A landscape of 2d string theories

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with **Scott Collier**, **Beatrix Mühlmann** and **Victor Rodriguez**

Motivation

- The landscape of string vacua is extremely complicated
- The situation for a 2d target space is much more controlled
- There are two well-explored classes:
  - (p, q) minimal string $\Longleftrightarrow$ Double-scaled (two) matrix-integral

    [Brezin Kazakov 90; Gross Migdal 90; Douglas Shenker 90; …]
  - $c = 1$ string $\Longleftrightarrow$ Matrix quantum mechanics

    [Moore Plesser Rangoolam ’91; Moore ’92, …]
- Can we classify the whole landscape of 2d string theories?

Motivation

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Focus on these today

- Can we classify the whole landscape of 2d string theories?

A two-dimensional string landscape

Persistent paradigm:

Worldsheet: matter CFT $\oplus$ Liouville CFT $\oplus$ b c ghosts

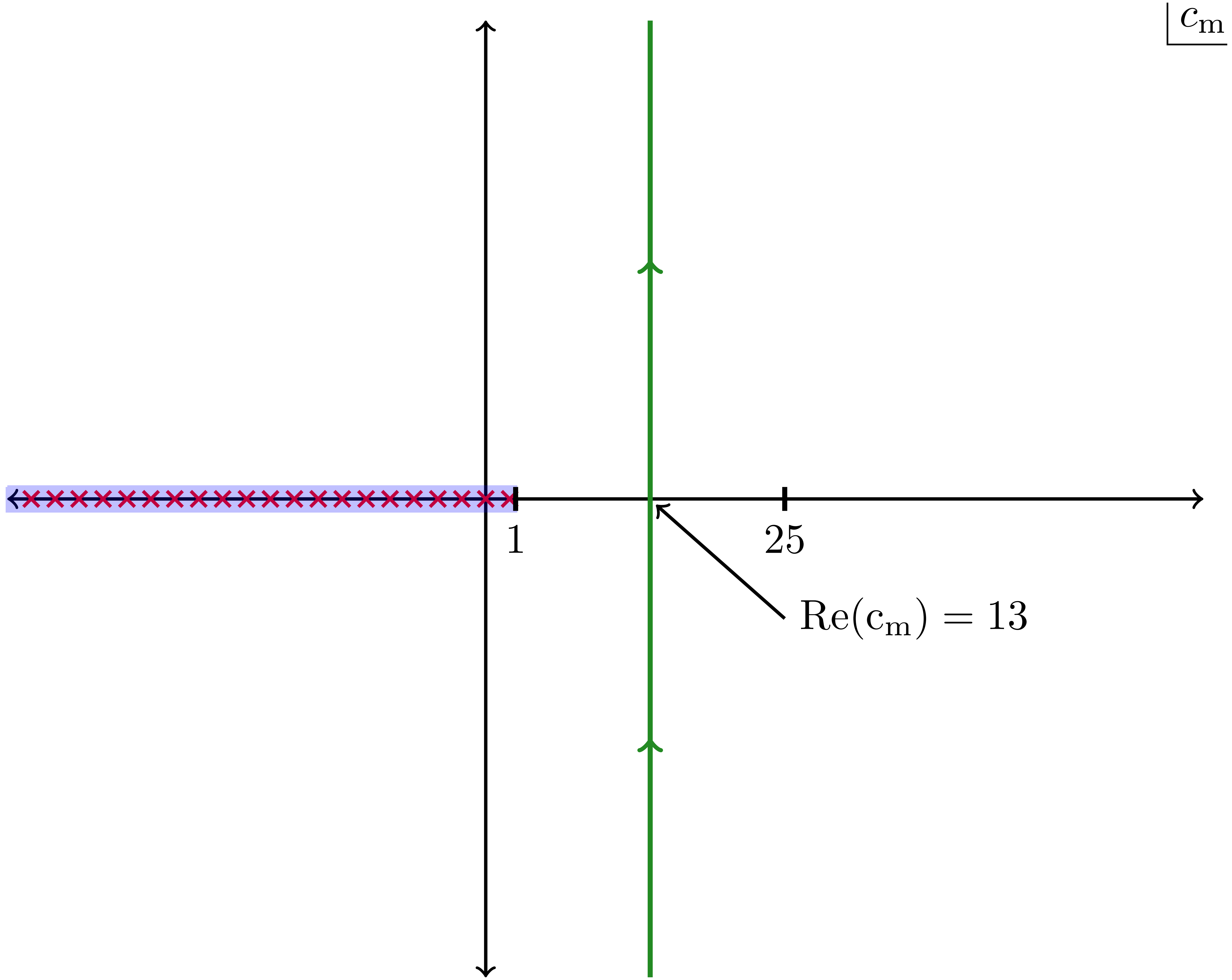
central charge: $c_{\rm m}$ $\quad c_{\rm L} = 26 - c_{\rm m}$ $\quad c_{\rm gh} = -26$

$\Updownarrow$

Double-scaled two-matrix integral

$$\int_{\mathbb{R}^{2N^2}} [dM_1][dM_2] e^{-N \,\mathrm{Tr}\left(V_1(M_1) + V_2(M_2) - M_1 M_2\right)}$$

A two-dimensional string landscape



A landscape of string/matrix model holographic dualities

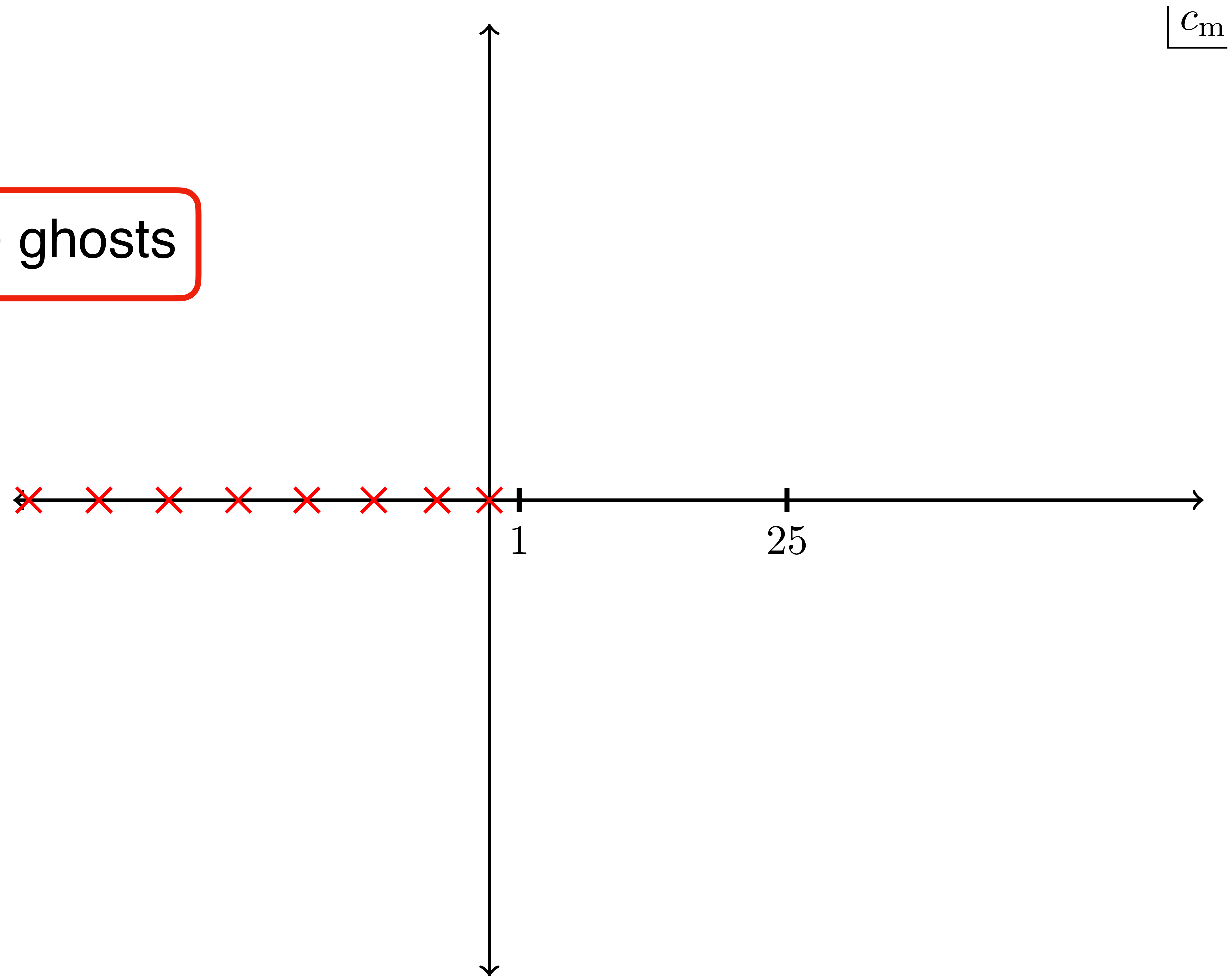
$(2,p)$ minimal string

$(2,p)$ minimal model $\oplus$ Liouville CFT $\oplus$ ghosts

$\updownarrow$

Double-scaled one-matrix integral

$$\rho_0(E) = \sinh\left(\frac{p}{2}\operatorname{arccosh}(1+E)\right)$$



[Brezin Kazakov 90; Gross Migdal 90; Douglas Shenker 90; …]

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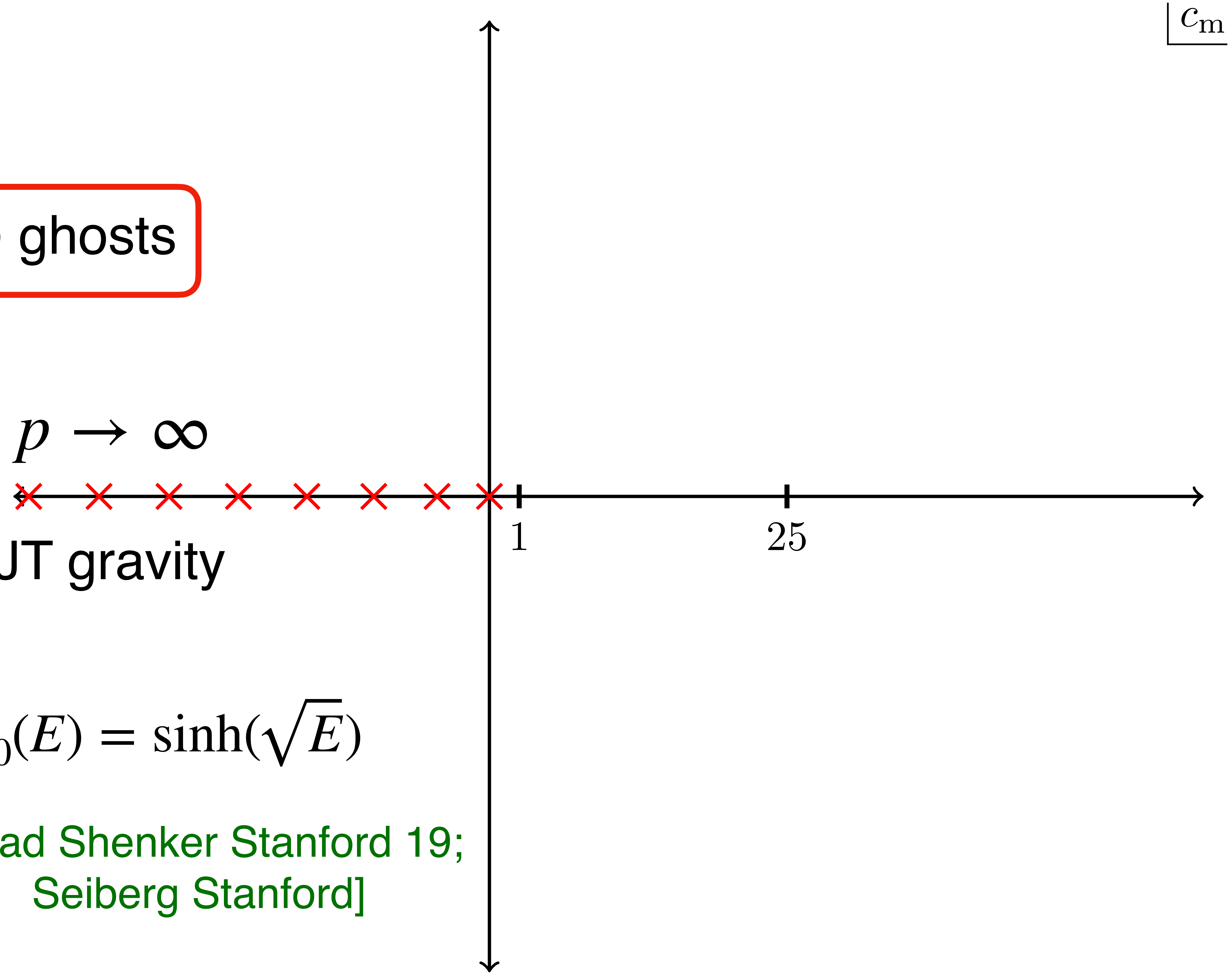


JT gravity

$$\rho_0(E) = \sinh(\sqrt{E})$$

[Saad Shenker Stanford 19; Seiberg Stanford]

[Brezin Kazakov 90; Gross Migdal 90; Douglas Shenker 90; …]

A landscape of string/matrix model holographic dualities

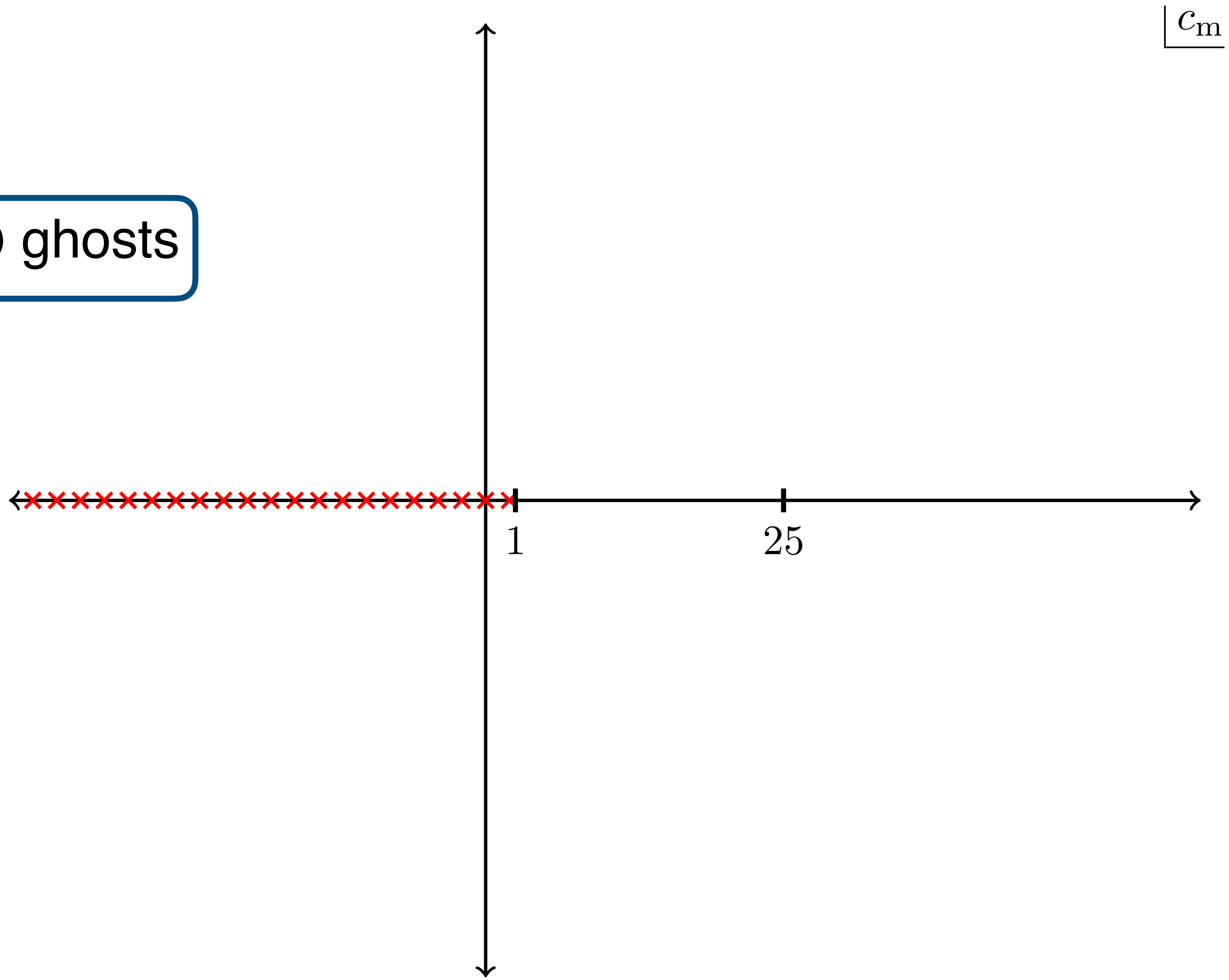
(p,q) minimal string

(p,q) minimal model $\oplus$ Liouville CFT $\oplus$ ghosts

$\updownarrow$

Double-scaled **two-matrix integral**

$$x(z) = T_p(z), \quad y(z) = T_q(z)$$



[Kazakov 86; Boulatov Kazakov 87; …; Eynard 02; Seiberg Shih 04; …]

A landscape of string/matrix model holographic dualities

“Virasoro minimal string (VMS)”

Liouville CFT ⊕ timelike Liouville CFT ⊕ ghosts

$c \geq 25$ $\quad$ $26 - c \leq 1$

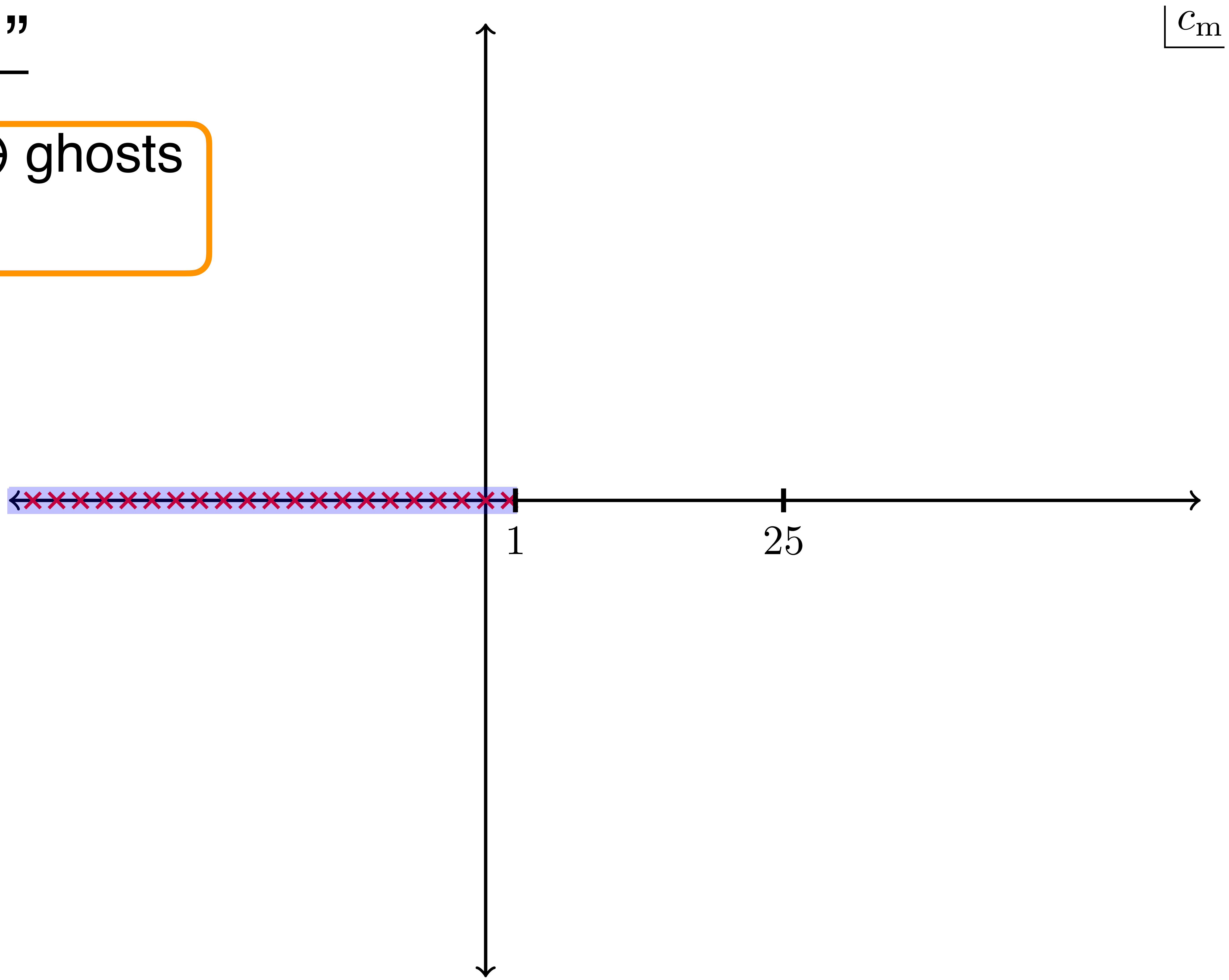
↕

Double-scaled one-matrix integral

$$\rho_0^{(b)}(E) = \frac{\sinh(2\pi b\sqrt{E})\sinh(2\pi b^{-1}\sqrt{E})}{\sqrt{E}}$$

