The Categorical Landau Paradigm

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String-Math 2024, ICTP, Trieste, June 7, 2024

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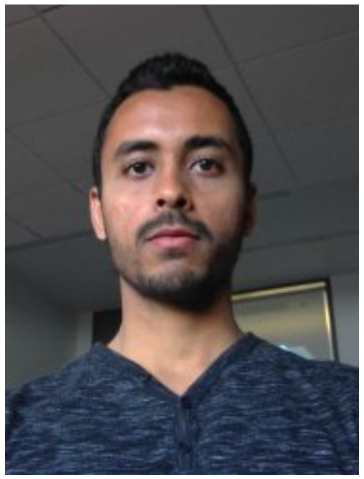
    

2310.03786: Categorical Landau Paradigm and SymTFT
2310.03784: Gapped Phases with Non-Invertible Symmetries
2312.17322: The Club Sandwich: Gapless Phases and Phase Transitions with Non-Invertible Symmetries
2403.00905: Hasse Diagrams for Gapless SPT and SSB Phases with Non-Invertible Symmetries
2405.05964: Phases of Lattice Models with Non-Invertible Symmetries

Related works – many in cond-mat:

Gapped phases:

[Thorngren, Wang][Inamura][Huang, Lin, Seifnashri][Cordova, Zhang][S. Huang, Meng Cheng]

Gapless phases:

[Chatterjee, $\pm$ Aksoy, Wen][Wen, Potter]

With fermions:

[S. Huang][Bhardwaj, Inamura, Tiwari]

Non-Invertible SPT phases from lattice models:

[Fechisin, Tantivasadakarn, Albert][Seifnashri, Shao][Jia]

Non-Invertible symmetries and phase transitions in lattice models (RepS_3):

[Eck, Fendley][Bhardwaj, Bottini, SSN, Tiwari][Chatterjee, Aksoy, Wen]

Symmetric Phases

Landau paradigm:
A continuous (2nd order) phase transition is a symmetry breaking transition.

G is a symmetry group, which is spontaneously broken to a subgroup H, resulting in $|G/H|$ vacua, which are acted upon by the broken symmetry (+SPT phases).

Example: $G = \mathbb{Z}_2$**.**
There are two gapped $\mathbb{Z}_2$-symmetric phases:

- Trivial phase ($H = \mathbb{Z}_2$): $\mathbb{Z}_2$ symmetric single vacuum.
- Spontaneously Symmetry Broken (SSB) Phase ($H = 1$): charged operator $\mathcal{O}_-$ gets a vacuum expectation value, two vacua, and the broken $\mathbb{Z}_2$ exchanges them.

Between these there is a 2nd order phase transition: in (1+1)d, the critical Ising CFT

$$\boxed{\mathbb{Z}_2 \text{ SSB Phase}} \quad \longleftarrow \quad \boxed{\text{Ising CFT}} \quad \longrightarrow \quad \boxed{\text{Trivial Phase}}$$

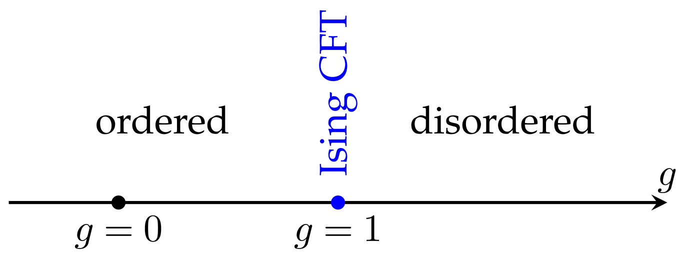
Lattice Model Realization: Ising model

Transverse field **Ising model**: $\mathcal{H} = (\mathbb{C}^2)^L$ with nearest neighbor Hamiltonian

$$H = -\sum_j Z_j Z_{j+1} - g \sum_j X_j \,.$$

There is a $\mathbb{Z}_2$ spin flip symmetry $\eta = \prod_j X_j$.

- $g = 0$: two ground states, $|\uparrow^L\rangle$ and $|\downarrow^L\rangle$: "ordered phase"
- $g \gg 1$: ground state preserves the $\mathbb{Z}_2$: "disordered phase"
- $g = 1$: critical Ising CFT at $c = 1/2$.



Kramers-Wannier duality:
$X_i \to Z_j Z_{j+1}$ and $Z_j Z_{j+1} \to X_{j+1}$, corresponds to $g \to g^{-1}$.
At $g = 1$: symmetry of the critical Ising CFT, which realizes the non-invertible defect:

$$N^2 = 1 + \eta$$

Non-Invertible Symmetries

Identifying global symmetries with topological defects [Gaiotto, Kapustin, Seiberg, Willett, **2014**] has led in the last 10 years to a vast generalization of the notion of symmetry. The most recent one is to so-called **Non-invertible** or **Categorical Symmetries**:
An example are fusion category symmetries $\mathcal{S}$ in 2d: for $a, b \in \mathcal{S}$

$$a \otimes b = N_1 c_1 \oplus \cdots \oplus N_k c_k \,, \quad c_i \in \mathcal{S}\,,\ N_i \in \mathbb{N}$$

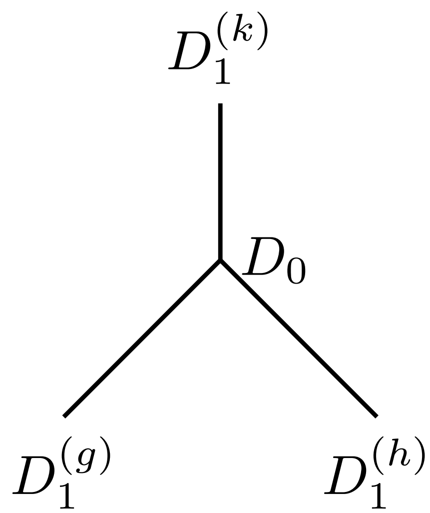
Long-history in 2d QFTs: [Fuchs, Runkel, Schweigert][Bhardwaj, Tachikawa][$\cdots$]
Many constructions in $d > 2$ QFTs [Starting in '21], which form **higher fusion category symmetries**

Fusion Category Symmetries

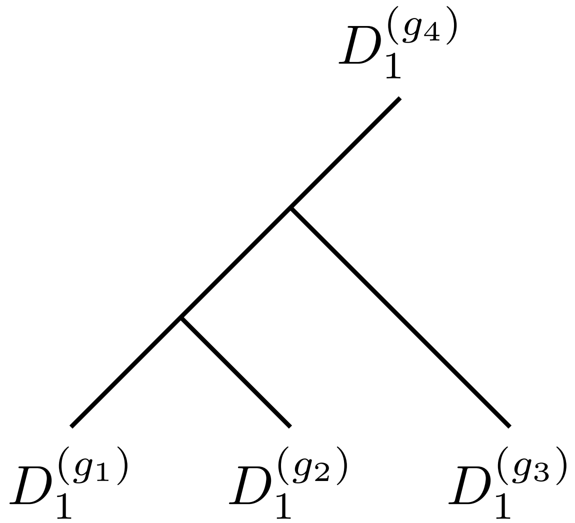
In 2d theories:

- Objects: topological lines $D_1^{(g)}$,
- Morphisms: topological point operators $D_0 \in \mathrm{Hom}(D_1^{(g)}, D_1^{(h)})$.
- Fusion:

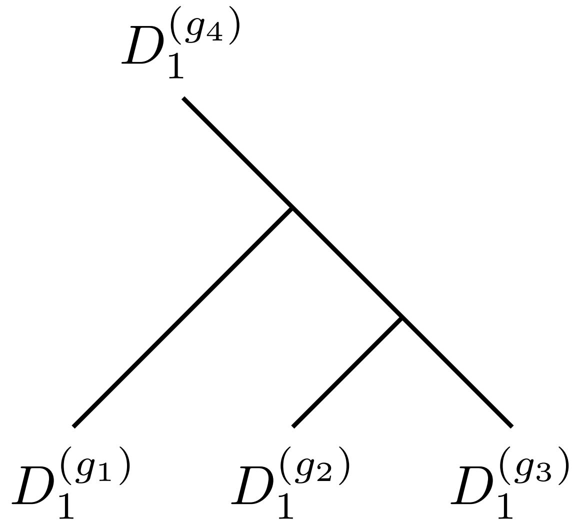
$$D_1^{(g)} \otimes D_1^{(h)} = \bigoplus_k N_k^{g,h} D_1^{(k)} \, .$$



- Associativity:



$$= F^{g_4}_{g_1,g_2,g_3}$$



and subsequent compatibility conditions (pentagon identity)

Two simple examples of Non-Invertible Symmetries in 2d:

- **Ising fusion category**:
  Generators are $1, \eta, N$, where $\eta \otimes \eta = 1$ is a $\mathbb{Z}_2$ group, and
  $N \otimes \eta = \eta \otimes N = N$, but N is non-invertible

$$N \otimes N = 1 \oplus \eta\,.$$

  N is the Kramers-Wannier self-duality of the critical Ising model.

- **Representations of a finite non-abelian group G**:
  e.g. permutation group on 3 elements S_3:

  $\text{Rep}(S_3) =$ representations of S_3 with the tensor product form a fusion category .

