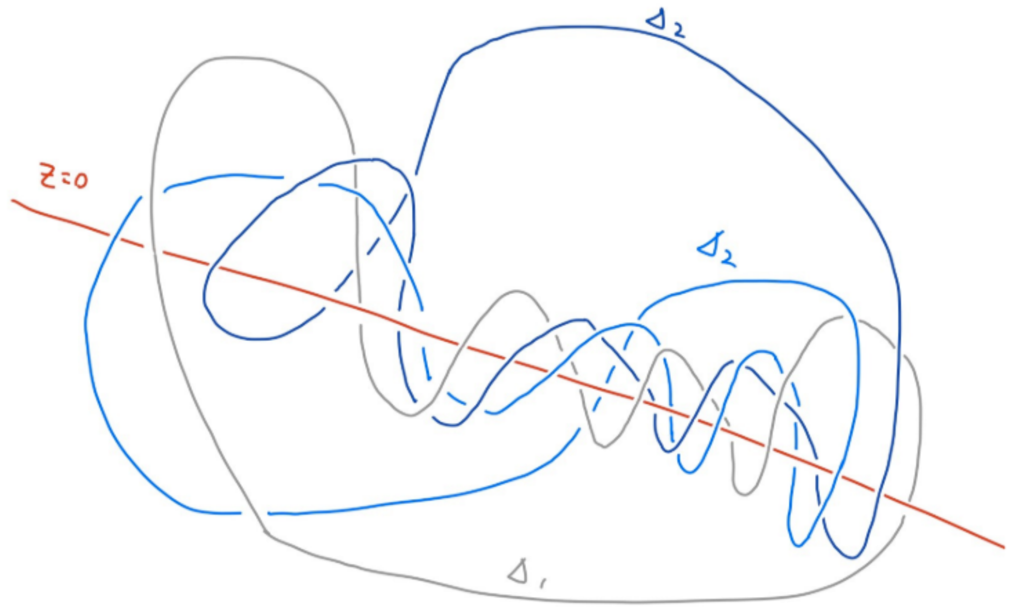
$$a_1 \to T_{S_X}, \qquad a_2 \to T_{\mathcal{O}_X}, \qquad b \to - \otimes \det{}^{-1} S_X$$

therefore we can decompose

$$T_Y = b^{-3}(T_{\mathcal{O}_X} b)^3 T_{S_X} T_{\mathcal{O}_X} b T_{S_X} b^{-1}$$

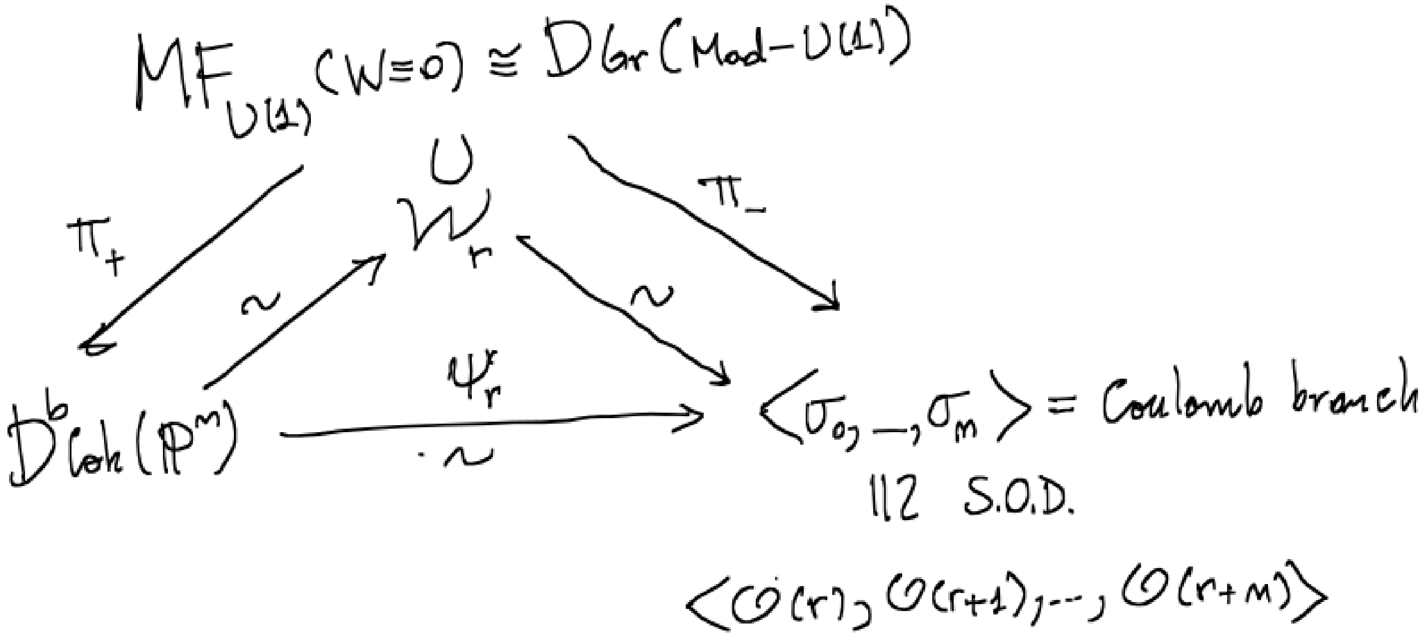
Two-parameter families

This structure for the 'Y-boundary' can be shown (in several examples), to generalize, at least in families where the singularities of $Z(A,k)$ are points. The other crossing ('X-boundary) is more complicated to analyze, in general



Future directions: Anomalous GLSMs

the existence of a RG-flow in the t-directions and a nontrivial Coulomb branch signals the existence of a semiorthogonal decomposition.



$$\mathcal{W}_r = \left\{ \mathcal{B} \in DGr(Mod-U(1)) \,\middle|\, \text{weights } w, \text{ satisfy } -\frac{(m+1)}{2} < w + \frac{\theta}{2\pi} < \frac{m+1}{2} \right\}$$

Future directions: Anomalous GLSMs

The function $Z_{D^2}(\mathcal{B}, L_t; t)$ satisfies a differential equation with irregular singularities and it is possible to compute its Stokes matrices by analyzing the overlap of the windows categories associated to Stokes sectors.

According to (part of) Dubrovin's conjecture the Stokes matrix coincides with the Gram matrix of the exceptional collection $\{E_i\}$ associated to a window category $\mathcal{W}_r$, i.e.

$$(S_{ij}) = \chi(E_i, E_j)$$

which can be checked explicitly for several Fano varieties.

Future directions: more puzzles

- Is there a way to 'systematize' the computation of nonabelian monodromies, or relate them to abelian ones?

- Mirror symmetry?

- Inclusion of twited masses/equivariant parameters

- Generalizations of Dubrovin's conjecture

Grazie mille!