String amplitudes $\mathsf{V}_{g,n}^{(b)}$: polynomials deformations of Weil-Petersson volumes

[Collier LE Mühlmann Rodriguez ’23]



A landscape of string/matrix model holographic dualities

"$|\text{Liouville}|^2$ string theory"

Liouville CFT ⊕ Liouville CFT ⊕ ghosts

$c_+ = 13 + i\mathbb{R}$ $c_- = 13 - i\mathbb{R}$

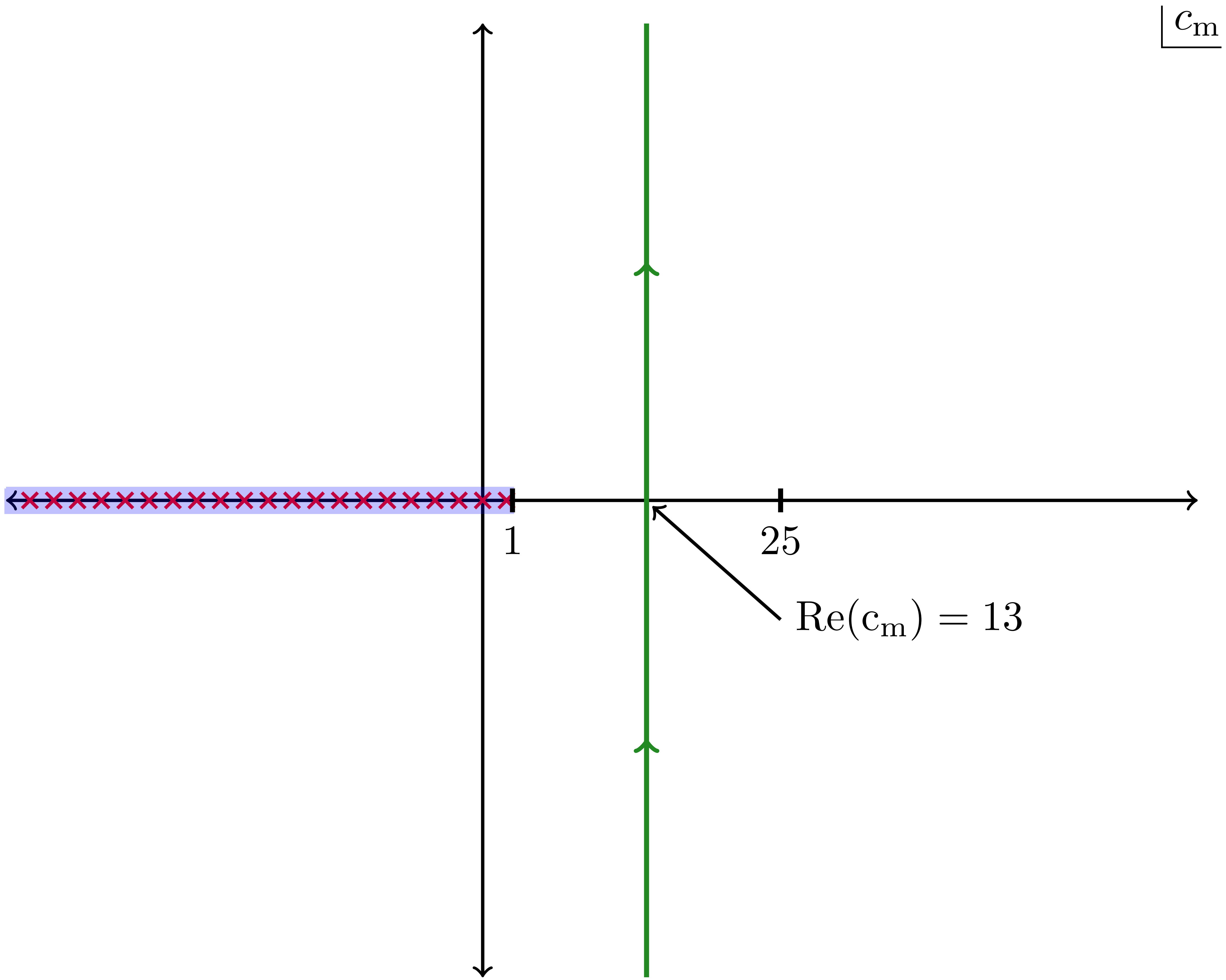
↕

Double-scaled **two-matrix integral**

$x(z) = -2\cos(\pi b^{-1}\sqrt{z}), \quad y(z) = 2\cos(\pi b\sqrt{z})$

String amplitudes $\mathsf{A}_{g,n}^{(b)}$ computed by topological recursion

[Collier LE Mühlmann Rodriguez WIP]



Today I will explain these new dualities.

They are irrational cousins of the (p, q) minimal string.

The string theory has less subtleties than the (p, q) minimal string since string amplitudes are analytic functions

Virasoro Minimal String

The worldsheet CFT

- The worldsheet theory is defined by the following non-perturbative CFT:

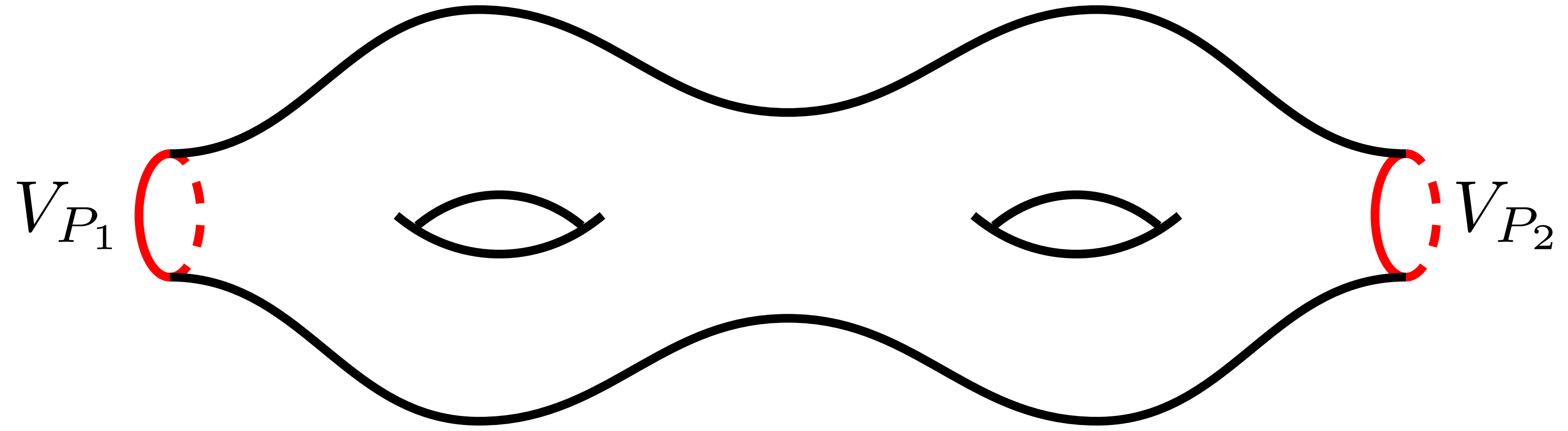
$$\begin{array}{c} c \geq 25 \\ \text{Liouville CFT} \end{array} \oplus \underbrace{\begin{array}{c} \hat{c} \leq 1 \\ \text{Liouville CFT} \end{array}}_{\text{“Timelike” Liouville CFT}} \oplus \; b, c \text{ ghosts}$$

with $\hat{c} = 26 - c$, this defines a critical worldsheet theory, which we may take to be a more precise definition of the theory

Liouville CFT

- Non-compact, unitary solution to the CFT crossing equations
  [Dorn Otto; Zamolodchikov^2 '95; Teschner '01; David, Kupiainen, Rhodes, Vargas '14, …]
  - $c = 1 + 6(b + b^{-1})^2 \geq 25, \quad b \in (0,1]$
  - Spectrum: continuum $\{V_P\}$
    $$h_P = \bar{h}_P = \frac{c-1}{24} + P^2 \geq \frac{c-1}{24}$$
  - OPE data:
    $\langle V_{P_1} V_{P_2} V_{P_3} \rangle_{g=0}^{(b)} = C_b(P_1, P_2, P_3)$, "DOZZ formula"
- With this set of CFT data, can compute any local observable

$$\left\langle \prod_{j=1}^{n} V_{P_j}(z_j) \right\rangle_g^{(b)}$$



e.g. via the conformal block decomposition

Timelike Liouville CFT

- Non-compact, non-unitary solution to CFT crossing equations [Zamolodchikov; Kostov Petkova '95; Harlow Maltz Witten '11; Ribault Santachiara '15] Imaginary Liouville theory: [Guillarmou, Kupiainen, Rhodes '23]

- **Not** simply the analytic continuation of Liouville CFT to $\hat{c} \leq 1$!

  - $\hat{c} = 1 - 6(\hat{b}^{-1} - \hat{b})^2 \leq 1, \quad \hat{b} \in (0,1]$

  - Spectrum: continuum $\{\hat{V}_{\hat{P}}\}$; $\hat{h}_{\hat{P}} = \bar{\hat{h}}_{\hat{P}} = \dfrac{\hat{c}-1}{24} + \hat{P}^2 \geq \dfrac{\hat{c}-1}{24}$

  - OPE data: $\langle \hat{V}_{\hat{P}_1} \hat{V}_{\hat{P}_2} \hat{V}_{\hat{P}_3} \rangle_{g=0}^{(i\hat{b})} = \hat{C}_{\hat{b}}(\hat{P}_1, \hat{P}_2, \hat{P}_3) = \dfrac{1}{C_{\hat{b}}(i\hat{P}_1, i\hat{P}_2, i\hat{P}_3)}$

- Correlation functions of local operators

$$\left\langle \prod_{j=1}^{n} \hat{V}_{\hat{P}_j} \right\rangle_g^{(i\hat{b})}$$

String theory observables

- Weyl anomaly:

$$c + \hat{c} = 26 \to \hat{b} = b$$

- Mass-shell condition

$$h_{P_j} + h_{\hat{P}_j} = 1 \to \hat{P}_j = iP_j$$

- On-shell vertex operators

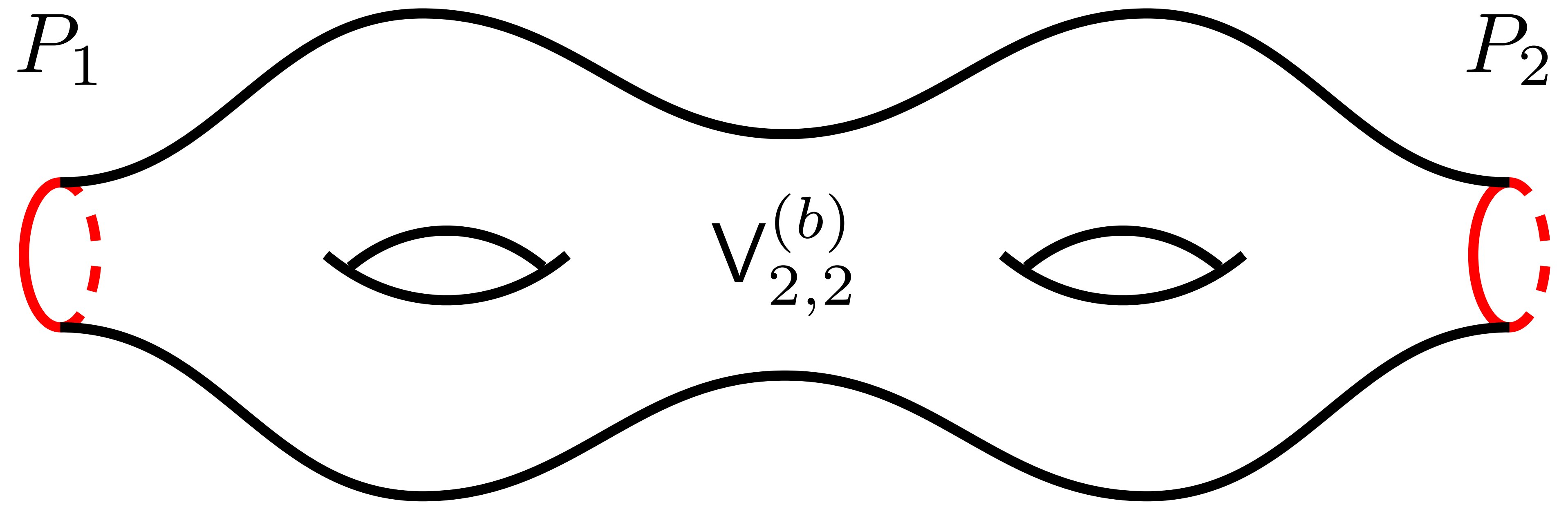
$$\mathscr{V}_P = c\bar{c}\, V_P\, \hat{V}_{iP}$$

- Observables are computed as usual by integrating CFT correlators over moduli space

$$\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) = \int_{\mathscr{M}(\Sigma_{g,n})} \left\langle \prod_{j=1}^{n} V_{P_j} \right\rangle_g^{(b)} \left\langle \prod_{j=1}^{n} \hat{V}_{iP_j} \right\rangle_g^{(ib)} \times \big(b, c \text{ ghosts}\big)$$

String theory observables

$$\mathsf{V}^{(b)}_{g,n}(P_1,\ldots,P_n) = \int_{\mathscr{M}(\Sigma_{g,n})} \left\langle \prod_{j=1}^{n} V_{P_j} \right\rangle_g^{(b)} \left\langle \prod_{j=1}^{n} \hat{V}_{iP_j} \right\rangle_g^{(ib)} \times \big(b, c \text{ ghosts}\big)$$



- **Absolutely convergent** integral over the moduli space of Riemann surfaces!
- “Quantum volumes”

Conformal blocks on Σ

- We can "derive" the matrix model description by thinking about Virasoro conformal blocks on Σ
- The **inner product** on the conformal block Hilbert space is tantalizingly similar to the worldsheet moduli integrals that appear in the Virasoro minimal string [H. Verlinde '89; Collier LE Zhang '23]

Conformal blocks

$$\left\langle \mathcal{F}^{(b)}_{\Sigma}(\mathbf{P}') \middle| \mathcal{F}^{(b)}_{\Sigma}(\mathbf{P}) \right\rangle = \int_{\mathcal{T}(\Sigma)} \overline{\mathcal{F}^{(b)}_{\Sigma}(\mathbf{P}'\,|\,\mathbf{m})}\, \mathcal{F}^{(b)}_{\Sigma}(\mathbf{P}\,|\,\mathbf{m}) \left\langle \prod_{j=1}^{n} \hat{V}_{iP_j} \right\rangle^{(ib)} \times (b, c \text{ ghosts})$$

Teichmuller space

$$= \frac{\delta^{3g-3+n}(\mathbf{P}-\mathbf{P}')}{\rho_{\Sigma}(\mathbf{P})}$$

"OPE density" of Liouville CFT on Σ

$$\rho_{\Sigma}(\mathbf{P}) = \prod_{a} \rho_0^{(b)}(\mathbf{P}_a) \prod_{(i,j,k)} C_b(\mathbf{P}_i, \mathbf{P}_j, \mathbf{P}_k)$$

Counting conformal blocks

- The quantum volumes formally count the number of linear combinations of conformal blocks that are invariant under the mapping class group

$$\#\text{conf. blocks} = \frac{1}{\left|\mathrm{MCG}(\Sigma)\right|}\,\mathrm{Tr}\int d\mathbf{P}\,\rho_\Sigma(\mathbf{P})\left|\mathcal{F}_\Sigma^{(b)}(\mathbf{P})\right\rangle\left\langle\mathcal{F}_\Sigma^{(b)}(\mathbf{P})\right|$$

$$= \frac{1}{\left|\mathrm{MCG}(\Sigma)\right|}\int_{\mathcal{T}(\Sigma)}\Big\langle\prod_j V_{P_j}\Big\rangle^{(b)}\Big\langle\prod_j \hat{V}_{iP_j}\Big\rangle^{(ib)}\times(b,c\ \text{ghosts})$$

$$= \mathsf{V}_{g,n}^{(b)}(P_1,\ldots,P_n)$$

An index theorem for the quantum volumes

- The number of conformal blocks may also be computed by an **index theorem**: it computes the dimension of the Hilbert space defined by quantization of the phase space $\mathscr{M}(\Sigma)$ [Maloney '15; LE '22]

$$\mathsf{V}^{(b)}_{g,n}(P_1,\ldots,P_n) = \int_{\overline{\mathscr{M}}(\Sigma)} \mathrm{td}(\mathscr{M}(\Sigma))\,\mathrm{e}^{\frac{c}{24}\kappa_1 + \sum_i P_i^2 \psi_i}$$

GRR

$$= \int_{\overline{\mathscr{M}}(\Sigma)} \exp\left(\frac{c-13}{24}\kappa_1 + \sum_{j=1}^{n} P_j^2 \psi_j - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)(2m)!}\kappa_{2m}\right)$$

- Hence the quantum volumes may be computed via intersection theory on $\overline{\mathscr{M}}(\Sigma)$
- Such intersection numbers may be computed using topological recursion = loop equations of the matrix model [Eynard, Orantin '07; Eynard '10]

Further properties of the quantum volumes

- Fom the intersection theory representation we learn that the quantum volumes are simply **polynomials**

$$\mathsf{V}^{(b)}_{g,n}(P_1,\dots,P_n) \in \mathbb{Q}\left[c, P_1^2, \dots, P_n^2\right] \text{ of degree } 3g-3+n$$

- **Duality symmetry** that roughly swaps the roles of the spacelike and timelike Liouville CFTs on the worldsheet
  - $b \to ib,\ P_j \to iP_j$

$$\mathsf{V}^{(ib)}_{g,n}(iP_1,\dots,iP_n) = (-1)^{3g-3+n}\mathsf{V}^{(b)}_{g,n}(P_1,\dots,P_n)$$

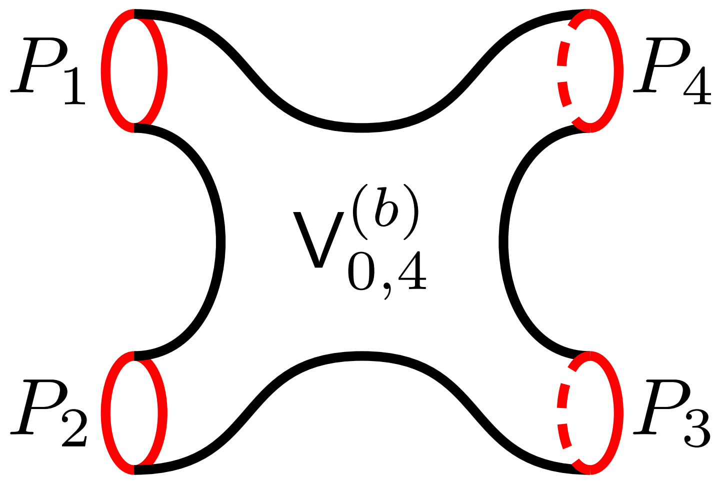
  - Intuitive, but cannot even be properly formulated on the worldsheet: timelike Liouville CFT only exists for $\hat{c} \leq 1$!