  The generators are the irreducible representations:
  the trivial (1), sign (U) and 2d representation E, respectively, with tensor product (fusion):

$$U \otimes U = 1\,, \qquad E \otimes U = U \otimes E = E\,, \qquad E \otimes E = 1 \oplus U \oplus E\,.$$

Non-Invertible (Higher-Fusion Category) Symmetries in $d = 4$

- 4d Kramers-Wannier duality defects:
  [Kaidi, Ohmori, Zheng][Choi, Cordova, Hsin, Lam, Shao]

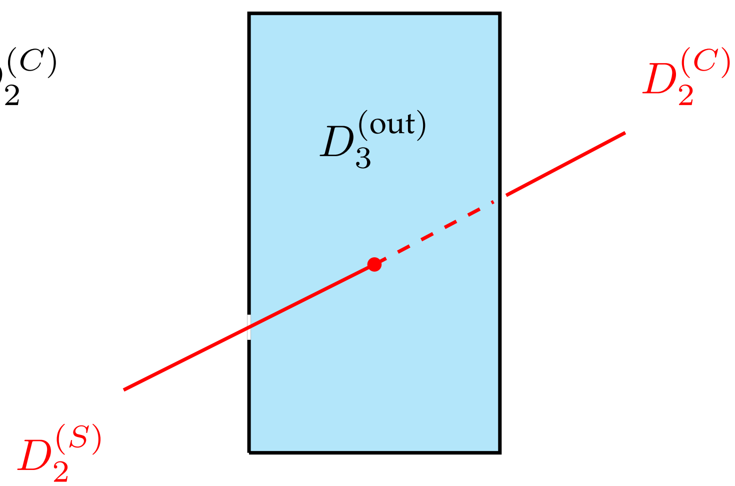
$$\text{QFT} \cong \text{QFT}/D \qquad \Rightarrow \qquad \text{non-invertible 0-form symmetry}$$

- Condensation defects from higher-gauging : [Roumpedakis, Seifnashri, Shao]

$$\mathcal{C}_d \sim \sum_{\Sigma \in H_q(M_d, \mathbb{Z}_N)} e^{i \int_\Sigma b}$$

- Gauging outer automorphisms [Bhardwaj, Bottini, SSN, Tiwari]:
  E.g. 1-form symmetry $\mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)}$ of Spin$(4n)$ exchanged by outer automorphism

$$D_2^{\text{inv}} = D_2^{(S)} \oplus D_2^{(C)}$$



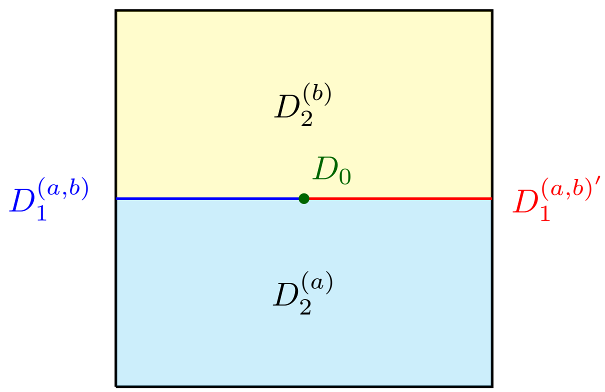
Higher Fusion Category Symmetries

In higher dimensions, higher-form symmetries (and generalizations thereof) need to be included. E.g. $(d-p-1)$-dimensional defect links in d dimensions with a p-dimensional charged operator. "p-form symmetry".

In d-spacetime dims non-invertible symmetries form a $(d-1)$-fusion category:

- Topological defects of dimension $(d-1)$, up to 0: $(d-1)$ objects, $(d-2)$ morphisms, $(d-3)$ 2-morphisms, etc.
- Fusion of defects in each dimension
- Compatibility/associativity conditions

$d=3$:



Classification of fusion 2-categories (up to Morita equivalence) [Decoppet].

Generalized Charges for Non-Invertible Symmetries

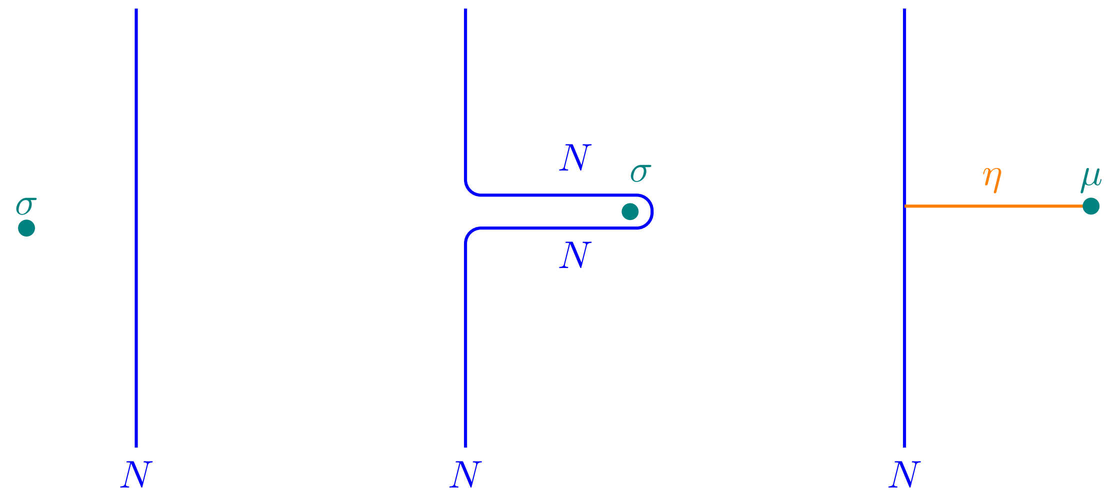
Generalized q-charge
= q-dim defect in a ”Representations of a Non-Invertible Symmetry”.

In 2d: tube algebra and lasso-action [Fröhlich, Fuchs, Runkel, Schweigert][Lin, Okada, Seifnashri, Tachikawa][Bhardwaj, SSN][Bartsch, Bullimore, Ferrari, Pearson]

Example: Ising fusion symmetry of the critical Ising model

$$\eta^2 = 1\,, \qquad N\eta = \eta N = N\,, \qquad N^2 = 1 \oplus \eta\,.$$

We can act on the spin operator σ ($1/16$ primary):



This is a hallmark of non-invertible symmetries: **they map genuine operators to non-genuine ones** (i.e. attached to topological defects).

Categorical Landau Paradigm

Conjecture/Hope: **Generalized (Categorical) Landau Paradigm:**
Explain (beyond Landau) phase transitions using a suitably generalized notion of symmetry.

Let $\mathcal{S}$ be a non-invertible symmetry. We develop a framework that determines:

- All $\mathcal{S}$-symmetric **gapped phases** including the order parameters, i.e. generalized charges acquiring vevs
- **Gapless phase transitions** between $\mathcal{S}$-symmetric gapped phases:

$$\boxed{\mathcal{S} \text{ Gapped Phase}} \quad \longleftarrow \quad \boxed{\text{CFT}} \quad \longrightarrow \quad \boxed{\mathcal{S} \text{ Gapped Phase'}}$$

Generalizes the Landau paradigm to $\mathcal{S}$ a categorical symmetry

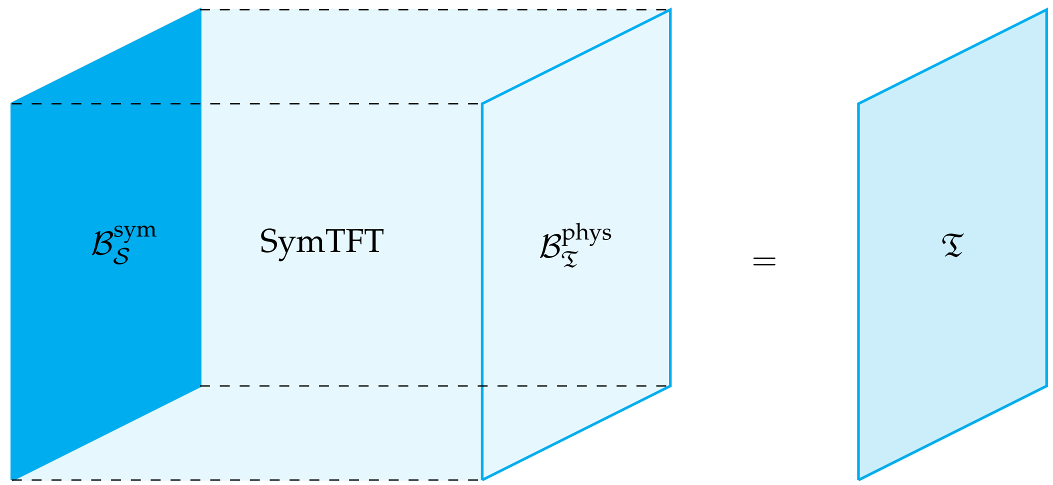
$\Rightarrow$ Categorical Landau Paradigm [Bhardwaj, Bottini, Pajer, SSN]

These can be classified using the so-called **Symmetry TFT**.