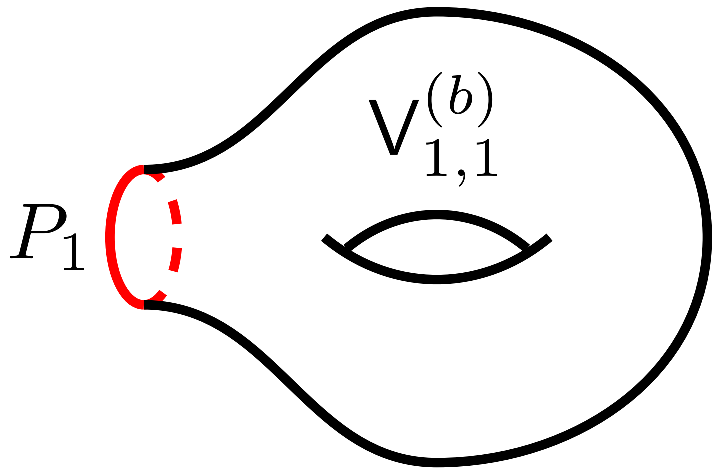
Quantum volumes

$$\mathsf{V}^{(b)}_{0,3}(P_1, P_2, P_3) = 1$$

$$\mathsf{V}^{(b)}_{0,4}(P_1, P_2, P_3, P_4) = \frac{c-13}{24} + P_1^2 + P_2^2 + P_3^2 + P_4^2$$

$$\mathsf{V}^{(b)}_{1,1}(P_1) = \frac{1}{24}\left(\frac{c-13}{24} + P_1^2\right)$$

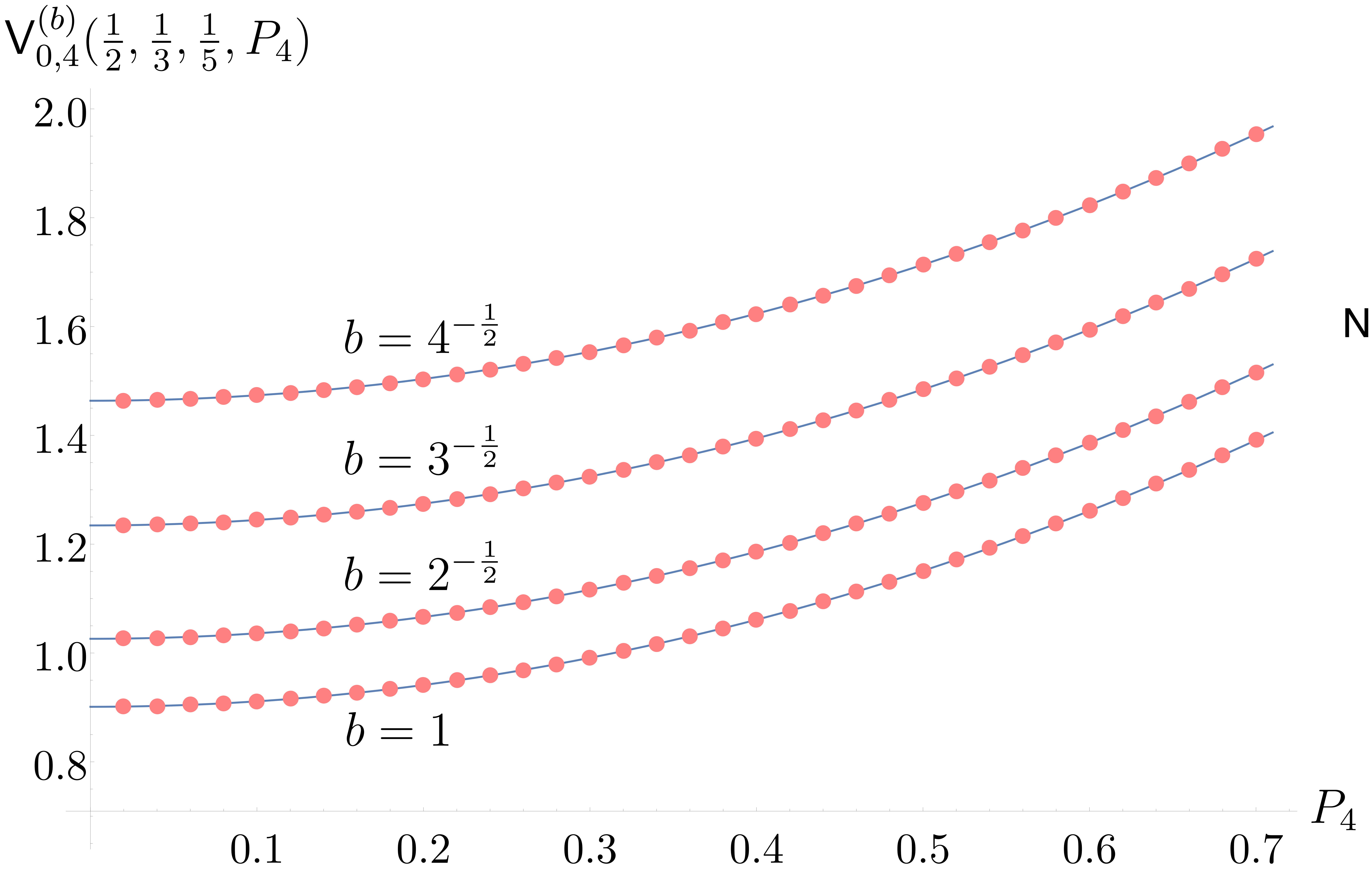




- **Absurdly simple** (recall origin: integrated DOZZ formulas, Virasoro conformal blocks over moduli space)
- Reduce to Weil-Petersson volumes in the ($b \to 0$) limit (= Jackiw-Teitelboim gravity):

$$\mathsf{V}^{(b)}_{g,n}(P_1, \ldots, P_n) \to (8\pi^2 b^2)^{3-3g-n} V_{g,n}(\ell_1, \ldots, \ell_n), \quad \ell_i = 4\pi b P_i$$

Quantum volumes: sphere four-point



Numerical error
$\lesssim 10^{-3}\,\%$

The Virasoro matrix integral

- The matrix integral is perturbatively fully fixed by its leading density of eigenvalues:

$$\varrho_0^{(b)}(E)\, dE = \frac{2\sqrt{2}\,\sinh(2\pi b\sqrt{E})\sinh(2\pi b^{-1}\sqrt{E})}{\sqrt{E}}\, dE$$

Universal **Cardy formula** for asymptotic density of states in 2d CFT

$\updownarrow$

Disk partition function is the **Virasoro vacuum character** in the dual channel

The Virasoro matrix integral

- The main observables of the matrix integral are **resolvents**

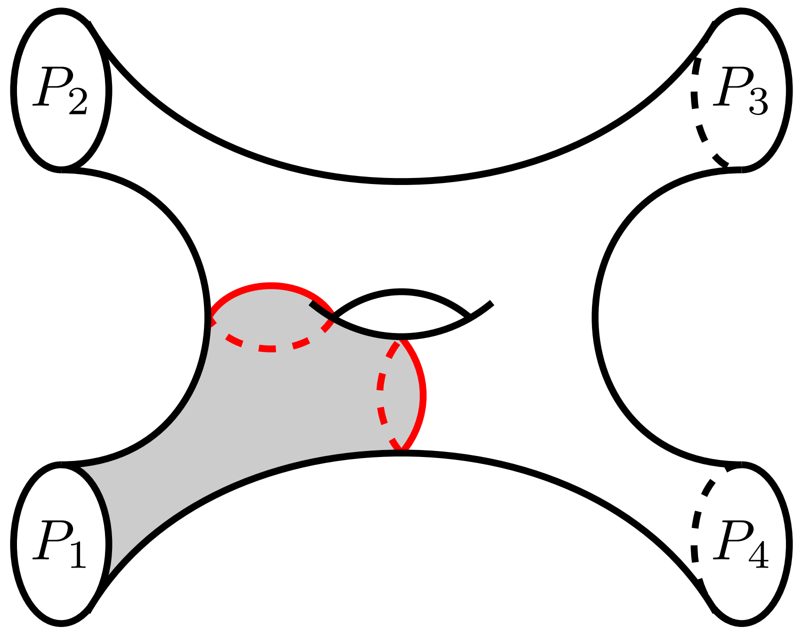
$$R^{(b)}(E_1, \ldots, E_n) = \left\langle \mathrm{Tr}\,\frac{1}{E_1 - H} \cdots \mathrm{Tr}\,\frac{1}{E_n - H} \right\rangle = \sum_{g=0}^{\infty} g_{\mathrm{s}}^{2g-2+n} R_{g,n}^{(b)}(E_1, \ldots, E_n)$$

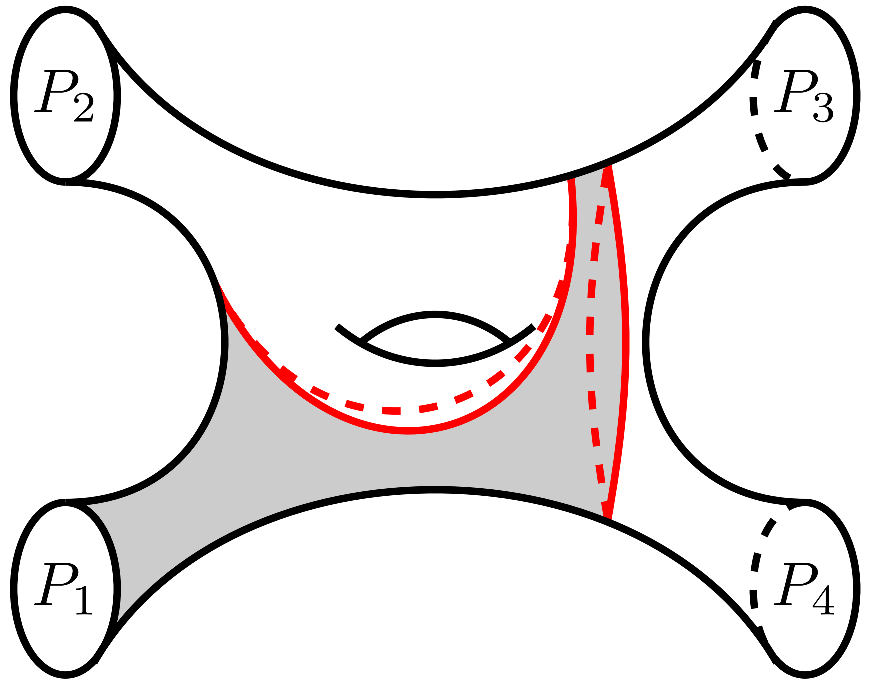
- They are related to the partition functions by Laplace transform

$$R_{g,n}^{(b)}(-z_1^2, \ldots, -z_n^2) = \int_0^\infty \left(\prod_j d\beta_j \, e^{-\beta_j z_j^2} \right) Z_{g,n}^{(b)}(\beta_1, \ldots, \beta_n)$$

  - which are in turn computed from the quantum volumes by gluing trumpets

Topological recursion and a deformed Mirzakhani recursion relation

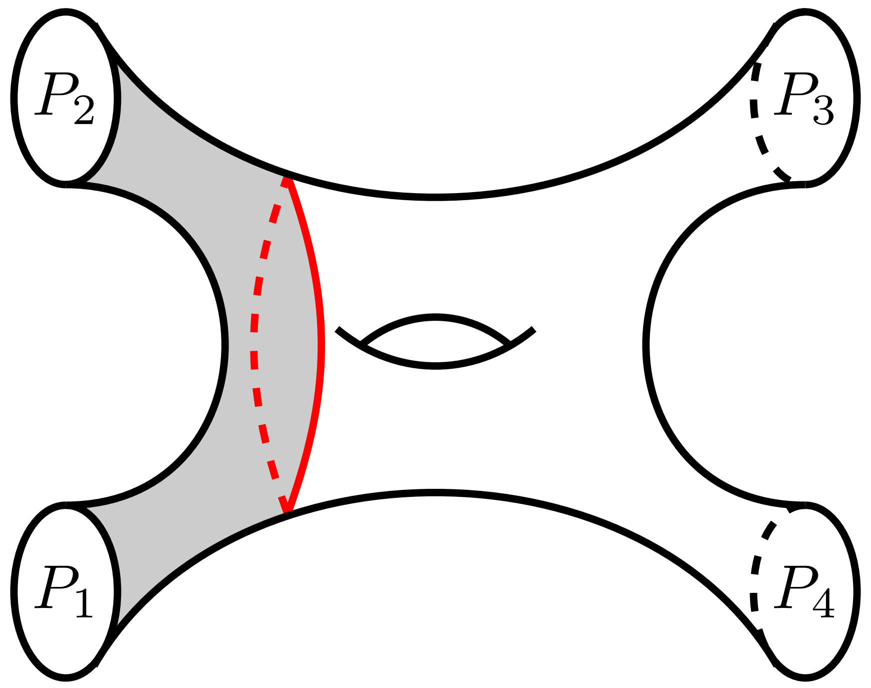




$$P_1 \mathsf{V}^{(b)}_{g,n}(P_1, \mathbf{P}) = \int_0^\infty (2P\,\mathrm{d}P)\,(2P'\,\mathrm{d}P')\,H(P+P', P_1)\left(\mathsf{V}^{(b)}_{g-1,n+1}(P, P', \mathbf{P}) + \sum_{h=0}^{g} \sum_{I \sqcup J = \{2,\ldots,n\}} \mathsf{V}^{(b)}_{h,|I|+1}(P, \mathbf{P}_I)\mathsf{V}^{(b)}_{g-h,|J|+1}(P', \mathbf{P}_J)\right)$$

$$+ \sum_{i=2}^{n} \int_0^\infty (2P\,\mathrm{d}P)\,\big(H(P, P_1 + P_i) + H(P, P_1 - P_i)\big)\mathsf{V}^{(b)}_{g,n-1}(P, \mathbf{P}\backslash P_i)$$

with $H(x,y) = \dfrac{y}{2} - \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} dt\, \frac{\sin(4\pi t x)\sin(4\pi t y)}{\sinh(2\pi b t)\sinh(2\pi b^{-1} t)}$



[Mirzakhani '06, Eynard Orantin '07]

$|\text{Liouville}|^2$ string theory

The worldsheet CFT

Liouville CFT $\oplus$ (Liouville CFT)* $\oplus$ b c ghosts

$c_+ = 13 + i\mathbb{R}$ $\quad$ $c_- = 13 - i\mathbb{R}$ $\quad$ $c_{\text{gh}} = -26$

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- The worldsheet theory is defined by the non-perturbative **CFT data**:

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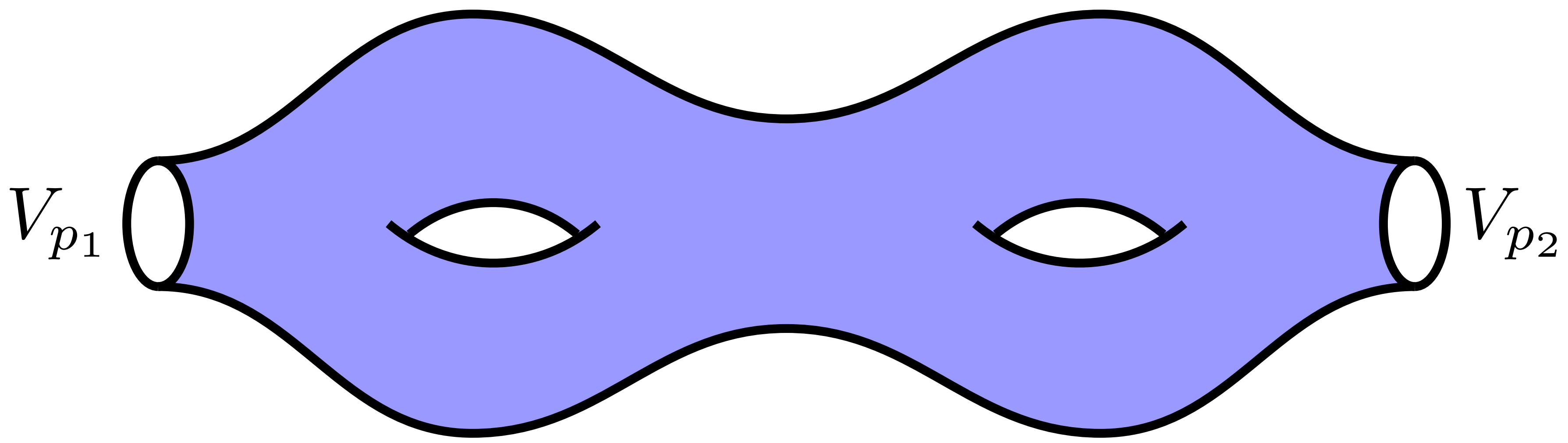
  - <u>Central charge</u>:
    $$c = 1 + 6(b + b^{-1})^2$$
    $$b \in e^{\frac{\pi i}{4}}\mathbb{R}$$
  - <u>Spectrum</u>: continuum $\{V_p\}$
    $$h_p = \bar{h}_p = \frac{c-1}{24} - p^2$$
  - <u>OPE data</u>:
    $$\langle V_{p_1} V_{p_2} V_{p_3} \rangle_{g=0}^{(b)} = C_b(p_1, p_2, p_3)$$
    "DOZZ formula"



[Dorn Otto 95; Zamolodchikov[2] 95; Teschner 01]

String amplitudes

- Reality of the worldsheet imposes

$$b \in e^{\frac{\pi i}{4}}\mathbb{R} \;\leftrightarrow\; \boxed{c \in 13 + i\mathbb{R}}$$

$$p_j \in e^{-\frac{\pi i}{4}}\mathbb{R} \;\leftrightarrow\; \boxed{h_j \in \frac{1}{2} + i\mathbb{R}}$$

- On-shell vertex operators:

$$\mathscr{V}_p = \mathrm{c}\,\bar{\mathrm{c}}\,V_p^{(b)}\,V_{ip}^{(-ib)}$$

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$$\boxed{\mathsf{A}_{g,n}^{(b)}(p_1, \ldots, p_n) = \left(\prod_{j=1}^{n} \mathcal{N}_b(p_j)\right) \int_{\mathscr{M}(\Sigma_{g,n})} \left| \left\langle \prod_{j=1}^{n} V_{p_j} \right\rangle_g^{(b)} \right|^2 \times \left(\mathsf{b}\,\mathsf{c}\text{ ghosts}\right)}$$

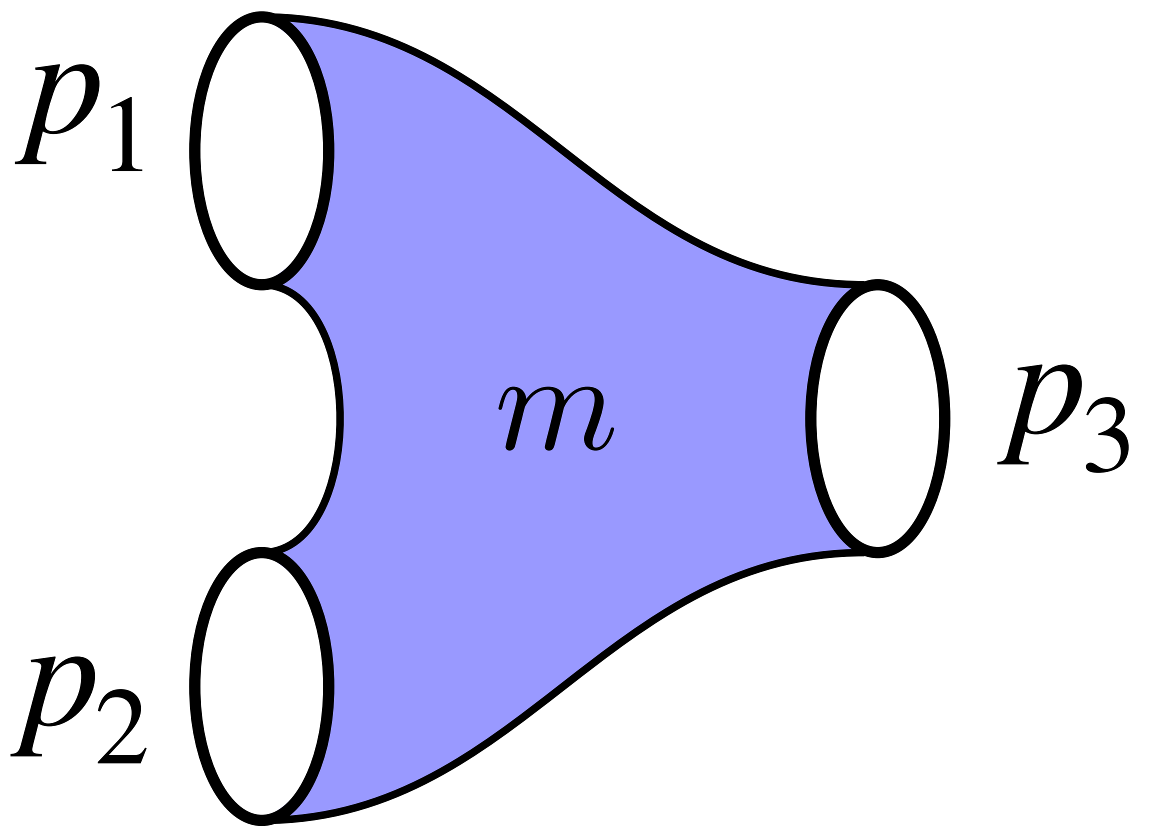
- **Absolutely convergent** integral over moduli space
- Invariant under **swap symmetry** $b \to -ib,\, p_j \to ip_j$

Sphere three-point amplitude

- The simplest observable in the theory is the sphere three-point amplitude

$$\mathsf{A}_{0,3}^{(b)}(p_1,p_2,p_3) = \left(\prod_{j=1}^{3} \mathcal{N}_b(p_j)\right) C_b(p_1,p_2,p_3) C_{-ib}(ip_1,ip_2,ip_3)$$

$$= \sum_{m=1}^{\infty} \frac{2b(-1)^m \sin(2\pi m b p_1)\sin(2\pi m b p_2)\sin(2\pi m b p_3)}{\sin(\pi m b^2)}$$



[Zamolodchikov 05]

General string amplitudes

$$\mathsf{A}^{(b)}_{g,n}(p_1,\ldots,p_n) = \left(\prod_{j=1}^{n} \mathcal{N}_b(p_j)\right) \int_{\mathcal{M}(\Sigma_{g,n})} \left| \left\langle \prod_{j=1}^{n} V_{p_j} \right\rangle_g^{(b)} \right|^2 \times (\mathsf{b}\,\mathsf{c}\text{ ghosts})$$

Complicated!

- Harness **analytic structure** & **swap symmetry** to bootstrap amplitude
  - **Poles** associated with resonances of Liouville CFT correlators
  - **Discontinuities** when moduli integral ceases to converge
  - More constraints from symmetry considerations, higher equations of motion, $\mathrm{SO}(8)$ triality symmetry for $(g,n)=(0,4)$, …

[Belavin, Zamolodchikov '06]

String amplitudes from Feynman diagrams

- Bootstrap systematically implemented with **simple diagrammatic rules**
  - Diagrams correspond to different **degenerations of the worldsheet** = stable graphs
  - **VMS quantum volume** $\mathsf{V}^{(b)}_{g,n}$ arises as a **string vertex**

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$$\mathsf{A}_{0,4}^{(b)}(p_1, p_2, p_3, p_4) =$$

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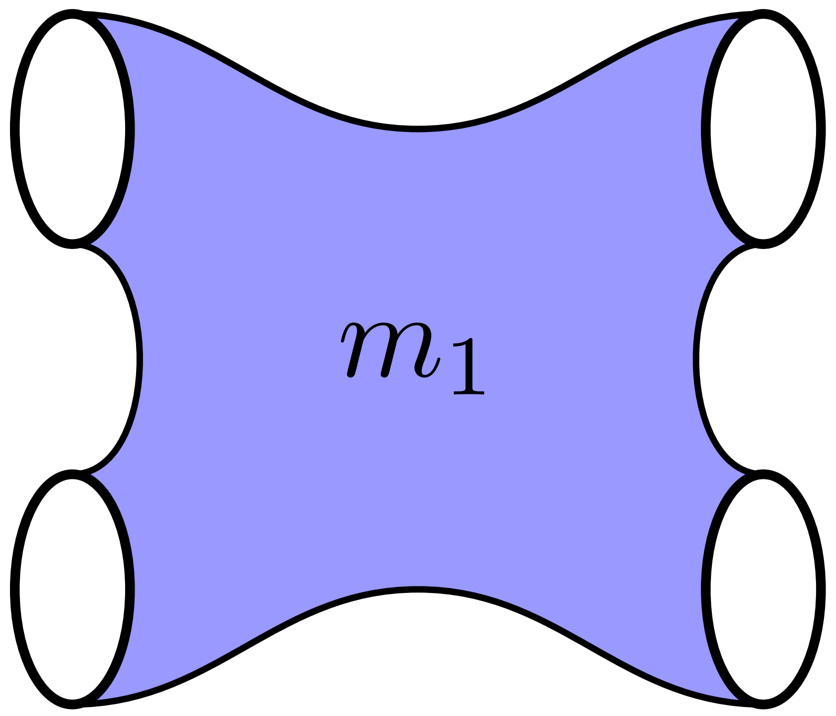
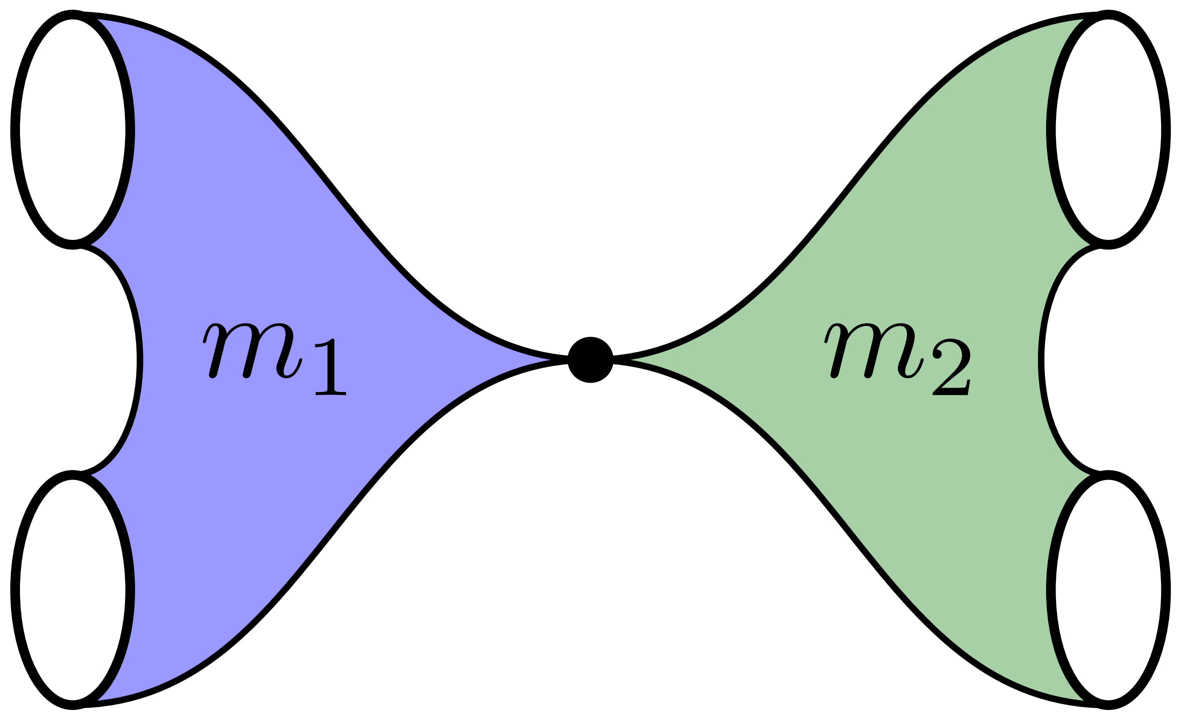


$$+ \sum_{m_1,m_2=1}^{\infty}$$



+ 2 permutations

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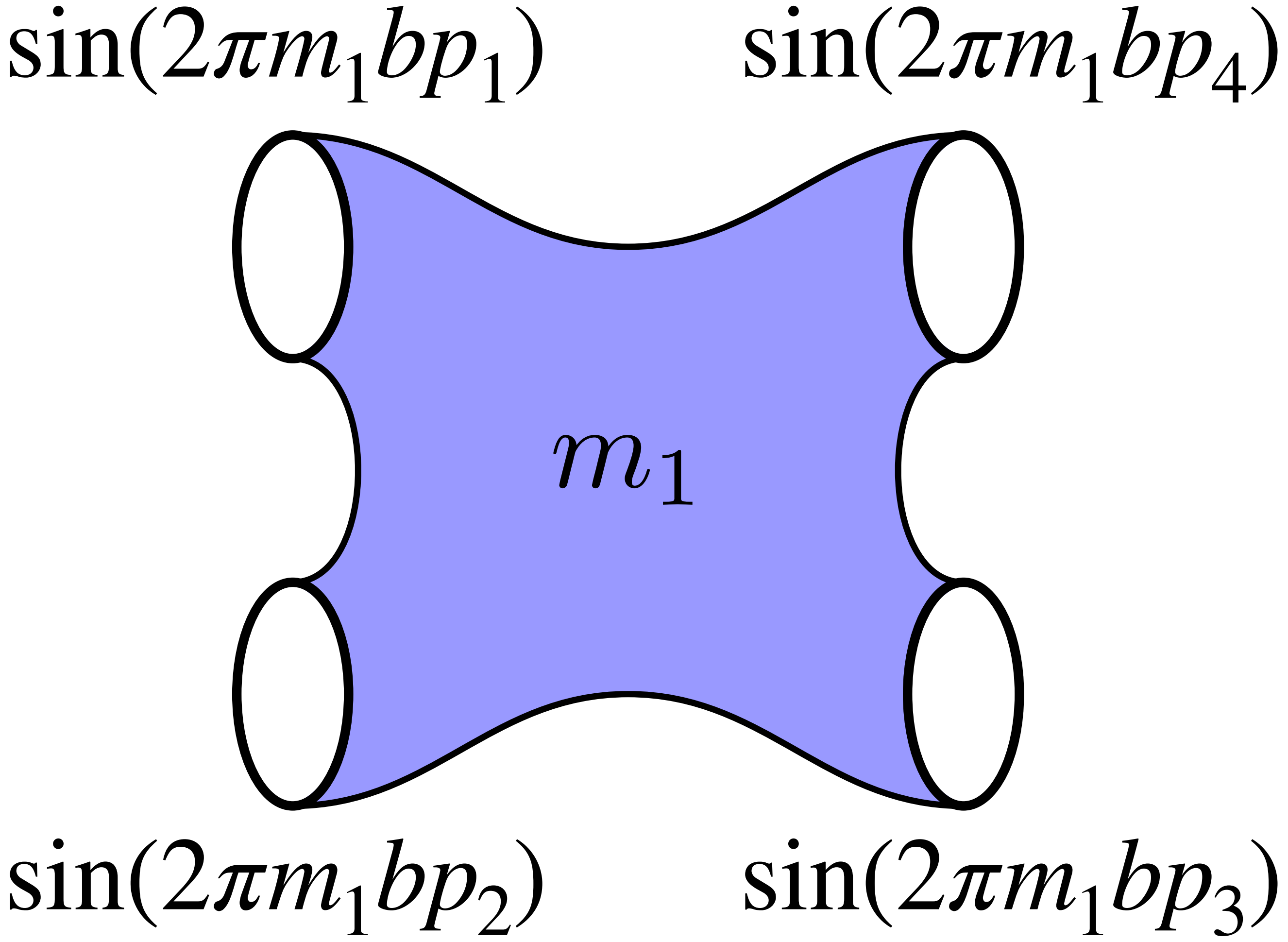
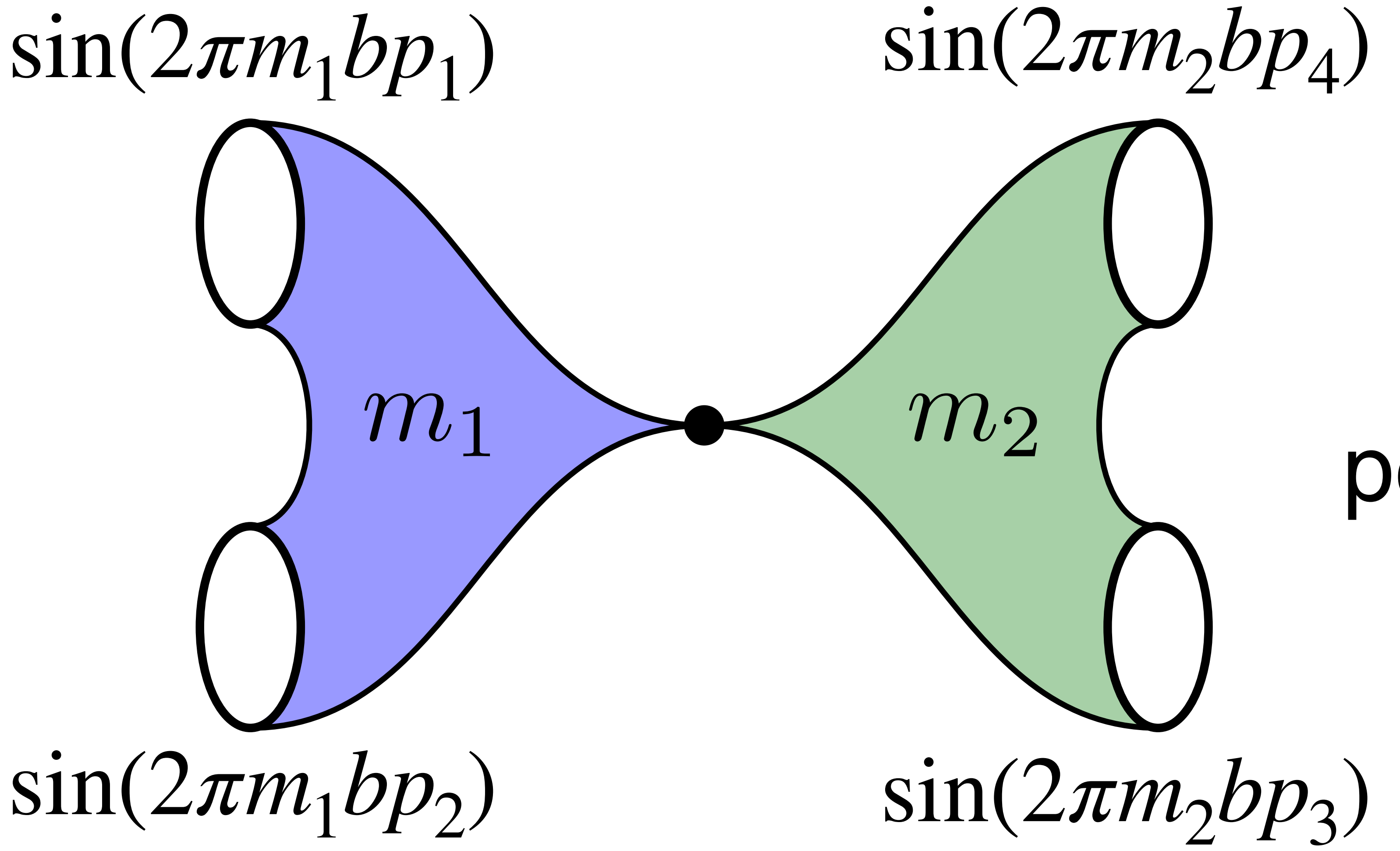
$$\mathsf{A}^{(b)}_{0,4}(p_1,p_2,p_3,p_4) =$$

$$\sum_{m_1=1}^{\infty}$$



$$+ \sum_{m_1,m_2=1}^{\infty}$$



+ 2 permutations

String amplitudes from Feynman diagrams

- Bootstrap systematically implemented with **simple diagrammatic rules**
  - Diagrams correspond to different **degenerations of the worldsheet** = stable graphs
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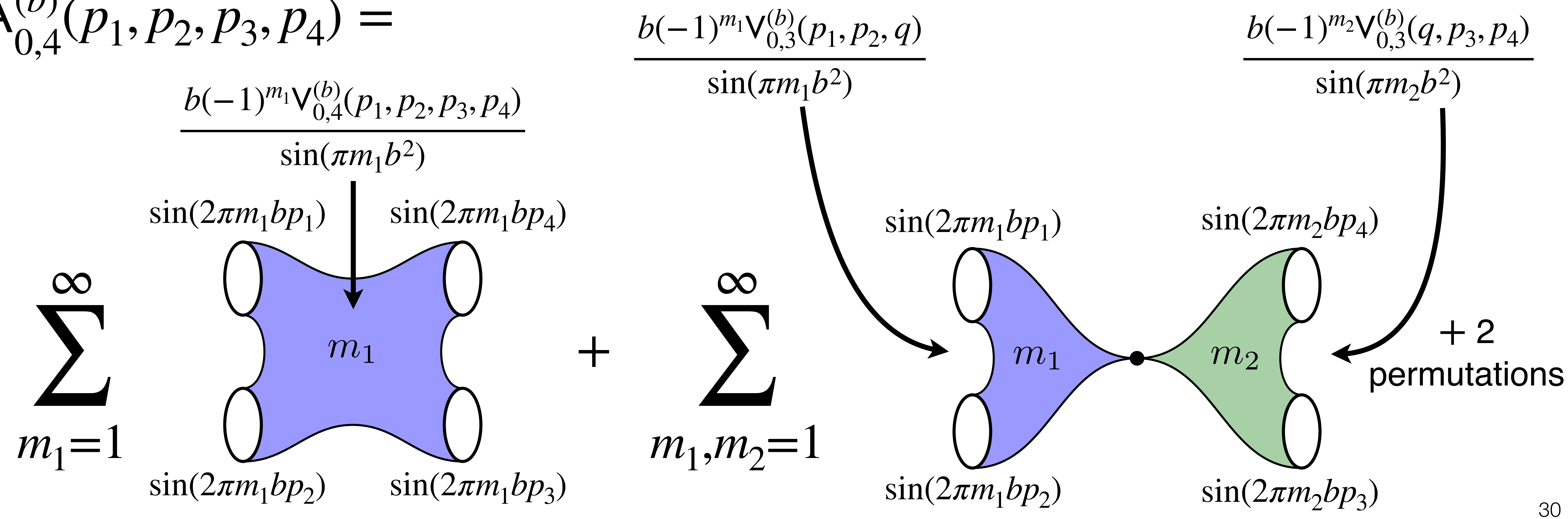

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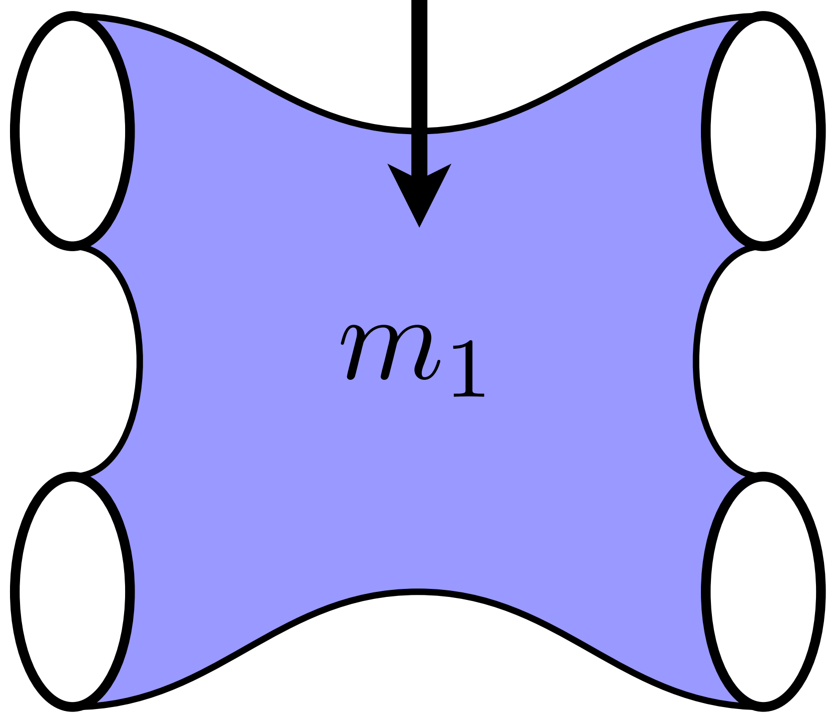
$$\sum_{m_1=1}^{\infty}$$

$$\frac{b(-1)^{m_1}\mathsf{V}^{(b)}_{0,4}(p_1,p_2,p_3,p_4)}{\sin(\pi m_1 b^2)}$$

$\sin(2\pi m_1 b p_1)$ $\sin(2\pi m_1 b p_4)$

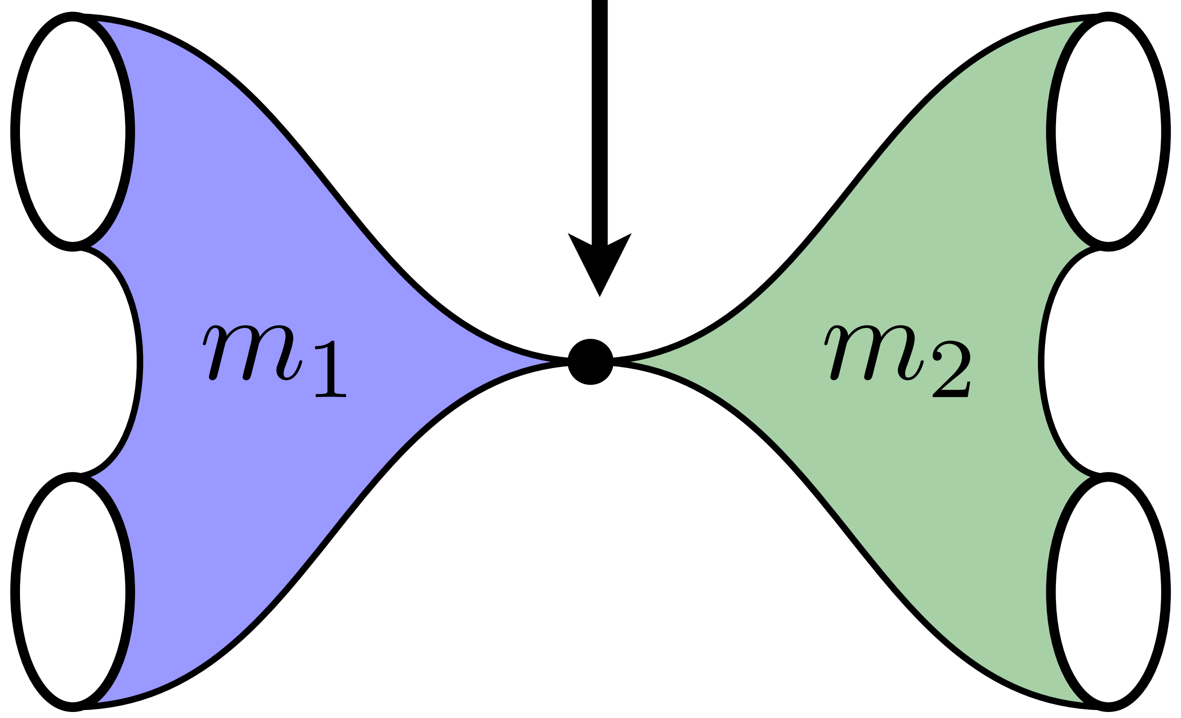


$\sin(2\pi m_1 b p_2)$ $\sin(2\pi m_1 b p_3)$

$$+ \sum_{m_1,m_2=1}^{\infty}$$

$$\frac{b(-1)^{m_1}\mathsf{V}^{(b)}_{0,3}(p_1,p_2,q)}{\sin(\pi m_1 b^2)} \quad \int 2q dq\ \sin(2\pi m_1 b q)\sin(2\pi m_2 b q) \quad \frac{b(-1)^{m_2}\mathsf{V}^{(b)}_{0,3}(q,p_3,p_4)}{\sin(\pi m_2 b^2)}$$