SymTFT ("Sandwich = Quiche^2"*)

[Gaiotto, Kulp][Apruzzi, Bonetti, Garcia-Extebarria, Hosseini, SSN] [*Freed, Moore, Teleman]

Given a physical QFT $\mathfrak{T}$ with (finite) symmetry $\mathcal{S}$ in d dimensions. The SymTFT is a $d+1$ dimensional TQFT $\mathfrak{Z}(\mathcal{S})$ by gauging $\mathcal{S}$ in $(d+1)$ dims:

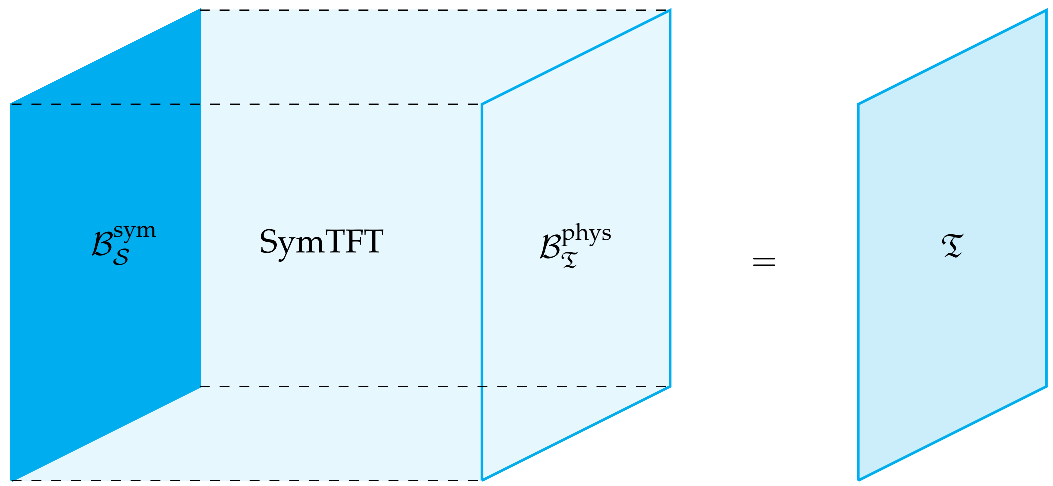


Examples:

- $\mathbb{Z}_N$ p-form symmetries (with anomalies): Dijkgraaf-Witten theory (with twist) $N\int b_{p+1}\wedge c_{d-p}$.
- Fusion category symmetries: Turaev-Viro TQFT.

The topological defects $\mathbf{Q}_p$ of the SymTFT form the Drinfeld center $\mathcal{Z}(\mathcal{S})$.

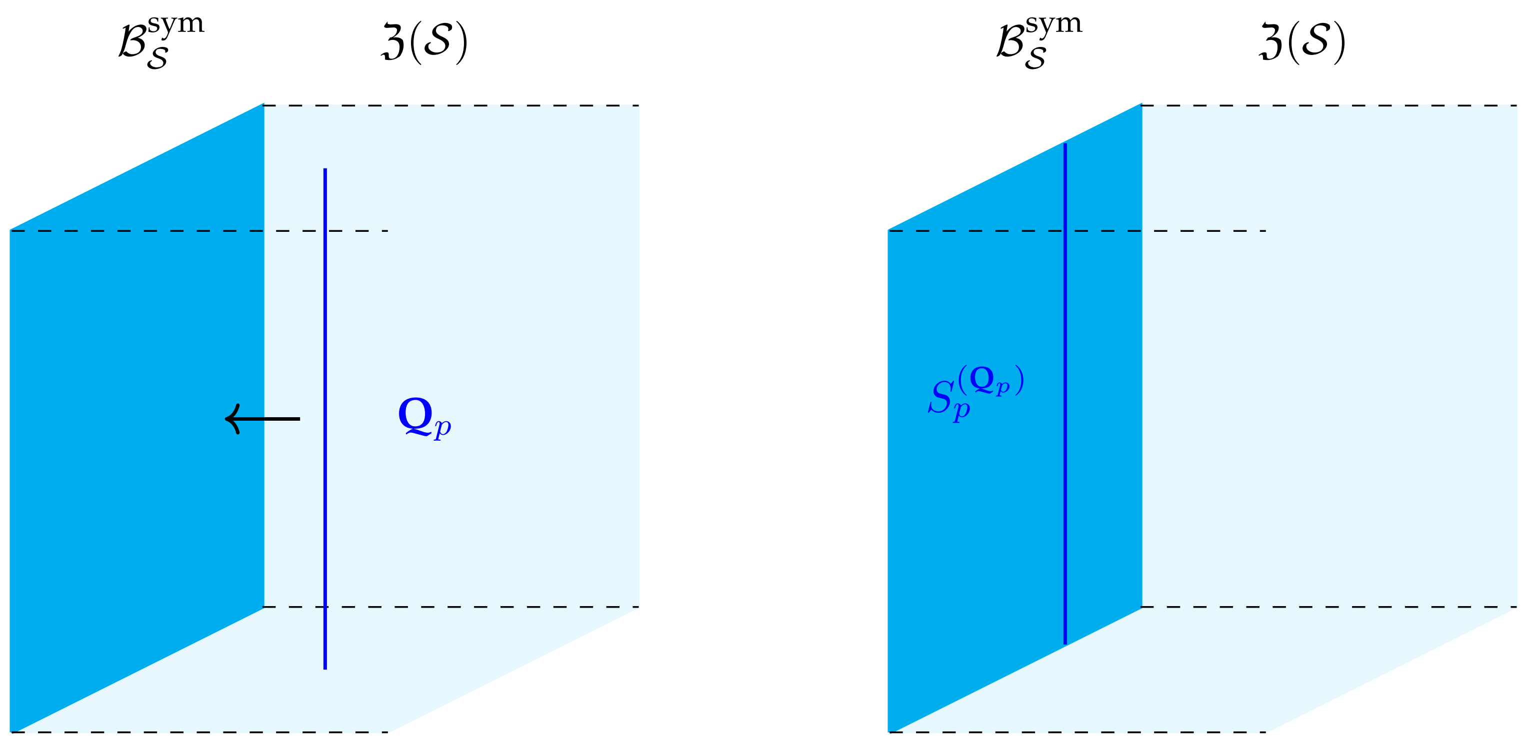
SymTFT ("Sandwich")



- $\mathcal{B}^{\text{sym}}_{\mathcal{S}}$ = Symmetry boundary (gapped boundary condition): condense a maximal number of mutually local topological defects. The remaining defects generate $\mathcal{S}$.
  These are classified by Lagrangian algebras of $\mathcal{Z}(\mathcal{S})$.
- $\mathcal{B}^{\text{phys}}_{\mathfrak{T}}$ = Physical boundary: condense a subset of mutually local defects (braiding trivially with each other, but not necessarily maximal)

The interval compactification gives $\mathfrak{T}$ with symmetry $\mathcal{S}$.

SymTFT: Recovering $\mathcal{S}$

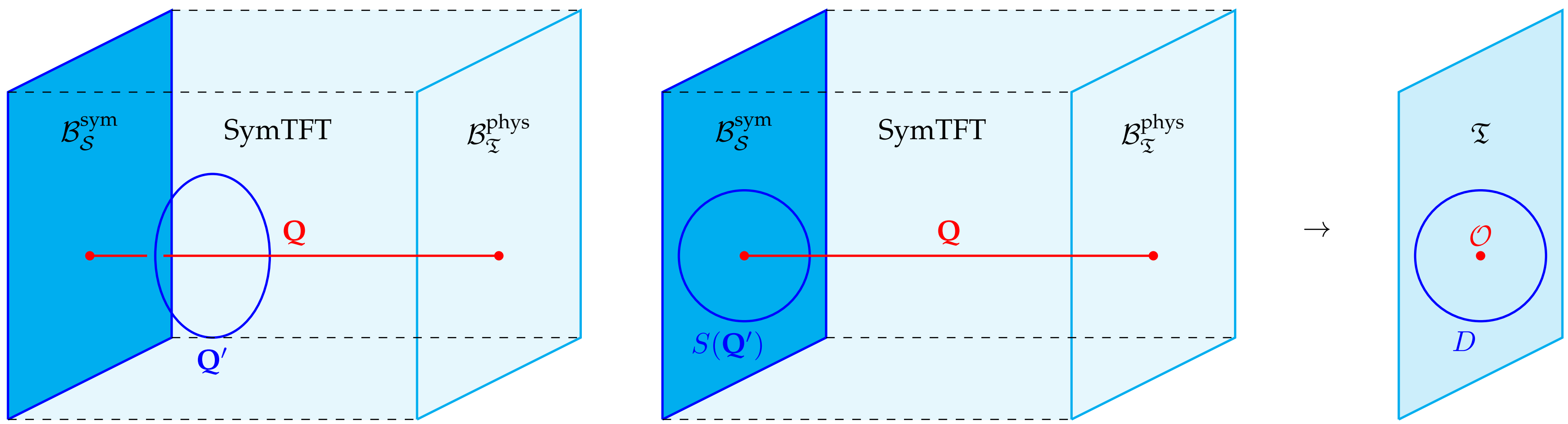


$\mathcal{B}_{\mathcal{S}}^{\text{sym}}$: gapped (topological) boundary conditions of the SymTFT:

$\Rightarrow$ Determined by a maximal set of mutually local topological defects, which form a Lagrangian algebra

$\mathbf{Q}_p$ with Neumann b.c.s give rise to symmetry generators $\mathcal{S}$.