$\sin(2\pi m_1 b p_1)$ $\sin(2\pi m_2 b p_4)$



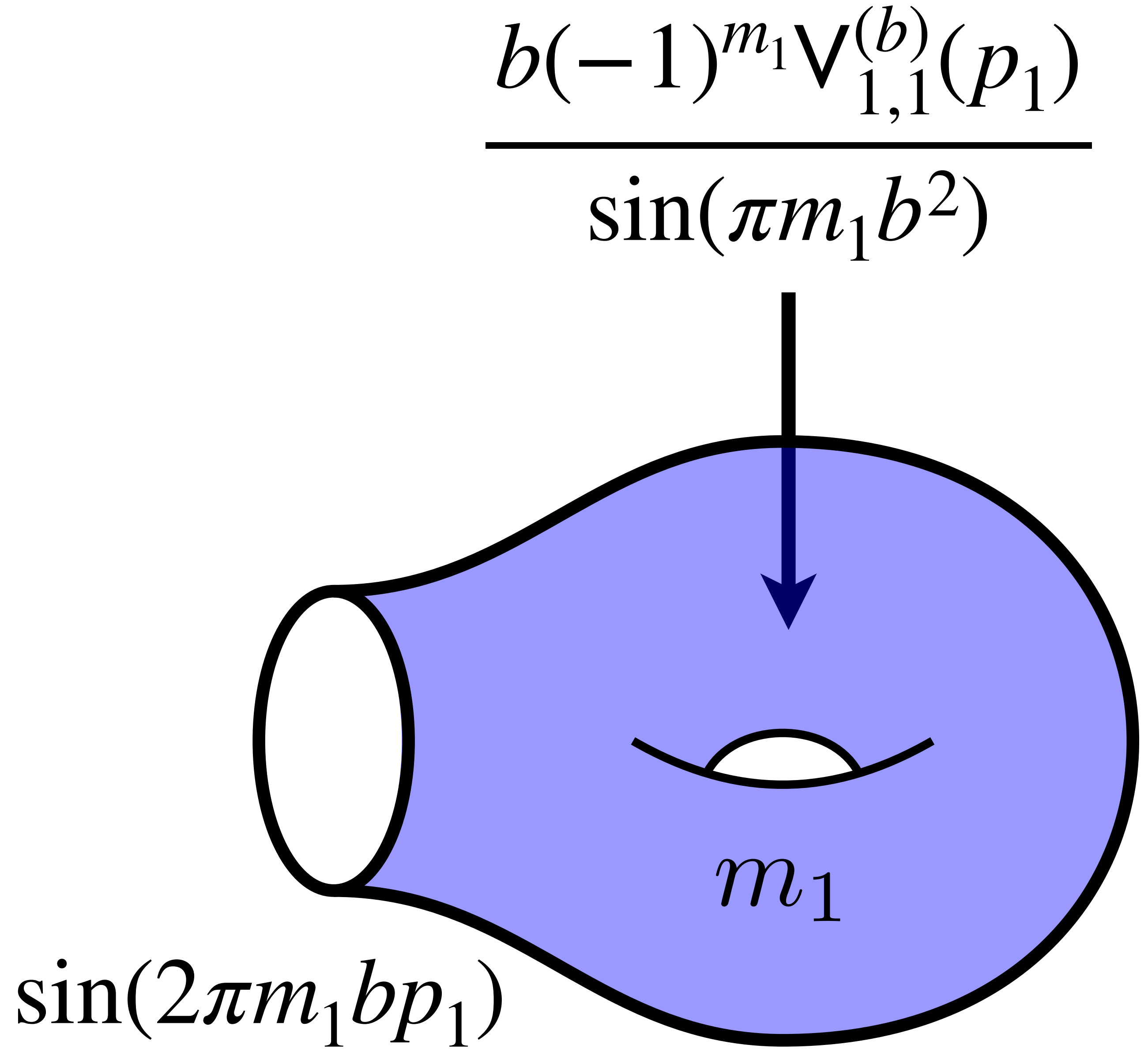
$\sin(2\pi m_1 b p_2)$ $\sin(2\pi m_2 b p_3)$

+ 2 permutations

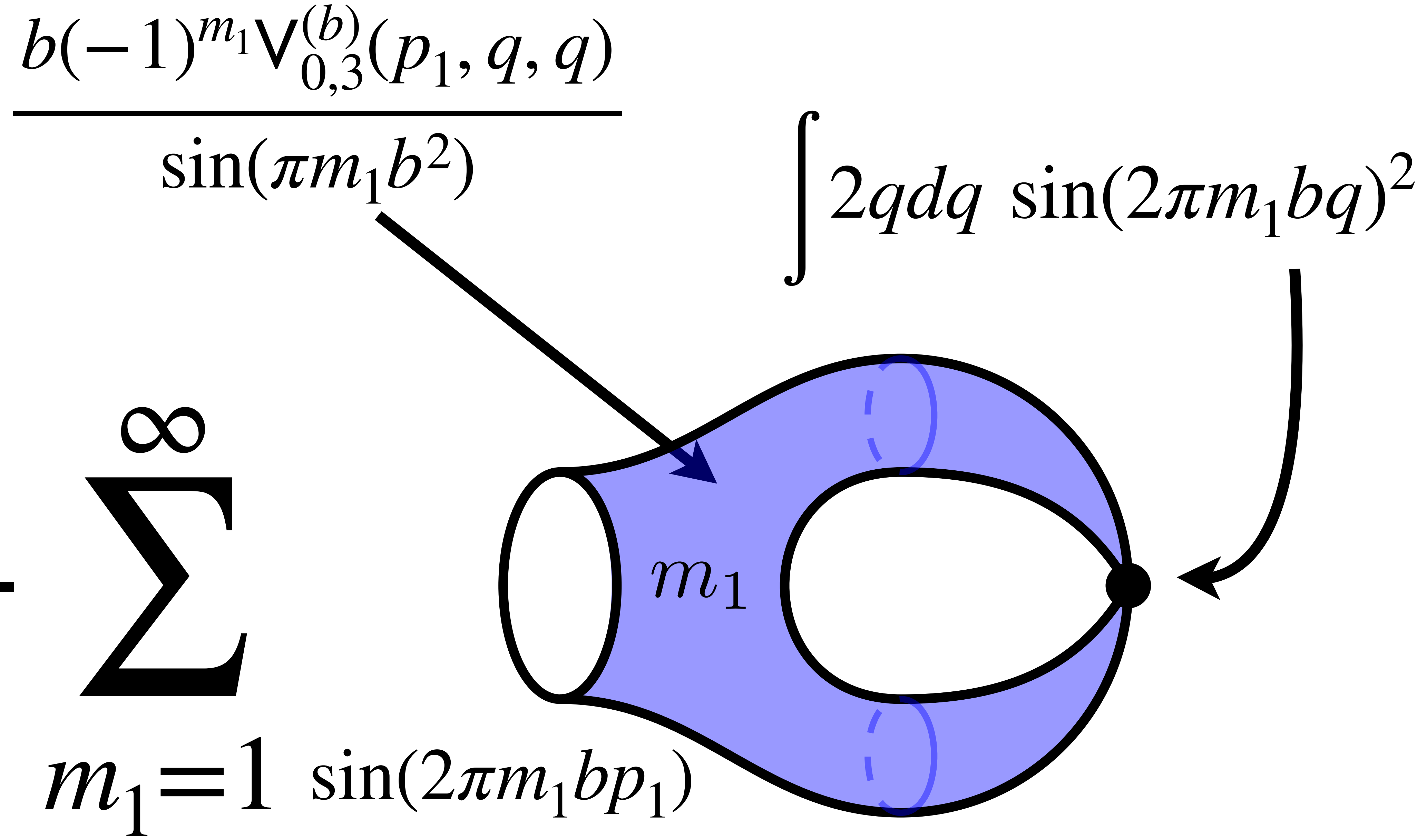
String amplitudes from Feynman diagrams

- Bootstrap systematically implemented with **simple diagrammatic rules**
  - Diagrams correspond to different **degenerations of the worldsheet** = stable graphs
  - **VMS quantum volume** $\mathsf{V}^{(b)}_{g,n}$ arises as a **string vertex**

$$\mathsf{A}^{(b)}_{1,1}(p_1) = \sum_{m_1=1}^{\infty}$$

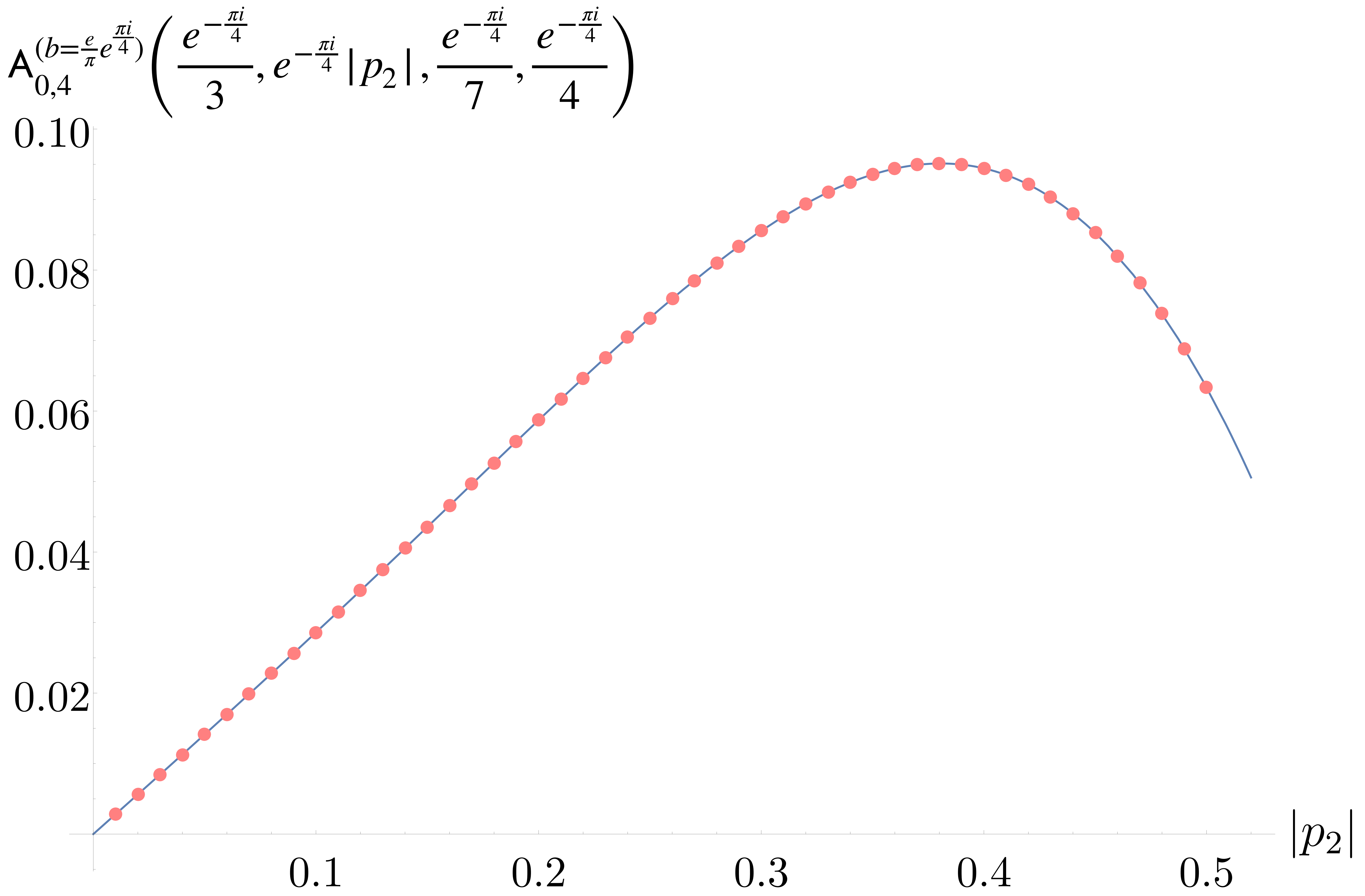


$$+ \frac{1}{2}\sum_{m_1=1}^{\infty}$$



Sphere four-point amplitude: numerical verification

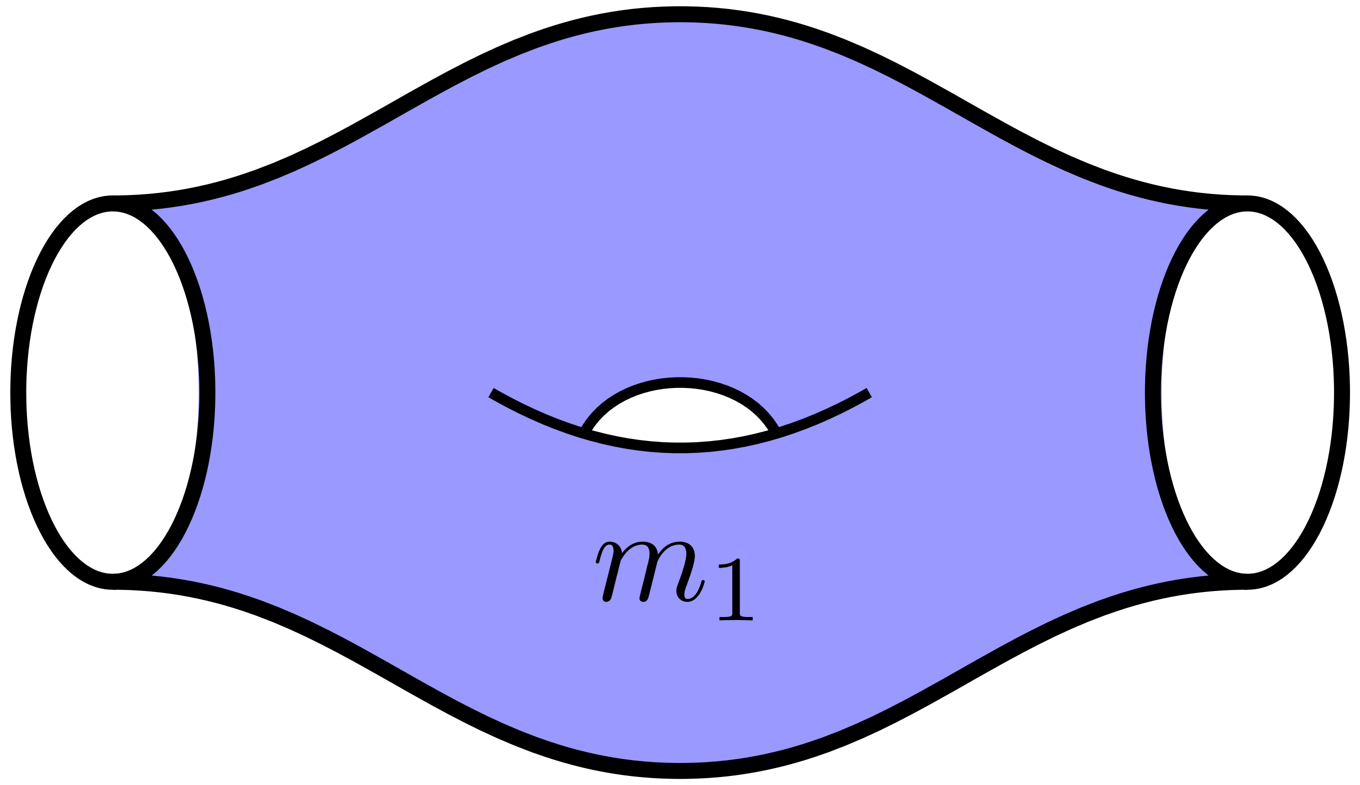
- Direct numerical integration over moduli space



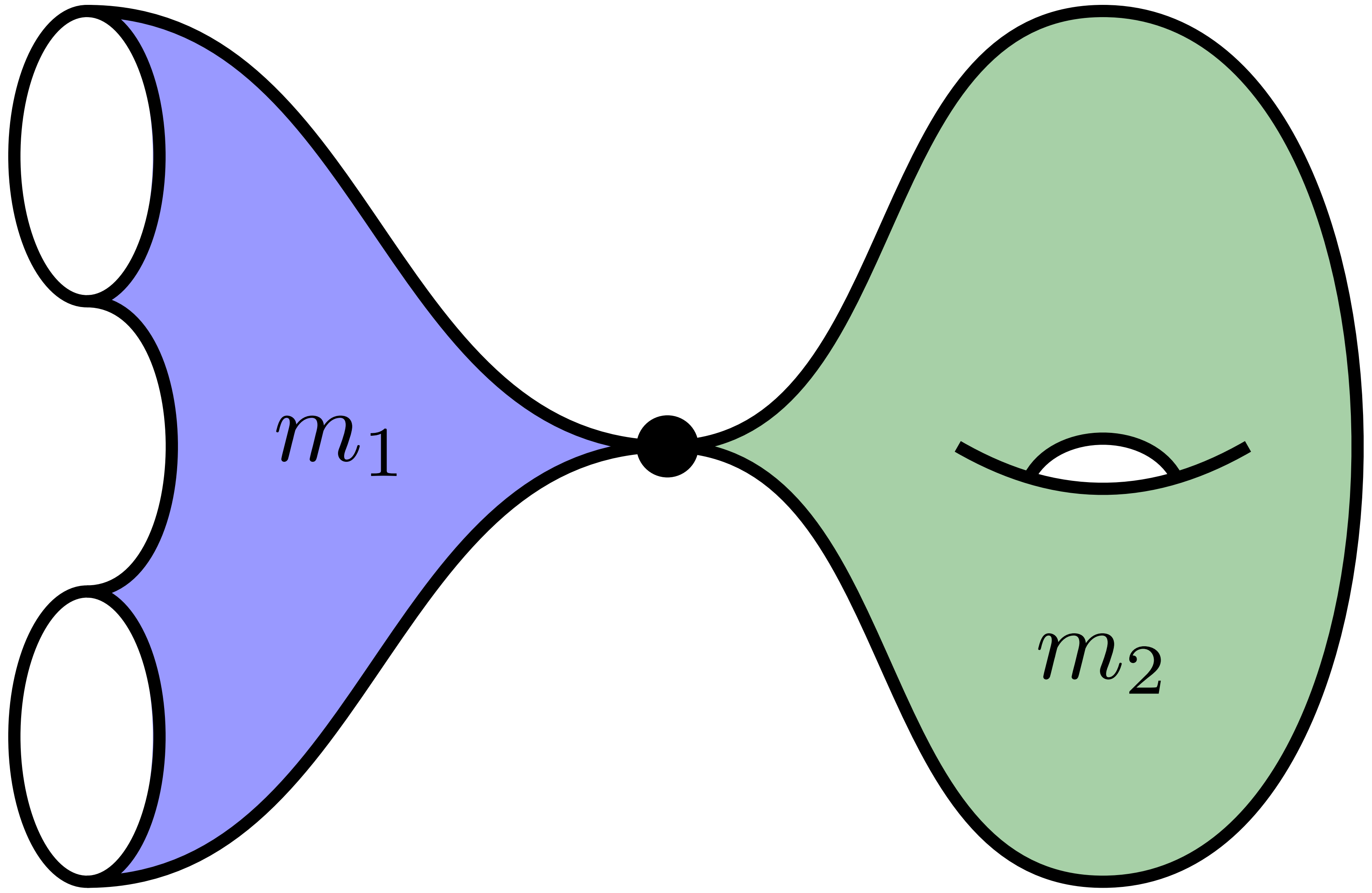
Error $< 10^{-5}\,\%$

Example: torus two-point amplitude

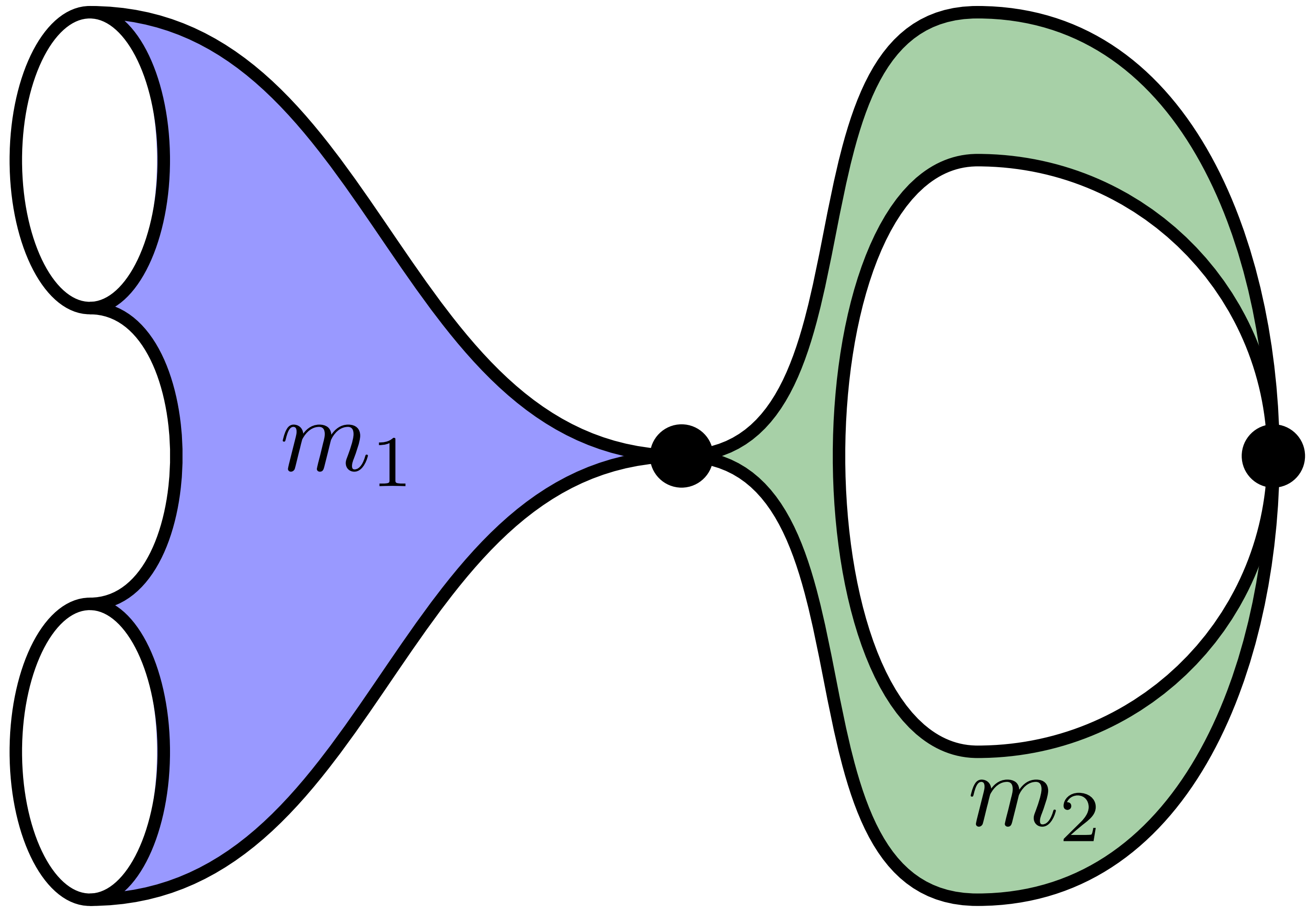
$$\mathsf{A}^{(b)}_{1,2}(p_1, p_2) =$$



$$+$$



$$+\frac{1}{2}$$



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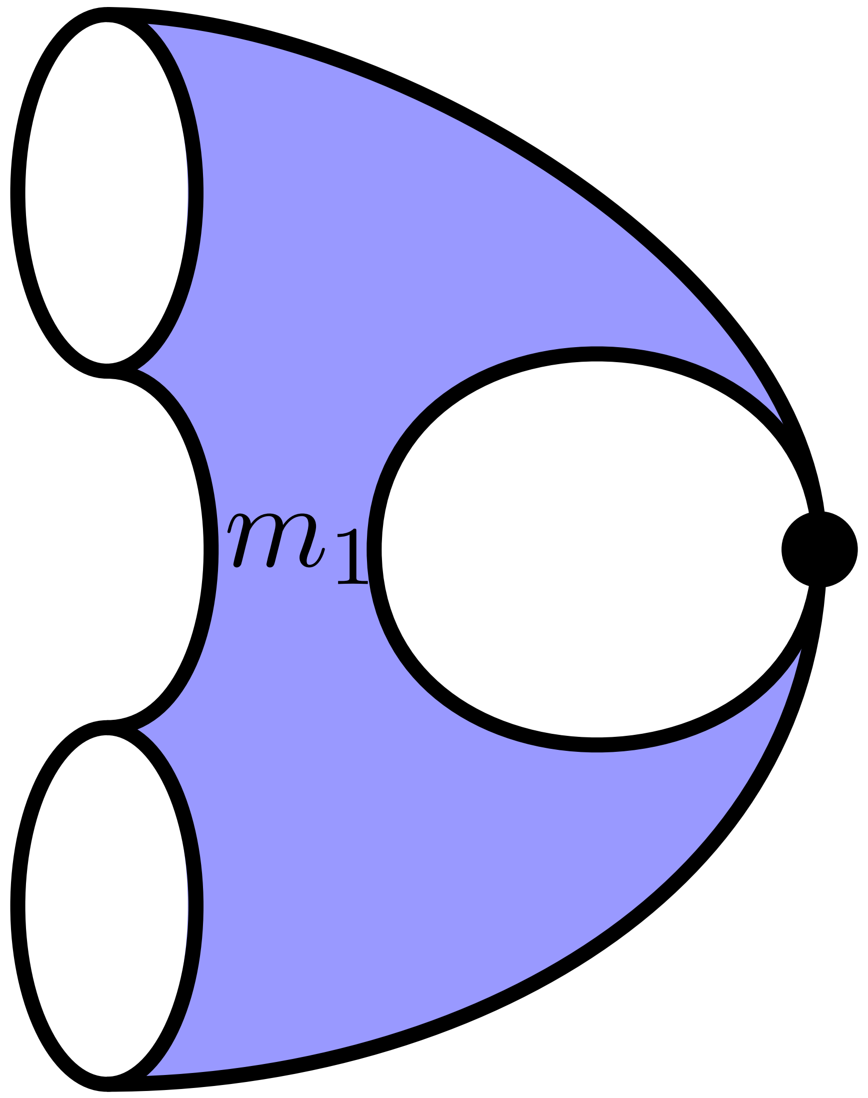

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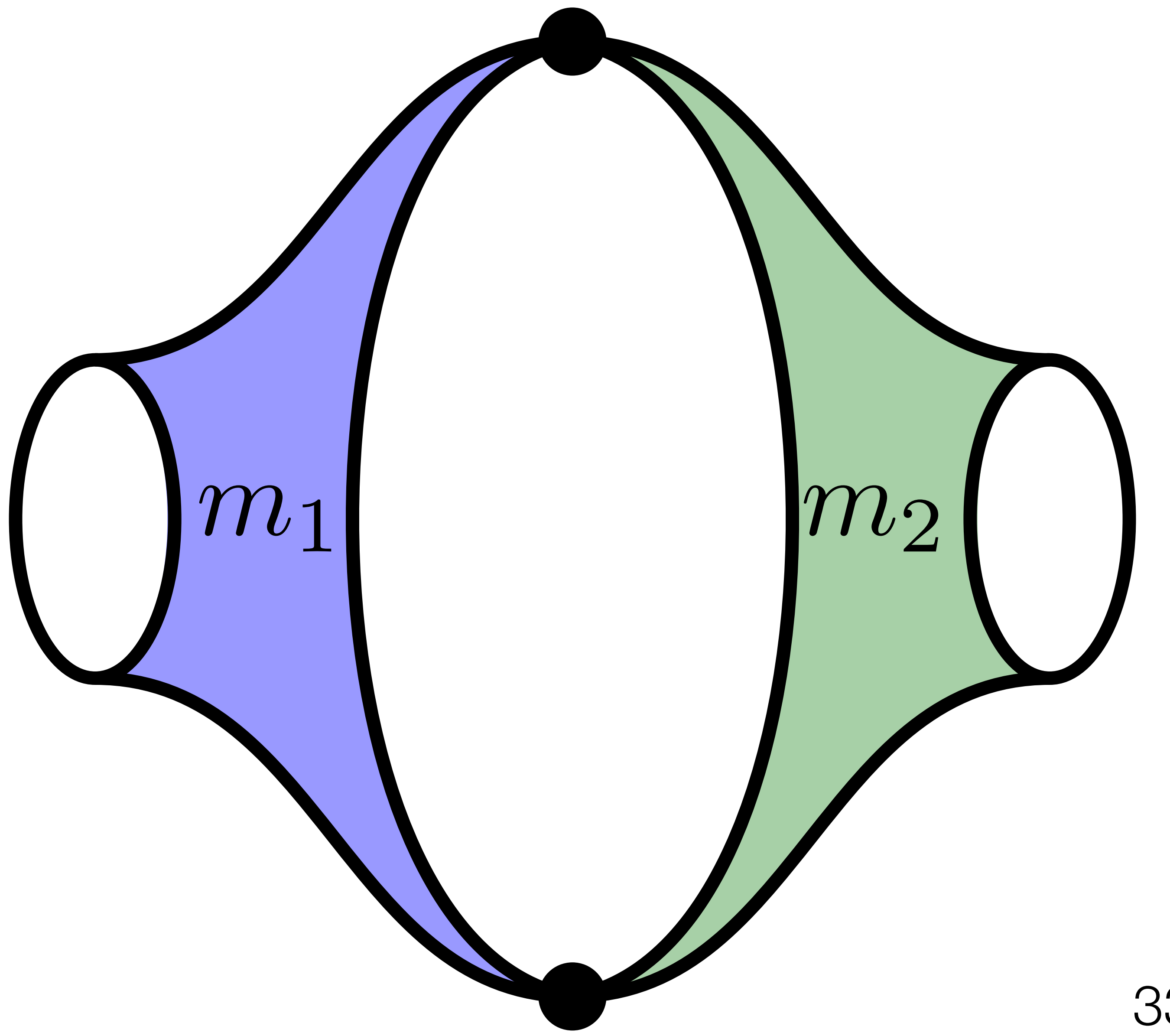


The dual matrix integral

- Claim: $|\text{Liouville}|^2$ string theory is precisely dual to a **double-scaled two-matrix integral**

$$\int_{\mathbb{R}^{2N^2}} [dM_1][dM_2]\, \mathrm{e}^{-N\,\mathrm{Tr}\left(V_1(M_1)+V_2(M_2)-M_1M_2\right)}$$

  - Characterized by the spectral curve:

$$x(z) = -\,2\cos(\pi b^{-1}\sqrt{z}), \quad y(z) = 2\cos(\pi b\sqrt{z})$$

Remarkably similar to the ordinary (p,q) minimal string!
cf. e.g. [Seiberg Shih 04]

  - Leading density of eigenvalues in M_1: $\rho_0(E) = \frac{2}{\pi}\sinh(-\pi i b^2)\, \sin\left(-ib^2\,\mathrm{arccosh}\left(\frac{E}{2}\right)\right)$

The spectral curve

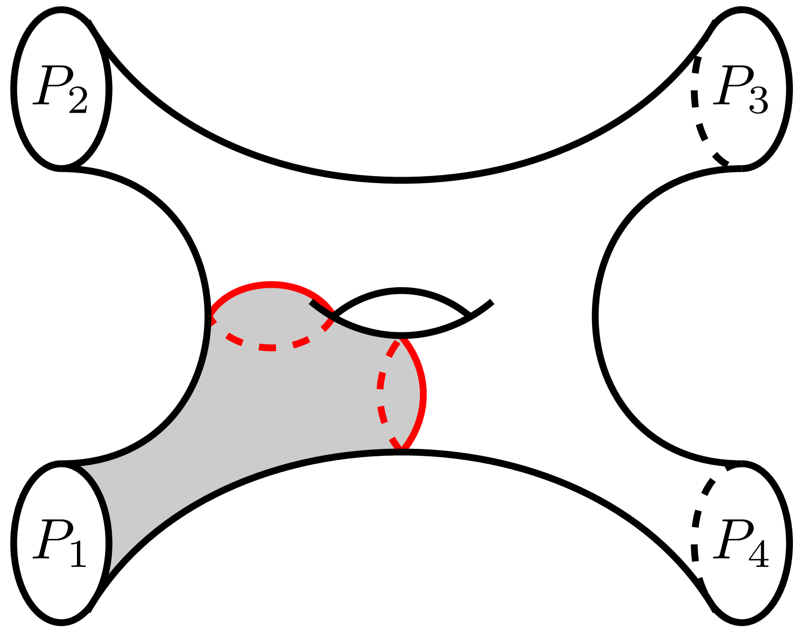
- Rational parametrization, $z \in \mathbb{C}$

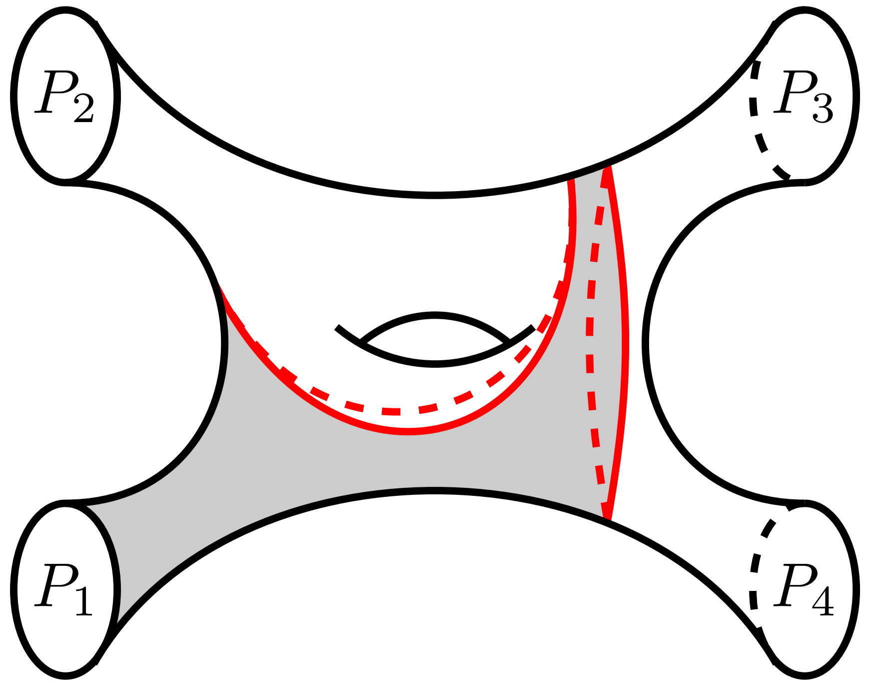
$$x(z) = -2\cos(\pi b^{-1}\sqrt{z}), \quad y(z) = 2\cos(\pi b\sqrt{z})$$

- Has infinite genus, infinitely many nodal singularities and infinitely branch points $\mathrm{d}x(z_*) = 0$
- Topological recursion involves a sum over all branch points. This sum converges absolutely and all $\omega_{g,n}$'s are still well-defined.

Topological recursion for string amplitudes

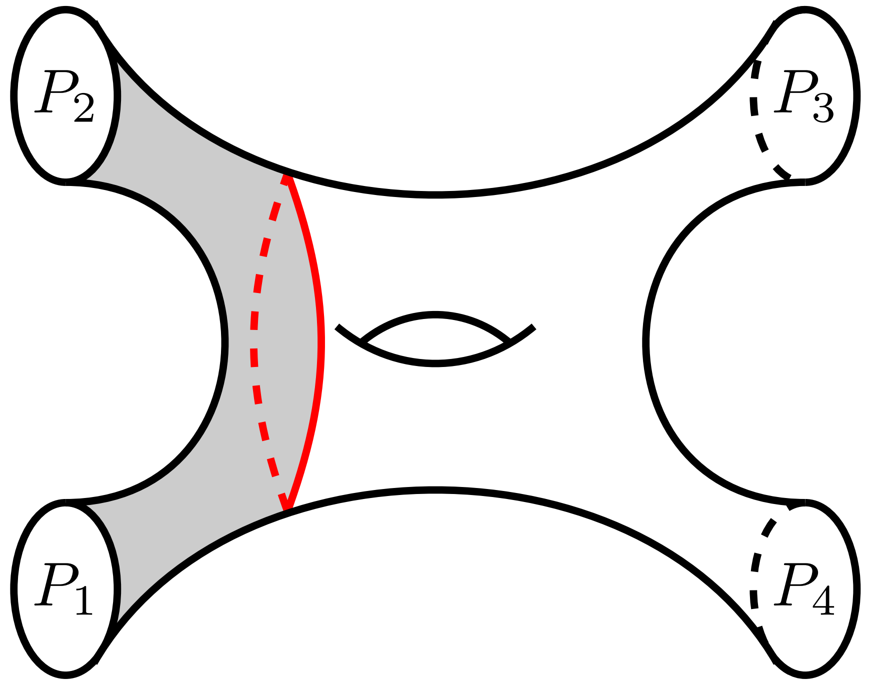
- The loop equations for the matrix integral [Chekhov Eynard Orantin 06] translate into a **recursion relation** for the string amplitudes (cf. [Mirzakhani 06; Eynard Orantin 07])





$$p_1 \mathsf{A}^{(b)}_{g,n}(p_1,\mathbf{p}) = \int 2q dq\, 2q' dq'\, H_b(q+q',p_1) \mathsf{A}^{(b)}_{0,3}(p_1,q,q') \left(\mathsf{A}^{(b)}_{g-1,n+1}(q,q',\mathbf{p}) + \sum_{h=0}^{g} \sum_{I \sqcup J = \{2,\dots,n\}} \mathsf{A}^{(b)}_{h,1+|I|}(q,\mathbf{p}_I) \mathsf{A}^{(b)}_{g-h,1+|J|}(q',\mathbf{p}_J) \right)$$

$$-\sum_{j=2}^{n} \int 2q dq \left(H_b(q,p_1+p_j) + H_b(q,p_1-p_j) \right) \mathsf{A}^{(b)}_{0,3}(p_1,p_j,q) \mathsf{A}^{(b)}_{g,n-1}(q,\mathbf{p} \backslash p_j)$$



$$H_b(x,y) = \frac{y}{2} - \frac{1}{2} \int_\Gamma du \frac{\sin(4\pi u x)\sin(4\pi u y)}{\sin(2\pi b u)\sin(2\pi b^{-1} u)}$$

← Essentially identical to recursion kernel for quantum volumes $\mathsf{V}^{(b)}_{g,n}$ in VMS!

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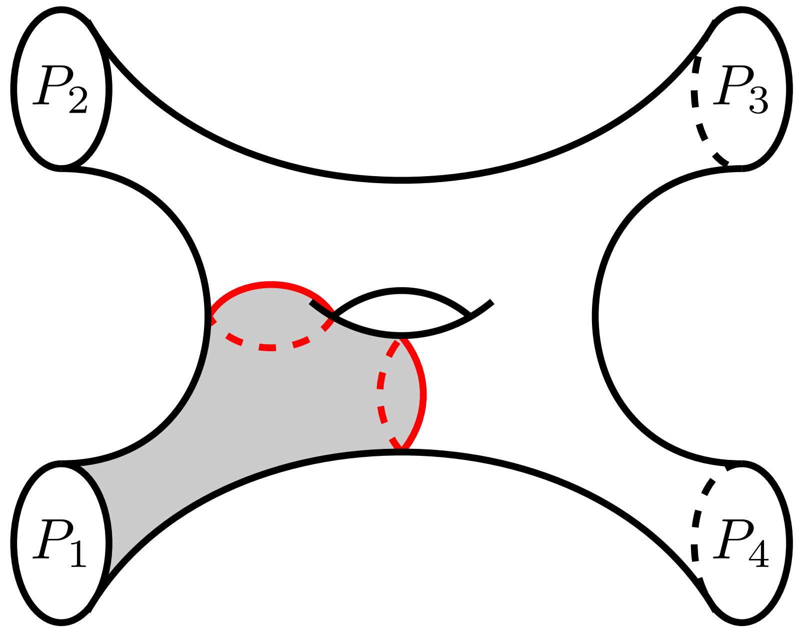
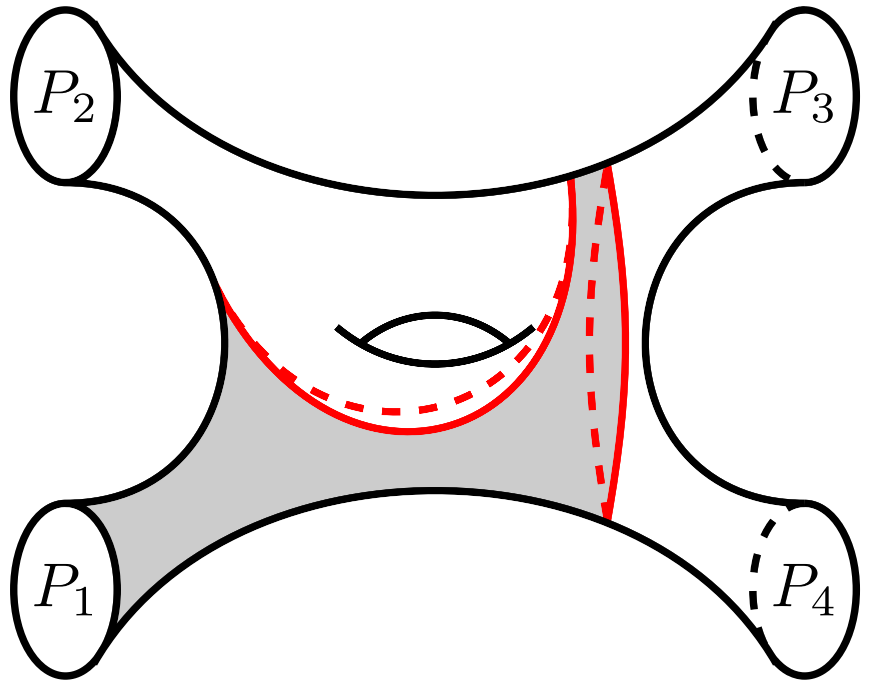
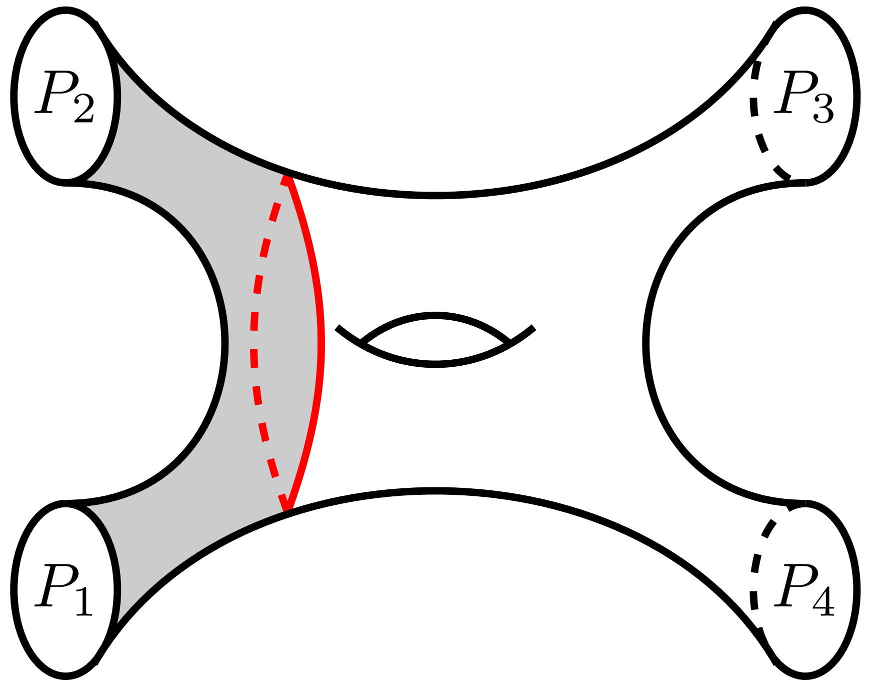




$$p_1 \mathsf{A}^{(b)}_{g,n}(p_1,\mathbf{p}) = \int 2q dq\, 2q' dq'\, H_b(q+q',p_1) \boxed{\mathsf{A}^{(b)}_{0,3}(p_1,q,q')} \left(\mathsf{A}^{(b)}_{g-1,n+1}(q,q',\mathbf{p}) + \sum_{h=0}^{g} \sum_{I \sqcup J = \{2,\dots,n\}} \mathsf{A}^{(b)}_{h,1+|I|}(q,\mathbf{p}_I)\mathsf{A}^{(b)}_{g-h,1+|J|}(q',\mathbf{p}_J) \right)$$

$$- \sum_{j=2}^{n} \int 2q dq \left(H_b(q,p_1+p_j) + H_b(q,p_1-p_j) \right) \mathsf{A}^{(b)}_{0,3}(p_1,p_j,q)\mathsf{A}^{(b)}_{g,n-1}(q,\mathbf{p}\backslash p_j)$$