Linking of Topological Defects is Action of Symmetry

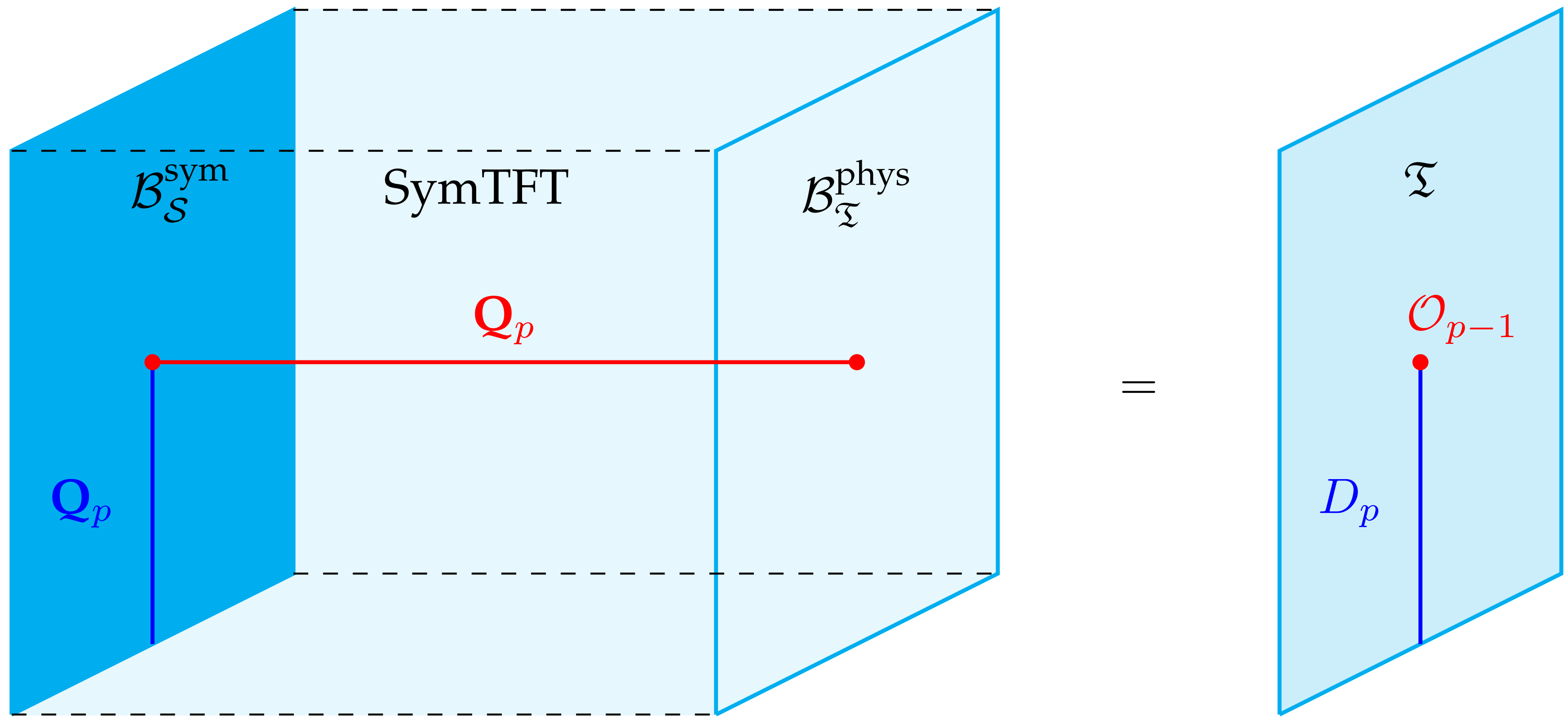


\# [Bhardwaj, SSN '23]:
The generalized charges are the topological defects $\mathbf{Q}$ of the SymTFT, which condense on both boundaries.

\# $\mathbf{Q}'$ that have Neumann b.c. on the $\mathcal{B}^{\text{sym}}_{\mathcal{S}}$ boundary are the generators of the symmetry $\mathcal{S}$.

\# Linking of $\mathbf{Q}$ and $\mathbf{Q}'$ determines the charge under the symmetry.

SymTFT: Non-genuine Operators



$\mathcal{O}$ is attached to a topological line, i.e. a non-genuine operator.

SymTFT for finite groups G

Drinfeld Center $\mathcal{Z}(\mathbf{Vec}_G)$:
For (non-abelian) $G^{(0)}$ the symmetry category is Vec_G, and elements of the center are

$$\mathbf{Q}^{[g],\mathbf{R}},$$

- conjugacy classes $[g]$
- representations of the stabilizer group H_g of $g \in [g]$.

Lagrangian algebras:
Gapped boundary conditions are given in terms of Lagrangian algebras:

$$\mathcal{L} = \oplus n_a \mathbf{Q}_1^a$$

such that

$$\dim(\mathcal{L}) \equiv \sum n_a \dim(\mathbf{Q}_1^a) = \dim(\mathcal{S}), \qquad \text{where } \dim(\mathcal{S})^2 = \sum \dim(\mathbf{Q}_1)^2$$

$$n_a n_b \leq \sum_{c\in\mathcal{L}} N_{ab}^c n_c$$

$$\frac{\sum_{b\in\mathcal{Z}} S^{ab} n^b}{\sum_{b\in\mathcal{Z}} S^{1b} n^b} = \text{ cyclotomic integer for all } a \in \mathcal{Z}$$

Example: S_3 in 2d

$S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2 = \{\mathrm{id}, a, a^2, b, ab, a^2b\}$
Irreps: $+$ (trivial), $-$ (sign), E (2d representations).
Conjugacy classes:

$$\begin{aligned} [\mathrm{id}]\,, &\quad H_{\mathrm{id}} = S_3 \\ [a]\,, &\quad H_a = \{\mathrm{id}, a, a^2\} = \mathbb{Z}_3 \\ [b]\,, &\quad H_b = \{\mathrm{id}, b\} = \mathbb{Z}_2\,. \end{aligned}$$

$H_a = \mathbb{Z}_3$ irreps: labeled by $1, \omega = e^{2\pi i/3}, \omega^2$.
$H_b = \mathbb{Z}_2$ irreps: labelled by $\pm$.

The lines in $\mathcal{Z}(\mathrm{Vec}_{S_3})$

$$\begin{aligned} \mathbf{Q}_1^{([\mathrm{id}],\mathbf{R})} : &\qquad \mathbf{R} = 1, 1_-, E \\ \mathbf{Q}_1^{([a],\mathbf{R})} : &\qquad \mathbf{R} = 1, \omega, \omega^2 \\ \mathbf{Q}_1^{([b],\mathbf{R})} : &\qquad \mathbf{R} = \pm\,. \end{aligned}$$

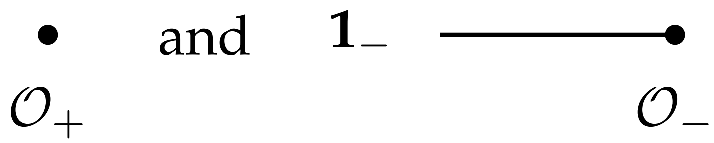
The topological b.c.s (Lagrangian algebras) are

$$\mathcal{L}_{S_3} = ([\mathrm{id}], 1) \oplus ([\mathrm{id}], 1_-) \oplus 2([\mathrm{id}], E) \qquad \mathcal{L}_{\mathrm{Rep}(S_3)'} = ([\mathrm{id}], 1) \oplus ([\mathrm{id}], 1_-) \oplus 2([a], 1)$$

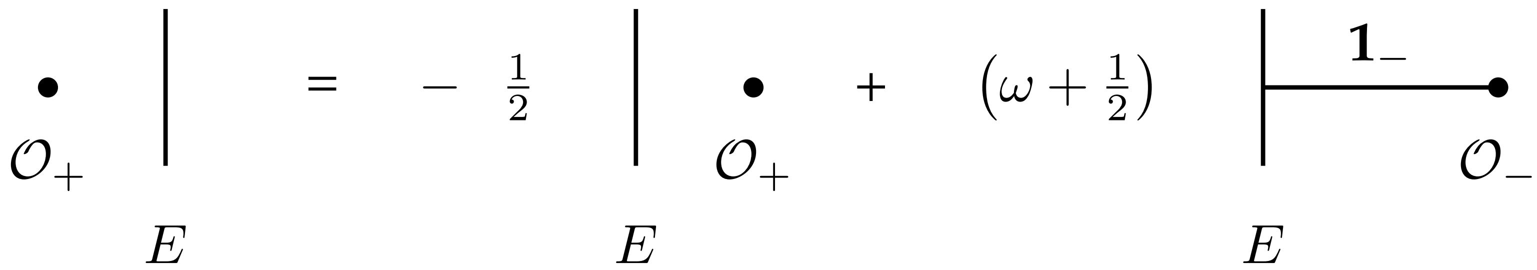
$$\mathcal{L}_{S_3'} = ([\mathrm{id}], 1) \oplus ([\mathrm{id}], E) \oplus ([b], 1) \qquad \mathcal{L}_{\mathrm{Rep}(S_3)} = ([\mathrm{id}], 1) \oplus ([a], 1) \oplus ([b], 1)$$

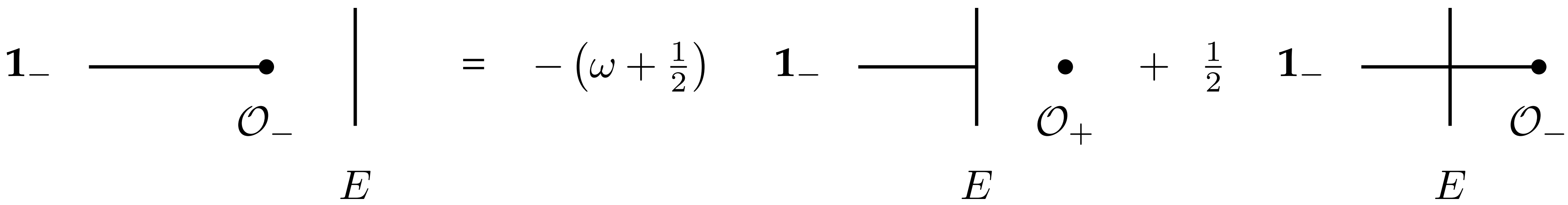
Multiplet structure:

- S_3: $\mathbf{Q}_1^{([\mathrm{id}],\mathbf{R})}$ are untwisted; $\mathbf{Q}_1^{([a])}$ and $\mathbf{Q}_1^{([b])}$ are twisted sector reps
- $\mathrm{Rep}(S_3)$: $\mathbf{Q}_1^{([\mathrm{id}],\mathbf{R})}$ are twisted (attached to $\mathbf{R}$ lines).
  $\mathbf{Q}_1^{([a],1)}$ contains two operators:



This can be derived from the action of the symmetry on defects in the SymTFT:





SymTFT

- Topological defects are the generalized charges
- Gauging $\mathcal{S}$ corresponds in the SymTFT to changing the symmetry b.c..
- If $\mathcal{S}$ and $\mathcal{S}'$ that are related by gauging, they have the same SymTFT.
- SymTFT exists for any higher-fusion category: the topological defects are the so-called Drinfeld Center. For 2-fusion categories see [Kong et al][Bhardwaj, SSN]

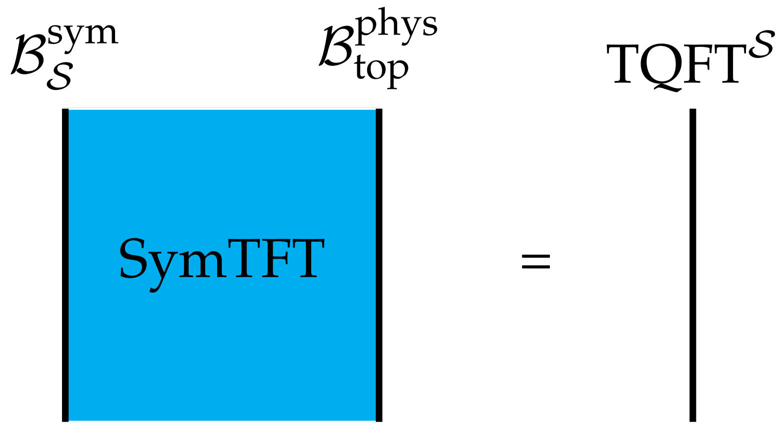
$$\mathfrak{Z}(2\text{Vec}_G^{\omega}) = \bigoplus_{[g]} 2\text{Rep}^{\omega_g}(H_g)$$

- Recently: SymTFT or SymT for continuous abelian and non-abelian symmetries [Antinucci, Benini][Apruzzi, Bedogna, Dondi][Bonetti, del Zotto, Minasian][Brennan, Sun]. This can be important for higher-group symmetries which mix continuous and finite symmetries.

Gapped $\mathcal{S}$-Symmetric Phases

Classification of gapped $\mathcal{S}$-symmetric phases

Gapped phases are obtained by choosing $\mathcal{B}^{\text{phys}}$ to be also a topological (gapped) boundary condition.



In the SymTFT:

$$\text{gapped b.c.s} \leftrightarrow \text{Lagrangian algebras } \mathcal{L} \text{ of the Drinfeld Center } \mathcal{Z}(\mathcal{S})\,.$$

Fix symmetry boundary to be $\mathcal{L}_{\mathcal{S}}$:
A gapped $\mathcal{S}$-symmetric phase is given by a Lagrangian algebra $\mathcal{L}$:

- SPT (symmetry protected topological phase): $\mathcal{L} \cap \mathcal{L}_{\mathcal{S}} = 1$
  Cannot deform to the trivial theory without breaking symmetry
- SSB (spontaneous symmetry breaking): $\mathcal{L} \cap \mathcal{L}_{\mathcal{S}} \supsetneq 1$

# of vacua = # of topological defects that condense on both boundaries, which are also the order parameters.

Gapped Phases with Group-Symmetry in 2d

Landau type classification: $\mathcal{S} = \text{Vec}_G$ then

- $H < G$ the unbroken symmetry
- $\omega \in H^2(H, U(1))$ cocycle/SPT phase.

Example: $G = \mathbb{Z}_4$

The SymTFT is a 3d topological order ($\mathbb{Z}_4$ Dijkgraaf-Witten-theory) $\int b_1 \cup \delta c_1$, with anyons $e = e^{i\int b_1}$ and $m = e^{i\int c_1}$:

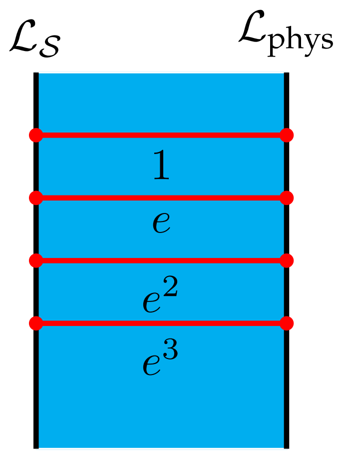
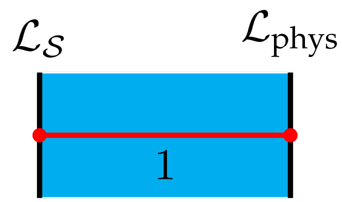
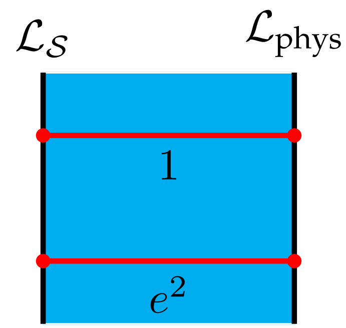
$$\text{Topological defects (anyons):} \qquad (e^i, m^j), \quad e^4 = 1,\ m^4 = 1\,.$$

e and m braid non-trivially. The Lagrangian, i.e. maximal, trivially braiding subsets of anyons are:

1. $\mathcal{L}_{\text{Dir}} = 1 \oplus e \oplus e^2 \oplus e^3$
2. $\mathcal{L}_{\text{Neu}} = 1 \oplus m \oplus m^2 \oplus m^3$
3. $\mathcal{L}_{\text{Neu}(\mathbb{Z}_2)} = 1 \oplus e^2 \oplus m^2 \oplus e^2 m^2$

The symmetry boundary is $\mathcal{B}^{\text{sym}}_{\mathcal{S}=\mathbb{Z}_4} = \mathcal{L}_{\text{Dir}}$.

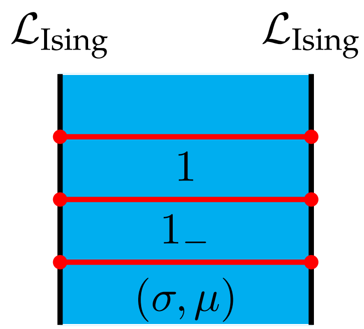
Gapped Phases with $\mathbb{Z}_4$ Symmetry via the SymTFT

$\mathcal{L}_{\text{phys}} = \mathcal{L}_{\mathcal{S}} = \mathcal{L}_{\text{Dir}}$	$\mathcal{L}_{\text{phys}} = \mathcal{L}_{\text{Neu}}$	$\mathcal{L}_{\text{phys}} = \mathcal{L}_{\text{Neu}(\mathbb{Z}_2)}$
		
$\mathbb{Z}_4$ SSB: 4 identical vacua, permuted by $\mathbb{Z}_4$	$\mathbb{Z}_4$ Trivial Phase: single vacuum with $\mathbb{Z}_4$ acting trivially	$\mathbb{Z}_2$ SSB: 2 identical vacua, permuted by $\mathbb{Z}_2$

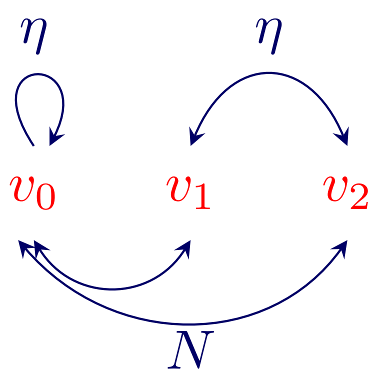
Gapped Phases with Non-Invertible Symmetry: Ising Category