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Some further comments

Semiclassical gravity

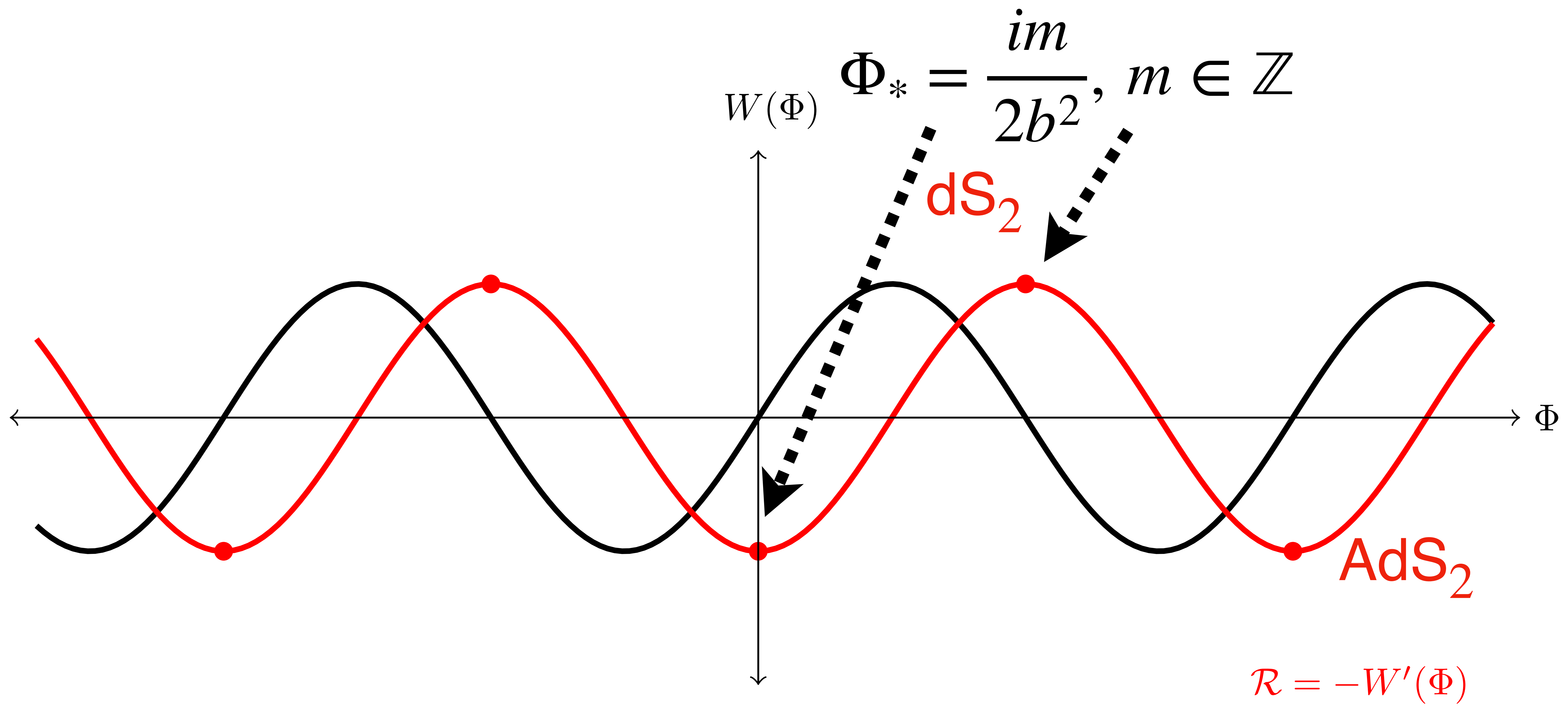
- The worldsheet theory is a theory of two scalar fields.
- There is a field redefinition mapping the worldsheet theory to dilaton gravity

[Seiberg Stanford ’19, Mertens Turiaci ’20]

$$S(\Sigma;\Phi,g) = -\frac{1}{2}\int_\Sigma d^2x\sqrt{g}\,\left(\Phi R + W(\Phi)\right)\ ,\quad W(\Phi) = \frac{\sinh(2\pi b^2\Phi)}{\sin(\pi b^2)}$$

- Sinh potential for VMS
- Sine potential for $|\text{Liouville}|^2$: has infinitely many vacua, both AdS and dS

$\Longrightarrow$ Rigorous model of 2d dS quantum gravity!



Asymptotic boundaries

- The string amplitudes $\mathsf{V}^{(b)}_{g,n}$ and $\mathsf{A}^{(b)}_{g,n}$ compute the gravitational path integral with boundary condition

$$g\,|_{\text{boundary}} = \text{fixed}\ , \quad K = \text{fixed}$$

- One can also consider asymptotic near (A)dS boundary conditions described by

$$g\,|_{\text{boundary}} = \frac{1}{\varepsilon}\,\text{fixed}\ , \quad \Phi = \frac{1}{\varepsilon}\text{fixed}$$

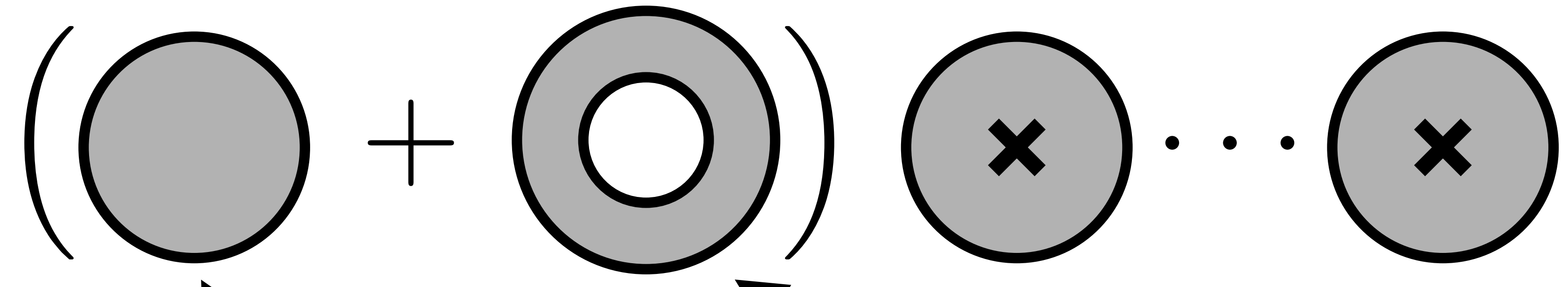
- They can be described on the worldsheet by an FZZT x ZZ boundary condition
- FZZT parameter is related to the boundary temperature by an integral transform

Non-perturbative effects

- The perturbative series (same for $\mathsf{V}^{(b)}_{g,n} \longrightarrow \mathsf{A}^{(b)}_{g,n}$)

$$\mathsf{V}^{(b)}_n(g_\mathrm{s}; P_1, \ldots, P_n) = \sum_{g=0}^{\infty} g_\mathrm{s}^{2g-2+n} \mathsf{V}_{g,n}(P_1, \ldots, P_n)$$ is **asymptotic**

- Leading non-perturbative contribution

$\pm \exp$ 

ZZ-instanton like boundary conditions

Sign ambiguous, depends on choice of contour in SFT field space

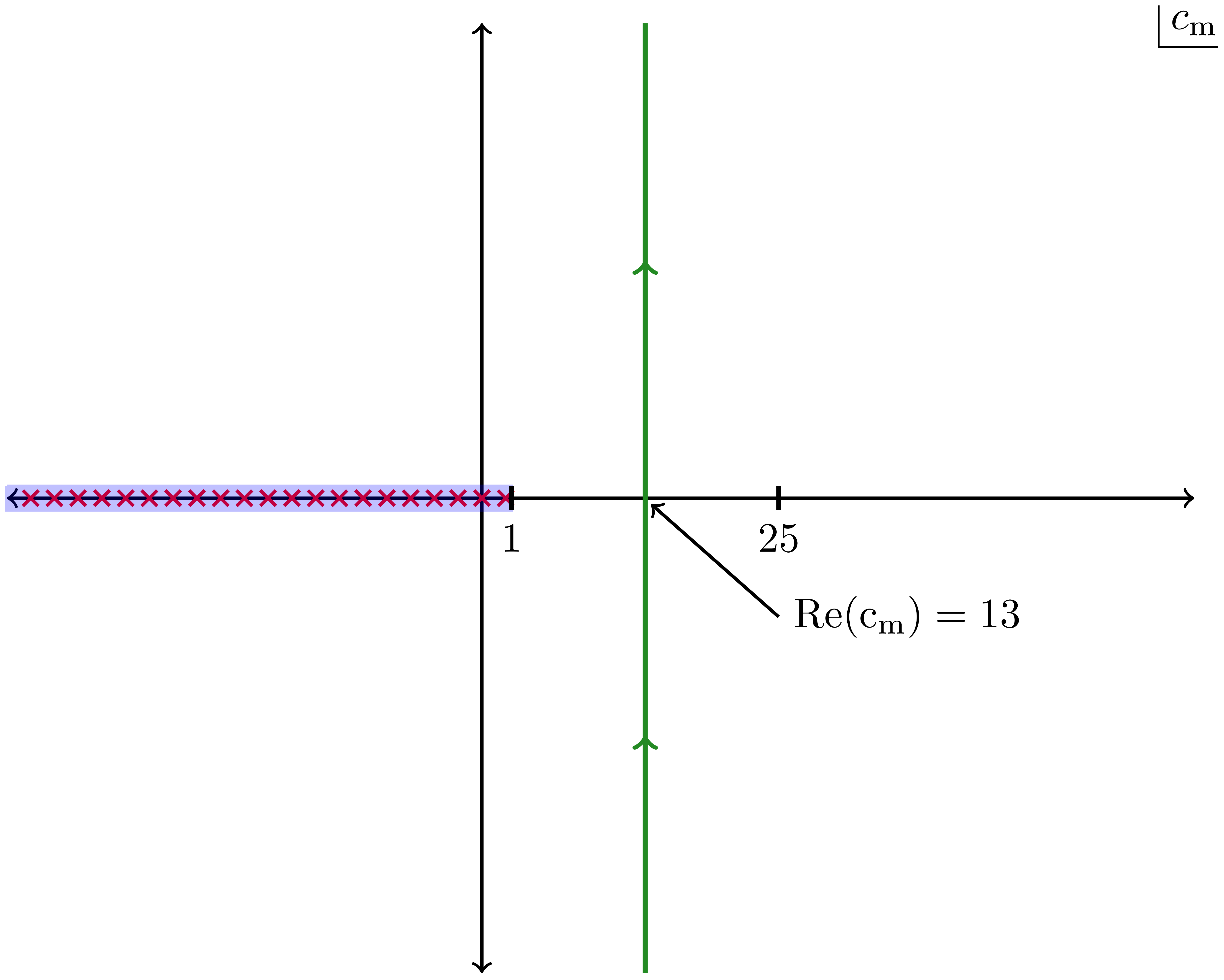
$\mathcal{O}(g_\mathrm{s}^{-1})$

Computation requires string field theory
[Sen '21 -, Eniceicu Mahajan Murdia Sen '22]

- Matches with matrix model computation (including ambiguity)

Summary

- The 2d string landscape is bigger than we thought
- Can we completely classify it?
- The irrational cousins of the minimal string are simpler and admit matrix integral duals
- The $|\text{Liouville}|^2$ string provides a rigorous model of 2d quantum gravity involving dS



Summary

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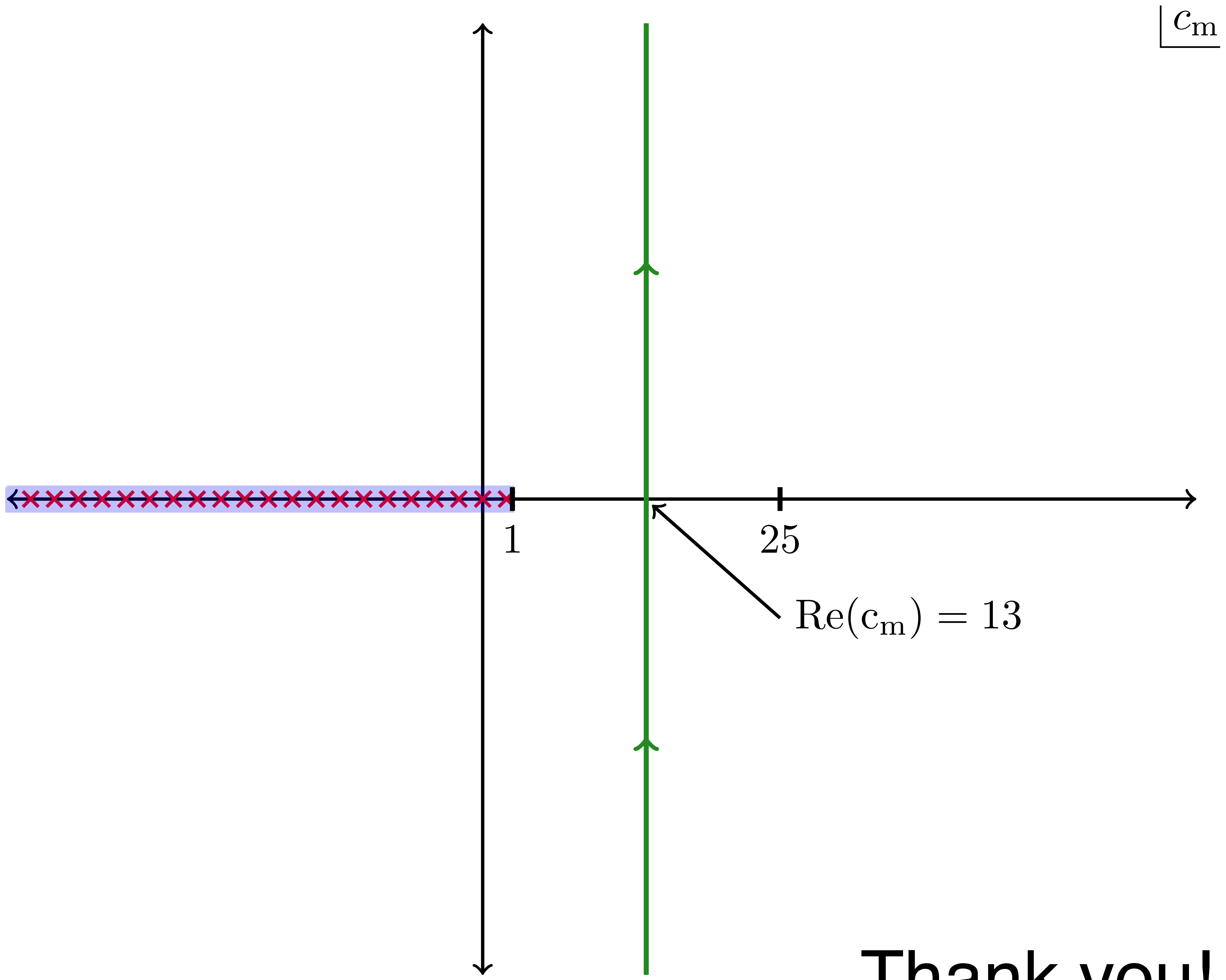


Thank you!

Bonus slides

Liouville theory and sine dilaton gravity

- Lagrangian formulation of the worldsheet theory:

$$S_L^+[\phi] = \frac{1}{4\pi}\int d^2x\sqrt{\tilde{g}}\left(\tilde{g}^{ij}\partial_i\phi\partial_j\phi + (b_+ + b_+^{-1})\tilde{R}\phi + \mu e^{2b_+\phi}\right)$$

$b_+ \in e^{\frac{\pi i}{4}}\mathbb{R}$

$\mu \in i\mathbb{R}$

$$S_L^-[\bar{\phi}] = \frac{1}{4\pi}\int d^2x\sqrt{\tilde{g}}\left(\tilde{g}^{ij}\partial_i\bar{\phi}\partial_j\bar{\phi} + (b_- + b_-^{-1})\tilde{R}\bar{\phi} + \bar{\mu} e^{2b_-\bar{\phi}}\right)$$

$b_- = -ib_+$

$\in e^{-\frac{\pi i}{4}}\mathbb{R}$

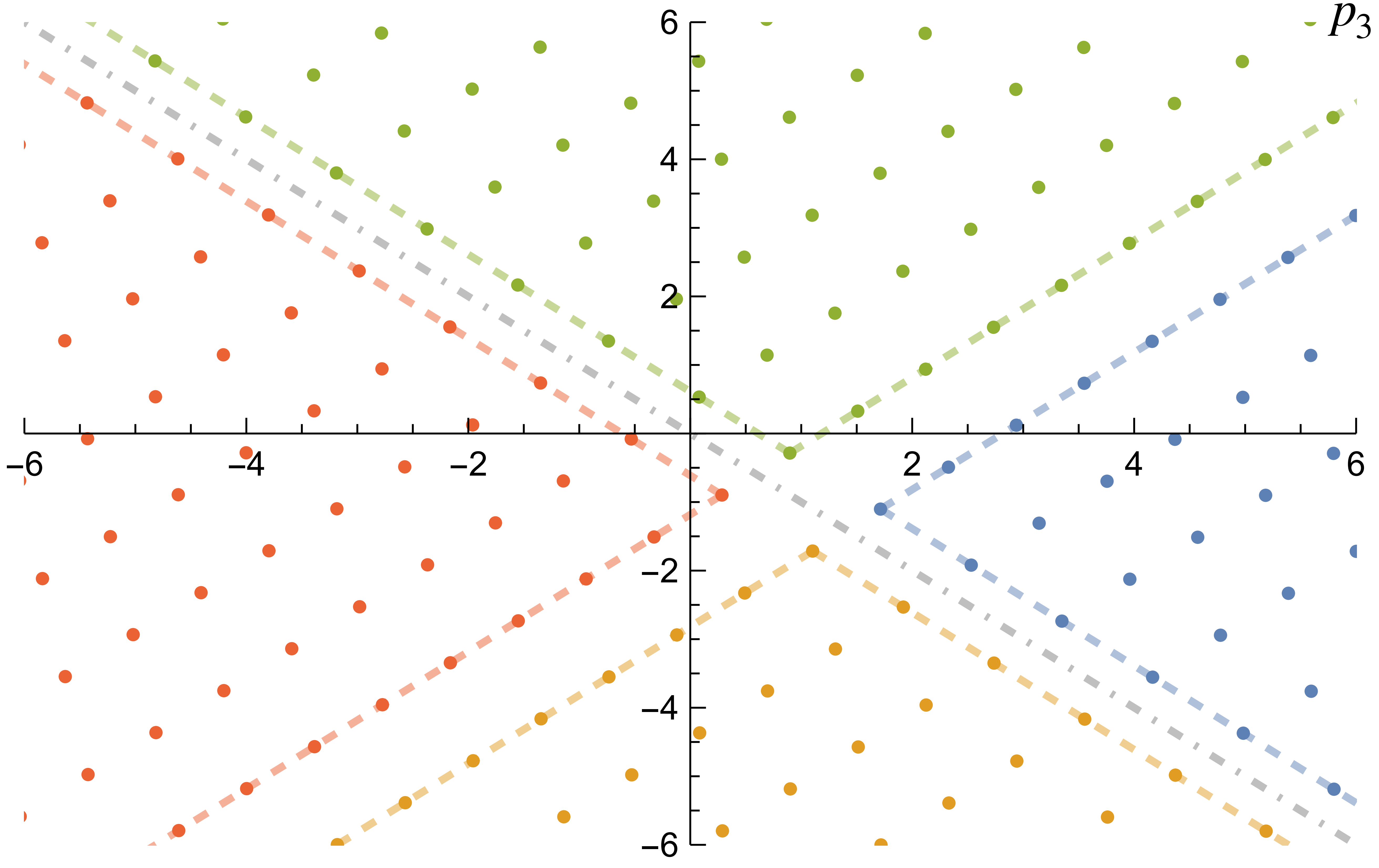
- Field redefinition:

$$\phi = b_+^{-1}\rho + \pi b_+\Phi\,, \qquad \bar{\phi} = b_-^{-1}\rho + \pi b_-\Phi\,, \qquad g = e^{2\rho}\tilde{g}$$