The SymTFT is Ising $\boxtimes\overline{\text{Ising}}$ and there is a unique subset of mutually local anyons (gapped b.c./Lagrangian algebra):

$$\mathcal{L}_{\text{Ising}} = 1 \oplus 1_- \oplus (\sigma, \mu)$$



Resulting in 3=2+1 vacua, with the symmetry acting as

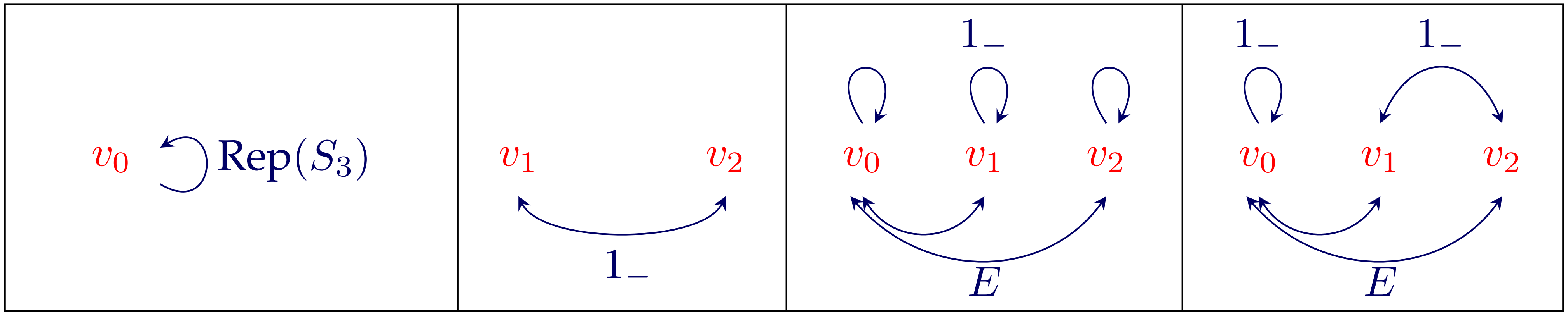


Unique Ising symmetric gapped phase: SSB phase with 3 vacua.

Gapped Phases with Non-Invertible Symmetry: Rep(S_3)

Repeating a similar SymTFT analysis now for the non-invertible symmetry Rep(S_3) ($1, 1_-, E$ irreps) we find four gapped phases:

Rep(S_3) trivial phase	$\mathbb{Z}_2$ SSB	Rep(S_3)/$\mathbb{Z}_2$ SSB	Rep(S_3) SSB



Gapless $\mathcal{S}$-Symmetric Phases and Phase Transitions

Phase Transitions

Consider two gapped $\mathcal{S}$-symmetric phases, how do we determine the $\mathcal{S}$-symmetric phase transitions?

$$\boxed{\mathcal{S} \text{ gapped}} \quad \longleftarrow \quad \boxed{\mathcal{S} \text{ gaplesss}} \quad \longrightarrow \quad \boxed{\mathcal{S} \text{ gapped}'}$$

- Gapped phase: determined by Lagrangians $\mathcal{L}_i$
- Gapless phase transition between $\mathcal{L}_1$ and $\mathcal{L}_2$ is characterized by

  $$\mathcal{A}_{12} = \mathcal{L}_1 \cap \mathcal{L}_2$$

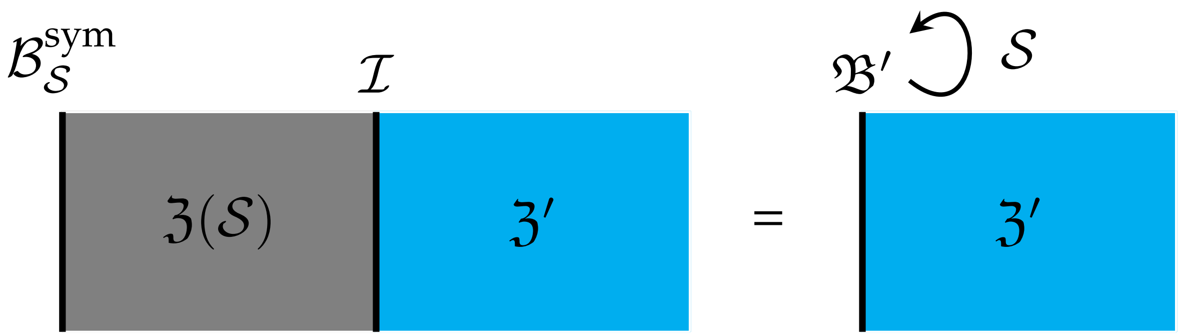
  i.e., a non-maximal set of mutually local topological defects.
- One can also tune and consider $\cap_i \mathcal{L}_i$ for any subset of Lagrangian algebras.

Gapless Phases and Phase Transitions

Now that we have all gapped phases, we expect to also be able to study transitions between gapped phases.

Requires promoting the SymTFT sandwich to a SymTFT **club-sandwich**.

The first step is to consider interfaces between topological orders: $\mathfrak{Z}(\mathcal{S})$ and $\mathfrak{Z}'$: **"club quiche"**



This constructs $\mathcal{S}$-symmetric boundary conditions of the topological oder $\mathfrak{Z}'$.

Condensable Algebras

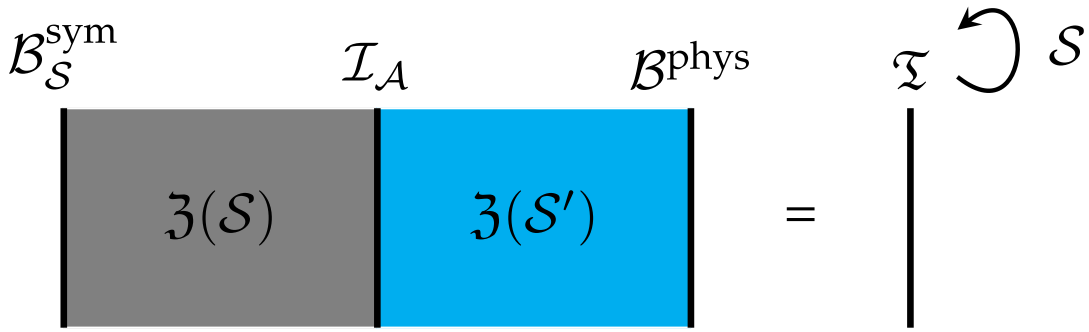
Such interfaces between topological orders are determined by condensable algebras $\mathcal{A}$ in $\mathcal{Z}(\mathcal{S})$:

- Example: $\mathcal{A}$ is Lagrangian: $\mathcal{Z}'$ is trivial
- $\mathcal{A}$ is not maximal, then $\mathcal{Z}(\mathcal{S})/\mathcal{A}$ is a non-trivial topological order $\mathcal{Z}(\mathcal{S}')$ for a reduced symmetry $\mathcal{S}'$.
- Equivalently, condensable algebras can be determined as Lagrangian algebras of the folded topological order $\mathcal{Z}(\mathcal{S}) \boxtimes \overline{\mathcal{Z}(\mathcal{S}')}$.

Club Sandwich and Phase Transitions

Consider the club quiche $\mathcal{S}$, $\mathcal{S}'$ with condensable algebra $\mathcal{A}$.

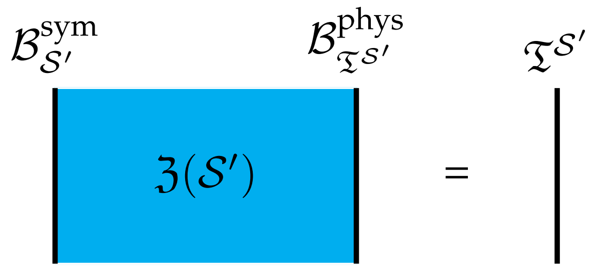
We can close it off with a physical boundary condition on the RHS, resulting in a **"club sandwich"**. The club quiche is a device to map $\mathcal{S}'$-symmetric b.c. of a topological order to $\mathcal{S}$-symmetric theories:



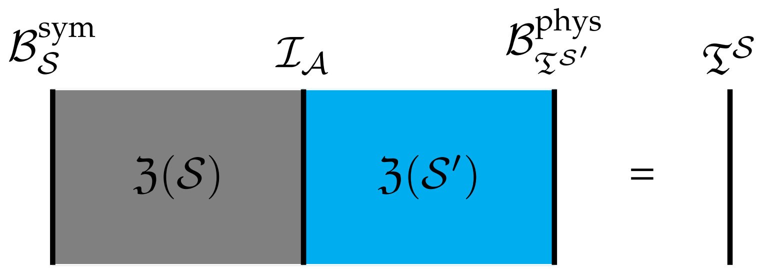
Concretely this can be used to make new phase transitions out of old:

$\Rightarrow$ Kennedy-Tasaki-transformations: $\mathcal{S}'$-symmetric to $\mathcal{S}$-symmetric theories

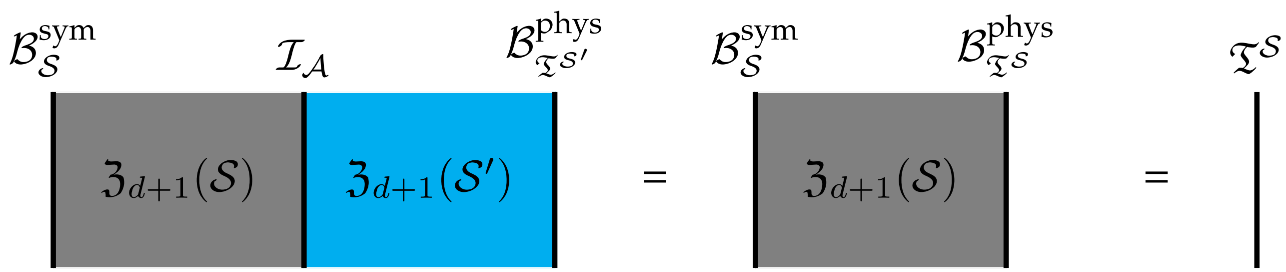
Start with an $\mathcal{S}'$-symmetric theory and its SymTFT:



Attaching the $\mathcal{S}$ to $\mathcal{S}'$ club quiche results in



Here the physical $\mathcal{S}$-symmetric boundary $\mathfrak{T}^{\mathcal{S}}$ is obtained by collapsing the second interval:



New Phase Transitions from Old

Consider an input phase transition between $\mathcal{S}'$-symmetric gapped phases

$$\mathfrak{T}_1^{\mathcal{S}'} \longleftarrow \mathcal{C}_{12}^{\mathcal{S}'} \longrightarrow \mathfrak{T}_2^{\mathcal{S}'}$$

The club sandwich produces a phase transition for the symmetry $\mathcal{S}$, which is the KT transformation of the initial input phase transition:

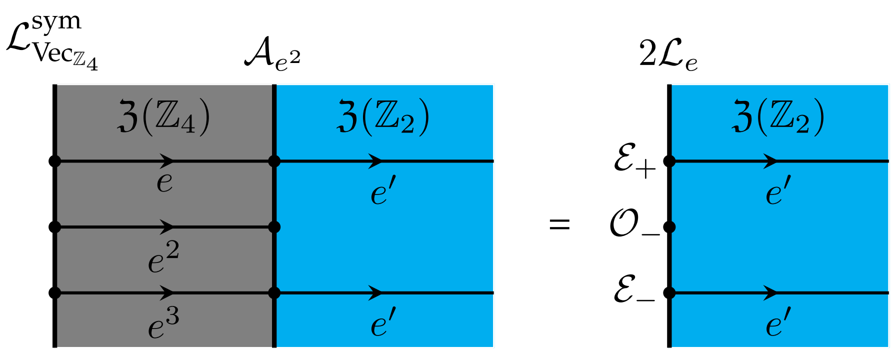
$$\mathfrak{T}_1^{\mathcal{S}} \longleftarrow \mathcal{C}_{12}^{\mathcal{S}} \longrightarrow \mathfrak{T}_2^{\mathcal{S}}$$

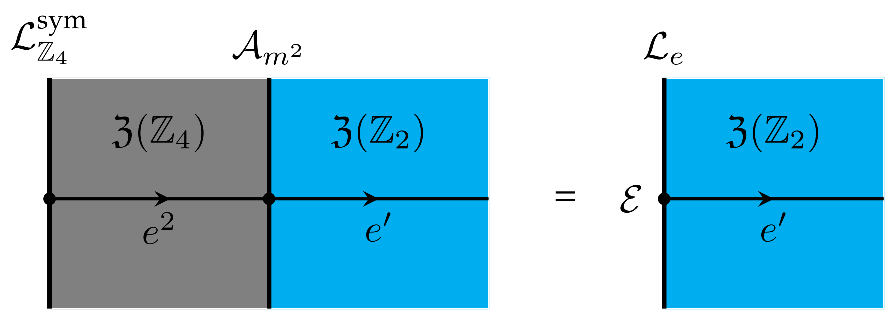
Club Quiches: $\mathbb{Z}_4$

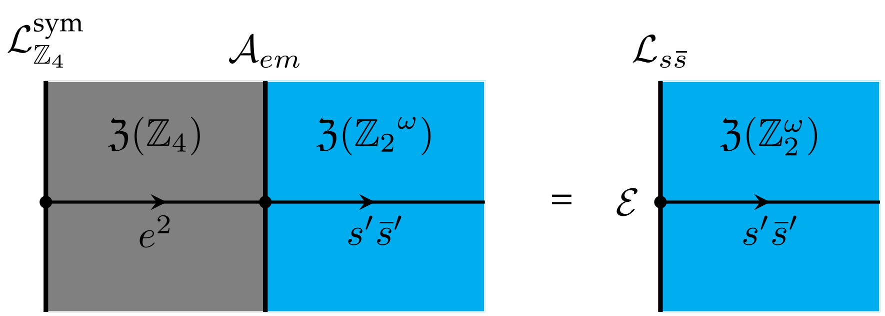
The condensable, not Lagrangian, algebras for $\mathcal{Z}(\mathbb{Z}_4)$ are

$$\mathcal{A}_1 = 1\,,\ \mathcal{A}_{e^2} = 1 \oplus e^2\ \ \mathcal{A}_{m^2} = 1 \oplus m^2\,,\ \mathcal{A}_{e^2m^2} = 1 \oplus e^2 m^2\,.$$

The reduced topological orders are determined from the club quiches:



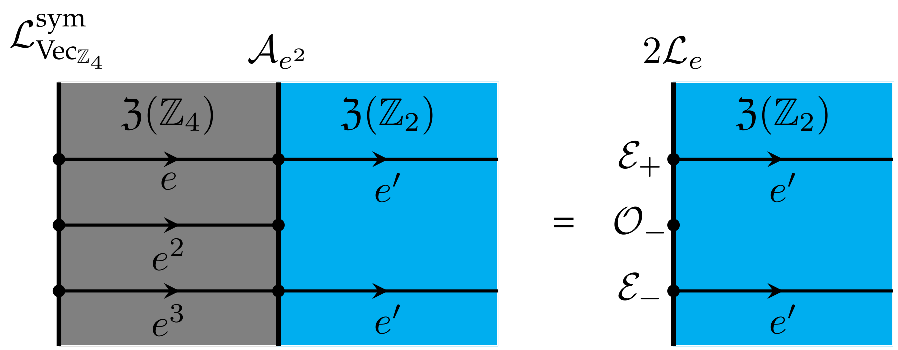




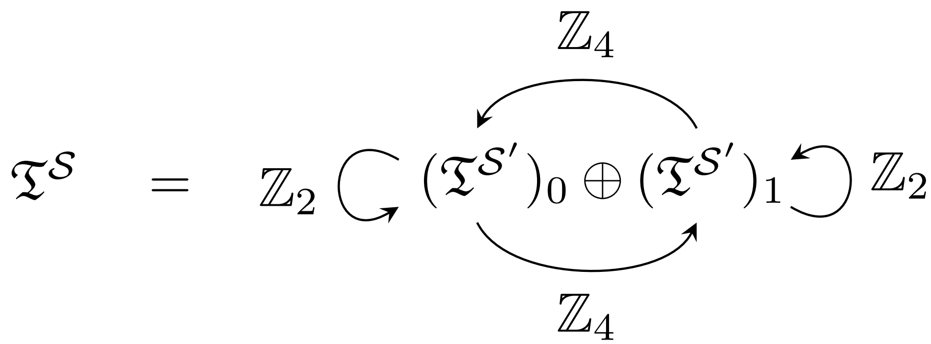
Note: $\mathbb{Z}_4$ acts by line operators on the boundary of $\mathcal{Z}(\mathcal{S}')$. E.g. in the first example it acts by permuting the two boundary conditions: $\mathcal{L}_e \oplus \mathcal{L}_e$.

$\mathbb{Z}_4$ Phase transitions from $\mathbb{Z}_2$

For the condensable algebra $\mathcal{A}_{e^2}$ the club quiche is:



This implies is the $\mathcal{S}$-symmetric gapless phase



$\mathbb{Z}_4$ Phase transitions from $\mathbb{Z}_2$

E.g. for $\mathcal{S}' = \mathbb{Z}_2$ the Ising transition, this constructs a $\mathbb{Z}_4$-symmetric transition

$$\mathfrak{T}^{\mathcal{S}} \quad = \quad \mathbb{Z}_2 \circlearrowright \text{Ising}_0 \underset{\mathbb{Z}_4}{\overset{\mathbb{Z}_4}{\rightleftarrows}} \oplus \, \text{Ising}_1 \circlearrowleft \mathbb{Z}_2$$

which models the transition between $\mathbb{Z}_4$ and $\mathbb{Z}_2$ SSB phases for $\mathbb{Z}_4$ symmetry.