Sphere three-point amplitude

- $\mathsf{A}_{0,3}^{(b)}(p_1, p_2, p_3)$ has simple poles for

$$p_1 \pm p_2 \pm p_3 = \left(r + \frac{1}{2}\right)b + \left(s + \frac{1}{2}\right)b^{-1},\ r, s \in \mathbb{Z}$$



$b \in e^{\frac{\pi i}{4}}\mathbb{R}$

$p_1, p_2 \in e^{-\frac{\pi i}{4}}\mathbb{R}$

Sphere three-point amplitude

$$\mathsf{A}_{0,3}^{(b)}(p_1, p_2, p_3) = \frac{ib\,\eta(b^2)^3\,\prod_{j=1}^{3}\vartheta_1(2bp_j\,|\,b^2)}{2\vartheta_3(bp_1 \pm bp_2 \pm bp_3\,|\,b^2)}$$

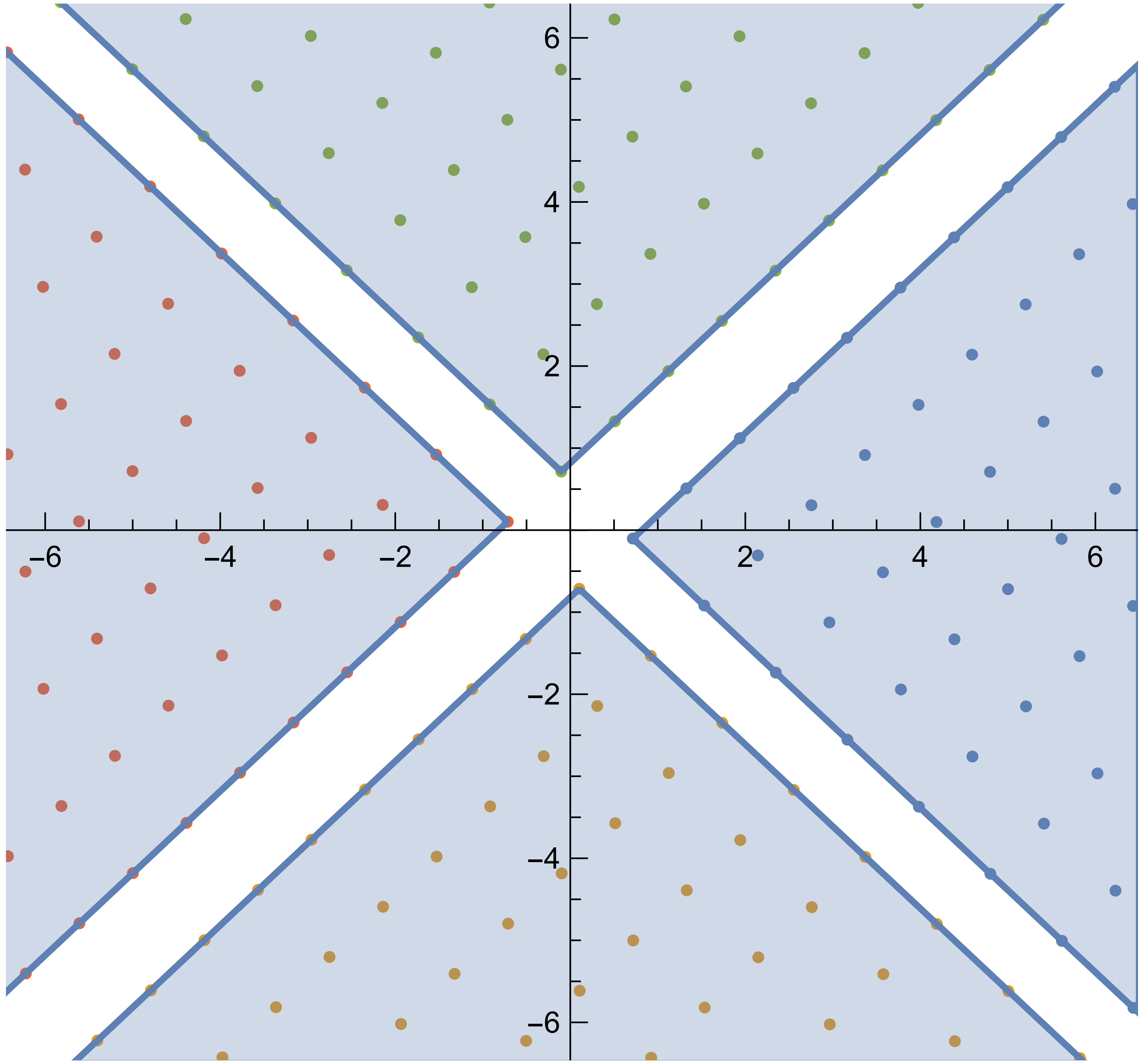
[Zamolodchikov 05]

- With a suitable identification of the parameters, this combination has recently appeared as the boundary two-point function in double-scaled SYK (cf. [Narovlansky, Verlinde, Zhang 23, 24])

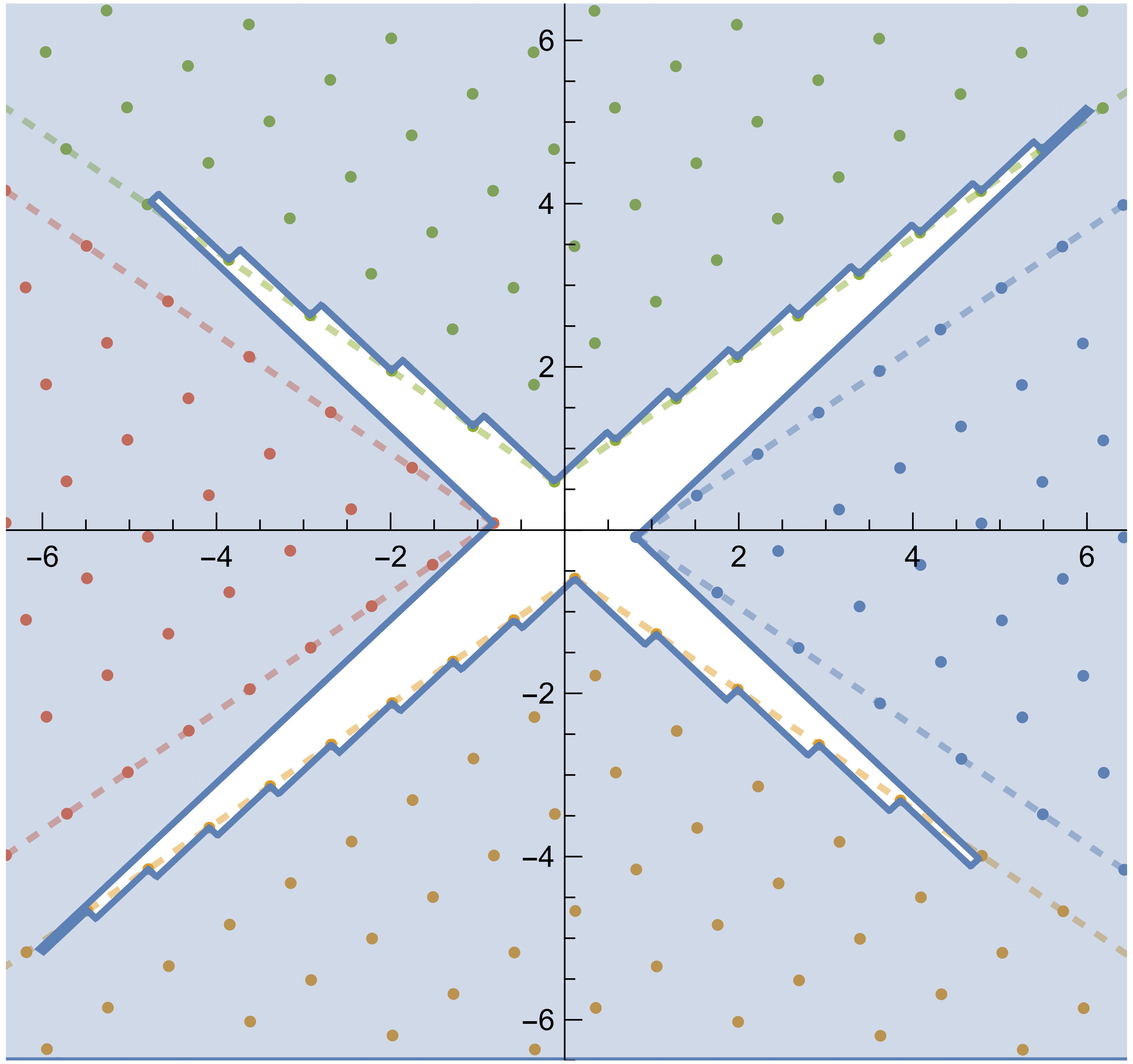
Sphere four-point amplitude: domain of analyticity

- The wedges of divergence associated with the branch cuts carve out a domain of analyticity in the $p_i \pm p_j$ plane that is non-compact if $b \in e^{\frac{\pi i}{4}}\mathbb{R}$, and compact otherwise

$p_i \pm p_j$



$b \in e^{\frac{\pi i}{4}}\mathbb{R}$



$b \in e^{\frac{\pi i}{5}}\mathbb{R}$

Sphere four-point amplitude: solution

- A solution is given by:

$$\mathsf{A}^{(b)}_{0,4}(p_1,p_2,p_3,p_4) = -\sum_{m_1,m_2=1}^{\infty} \frac{(-1)^{m_1+m_2}\sin(2\pi m_1 b p_1)\sin(2\pi m_1 b p_2)\sin(2\pi m_2 b p_3)\sin(2\pi m_2 b p_4)}{\pi^2 \sin(\pi m_1 b^2)\sin(\pi m_2 b^2)}$$

$$\times\left(\frac{1}{(m_1+m_2)^2} - \frac{1-\delta_{m_1,m_2}}{(m_1-m_2)^2}\right) + \text{2 perms}$$

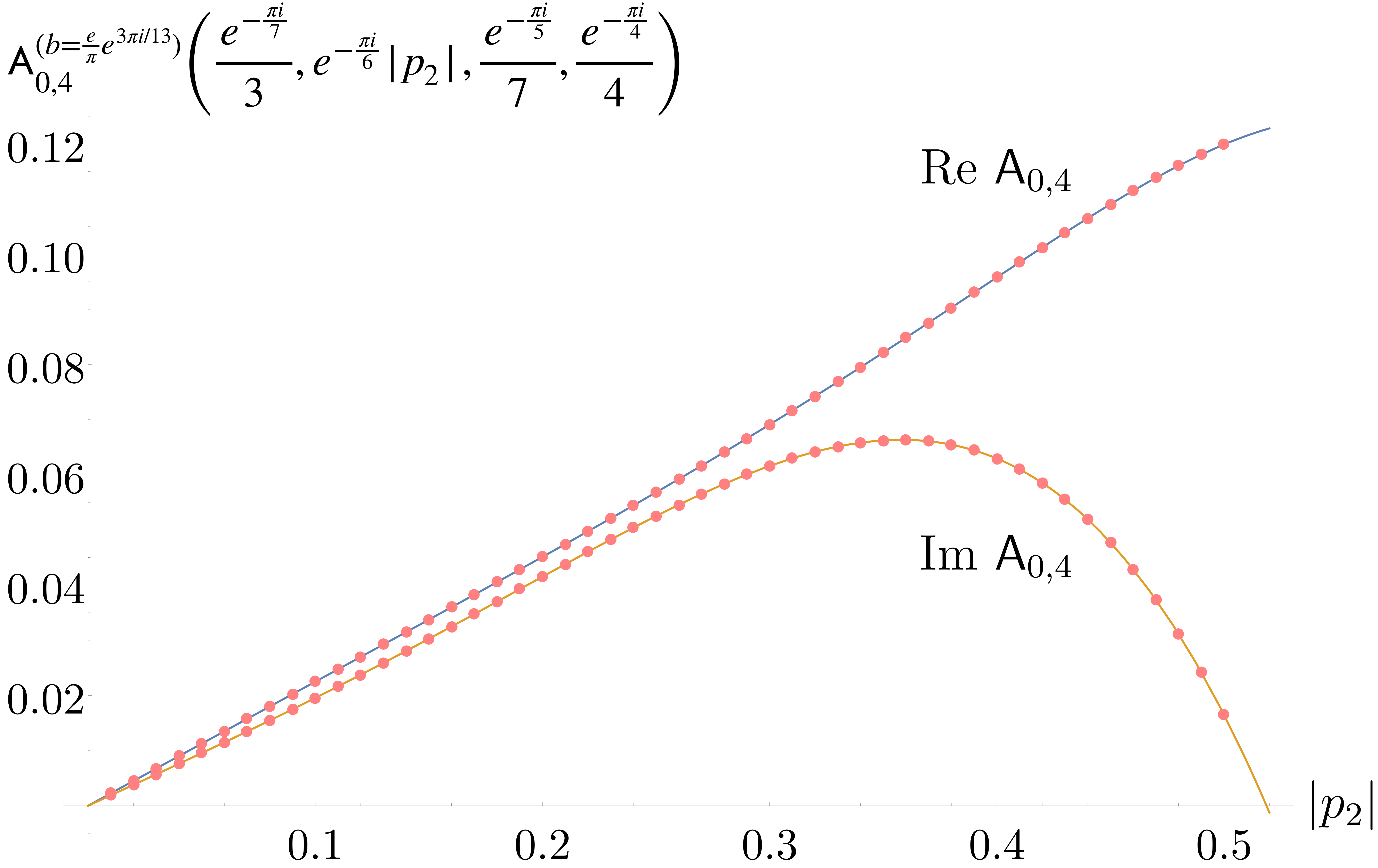
$$+\sum_{m_1=1}^{\infty} \frac{2b^2\left(\prod_{j=1}^4 \sin(2\pi m_1 p_j)\right)\mathsf{V}^{(b)}_{0,4}(p_1,p_2,p_3,p_4)}{\sin(\pi m_1 b^2)^2}$$

Discontinuities

$$\sim \int' 2p\,dp\,\mathsf{A}^{(b)}_{0,3}(p_1,p_2,p)\mathsf{A}^{(b)}_{0,3}(p,p_3,p_4)$$

Sphere four-point amplitude: numerical verification

- Direct numerical integration over moduli space for other values of the parameters

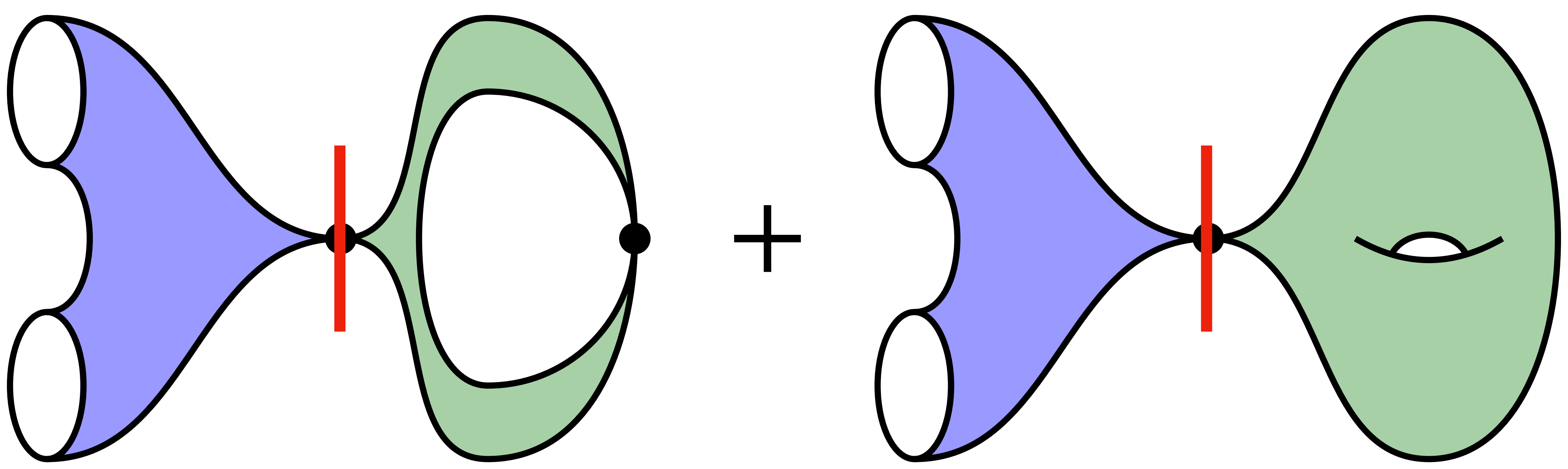


Error $< 10^{-5}\,\%$

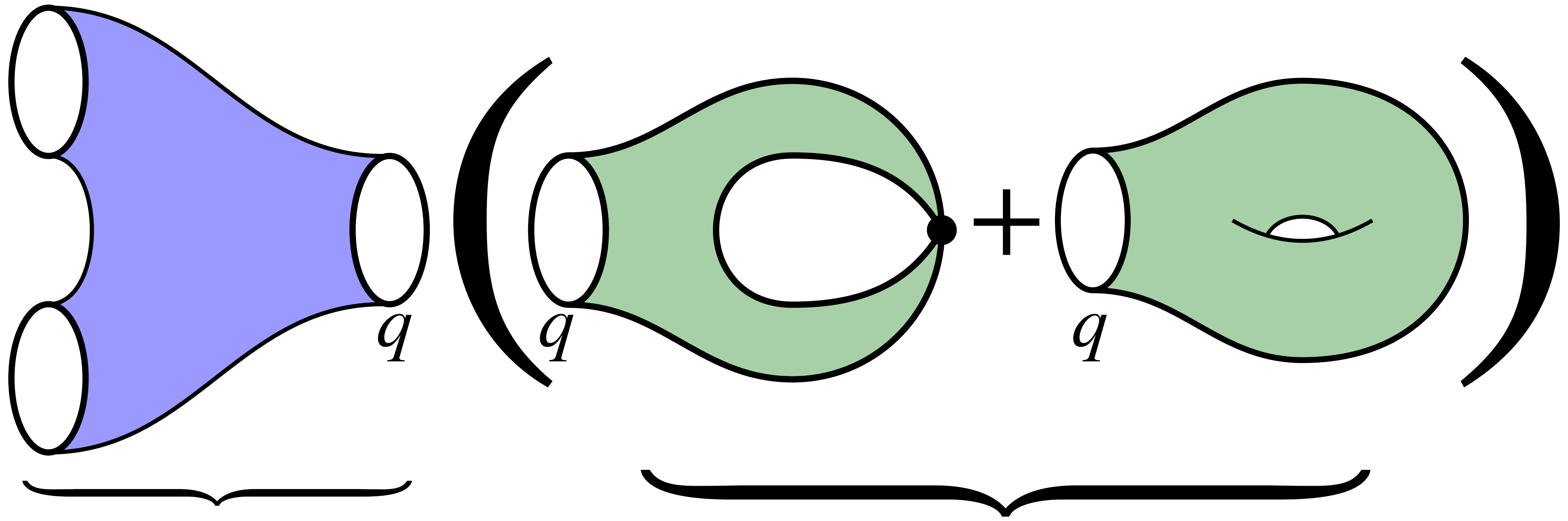
Discontinuities of string amplitudes from cutting Feynman diagrams

$$\operatorname*{Disc}_{q_*=0} \mathsf{A}^{(b)}_{g,n}(\mathbf{p}) = 2\pi i q_* \left(\operatorname*{Res}_{q=\frac{1}{2}q_*} \mathsf{A}_{g-1,n+2}(q,q,\mathbf{p}) + 2\sum_{h=0}^{g} \sum_{I\sqcup J=\{p_1,\ldots,p_n\}} \operatorname*{Res}_{q=q_*} \mathsf{A}^{(b)}_{h,1+|I|}(q,\mathbf{p}_I)\mathsf{A}^{(b)}_{g-h,1+|J|}(q,\mathbf{p}_J) \right)$$

e.g.



$= \; 4\pi i q_* \operatorname*{Res}_{q=q_*}$



$\mathsf{A}^{(b)}_{0,3}$ $\mathsf{A}^{(b)}_{1,1}$

Topological recursion and string amplitudes

- Initial data for topological recursion:

$$\omega_{0,1}^{(b)}(z) = -\frac{2\pi \sin(\pi b^{-1}\sqrt{z})\cos(\pi b\sqrt{z})}{b\sqrt{z}}dz \qquad \omega_{0,2}^{(b)}(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}$$

- String amplitudes are simply related to the $\omega_{g,n}^{(b)}(z_1, \ldots, z_n)$ differentials by inverse Laplace transform

$$\mathsf{A}_{g,n}^{(b)}(p_1, \ldots, p_n) = \int_\gamma \left(\prod_{j=1}^{n} \frac{1}{4\pi i} \frac{e^{2\pi i p_j w_j}}{p_j} \right) \omega_{g,n}^{(b)}(w_1, \ldots, w_n) \qquad \left(w_j = \sqrt{z_j}\right)$$

$$= \sum_{m_1,\ldots,m_n=1}^{\infty} \operatorname*{Res}_{z_1=m_1^2 b^2} \cdots \operatorname*{Res}_{z_n=m_n^2 b^2} \prod_{j=1}^{n} \frac{\cos(2\pi p_j\sqrt{z_j})}{p_j} \omega_{g,n}^{(b)}(z_1, \ldots, z_n)$$