Similarly we find for the $\mathbb{Z}_4$ trivial and $\mathbb{Z}_2$ SSB transition of $\mathbb{Z}_4$:

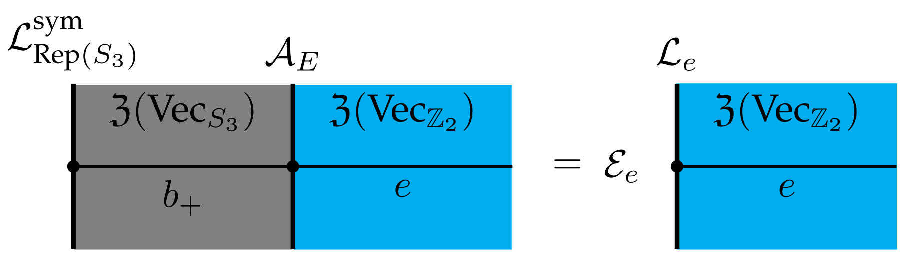
$$\mathfrak{T}^{\mathcal{S}} = \text{Ising} \circlearrowleft \mathbb{Z}_4$$

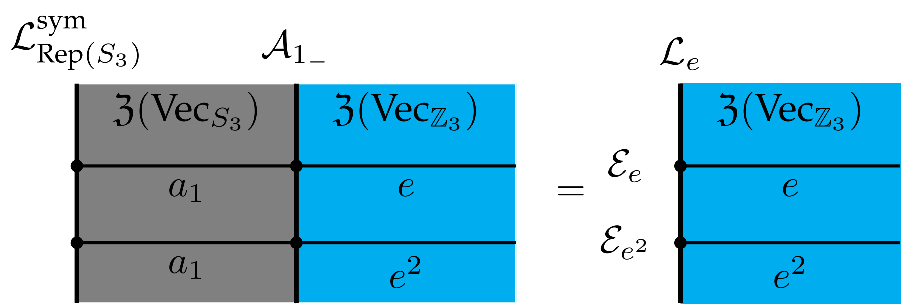
Club Quiches: $\mathrm{Rep}(S_3)$

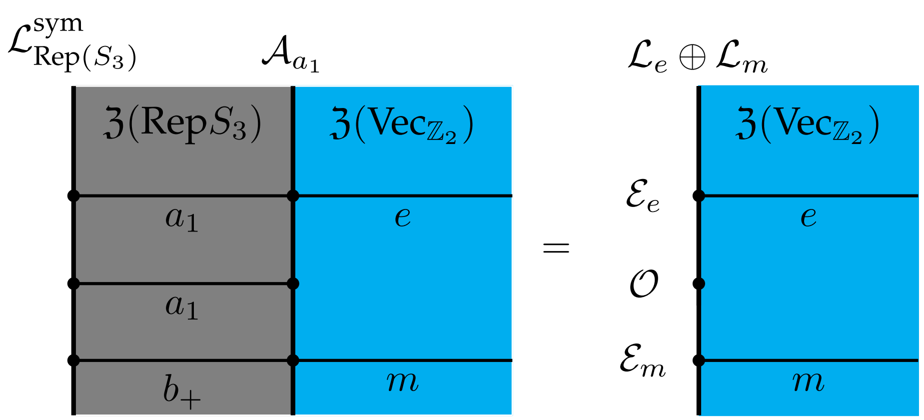
The non-Lagrangian, condensable algebras are

$$\mathcal{A}_{1_-} = 1 \oplus 1_-\,, \quad \mathcal{A}_E = 1 \oplus E\,, \quad \mathcal{A}_{a_1} = 1 \oplus a_1\,.$$

And the reduced topological orders are







$\text{Rep}(S_3)$ Phase transitions from $\mathbb{Z}_3$

For $\text{Rep}(S_3)$ we have input transitions that are $\mathbb{Z}_3$-symmetric, which is the 3-state Potts model.

The $\text{Rep}(S_3)/\mathbb{Z}_2$ SSB – $\text{Rep}(S_3)$ SSB transition is obtained to be:

$$\mathfrak{T}^{\mathcal{S}} \quad = \quad E \circlearrowright \underbrace{\text{Ising}_e \oplus (\text{Ising}_m)_{\sqrt{2}}}_{E} \circlearrowleft 1_-$$

where the $\text{Rep}(S_3)$ acts as

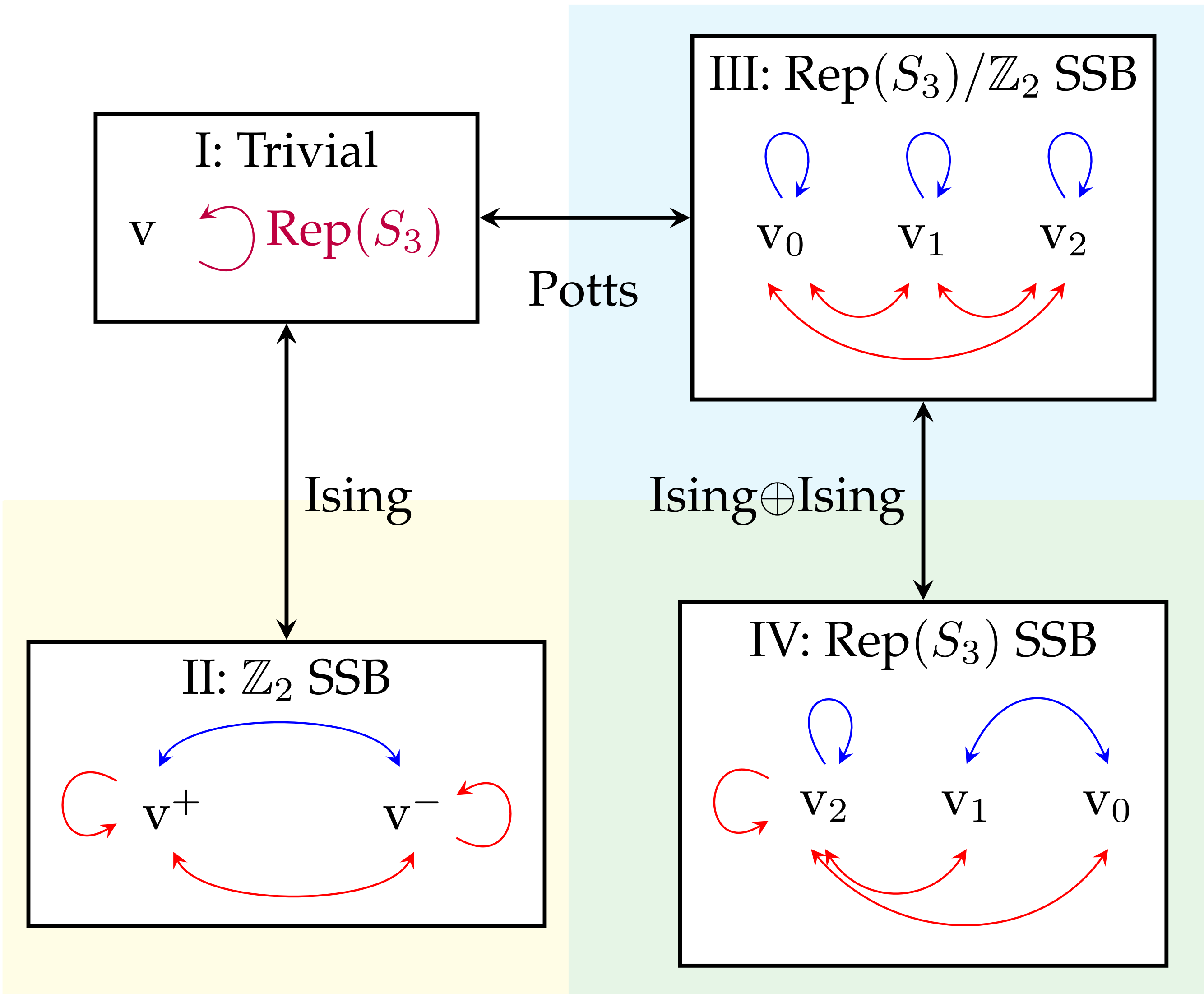
$$1_- = 1_{ee} \oplus \eta_{mm} \,, \quad E = S_{em} \oplus S_{me} \oplus \eta_{ee}$$

For the full list of such transitions see [Bhardwaj, Bottini, Pajer, SSN]

Phase diagram for Rep(S_3) in 2d

Rep(S_3)= $\{1, \sigma, E\}$. Both from continuum and from spin-chain models

[Bhardwaj, Pajer, SSN, Warman][Bhardwaj, Bottini, SSN, Tiwari][Chatterjee, Aksoy, Wen]

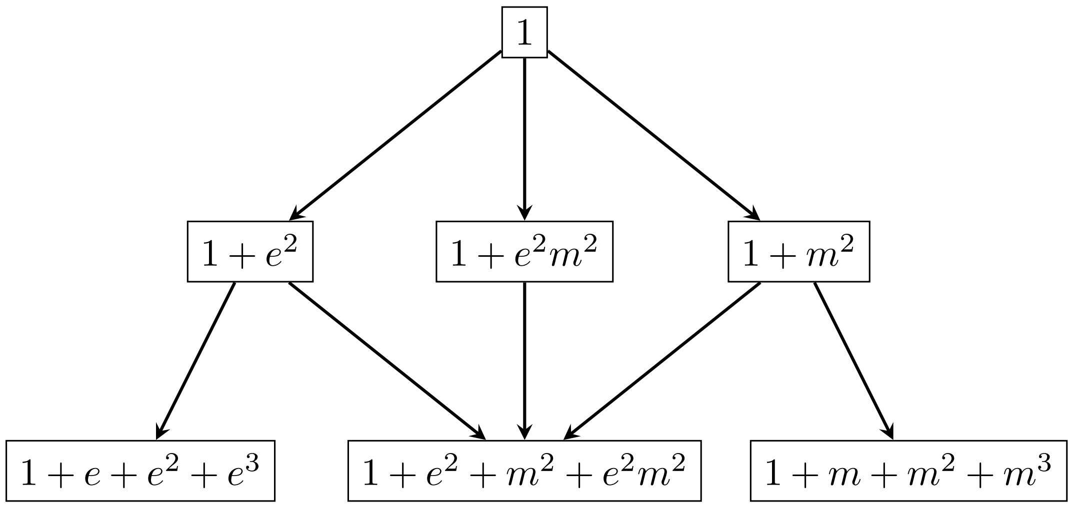


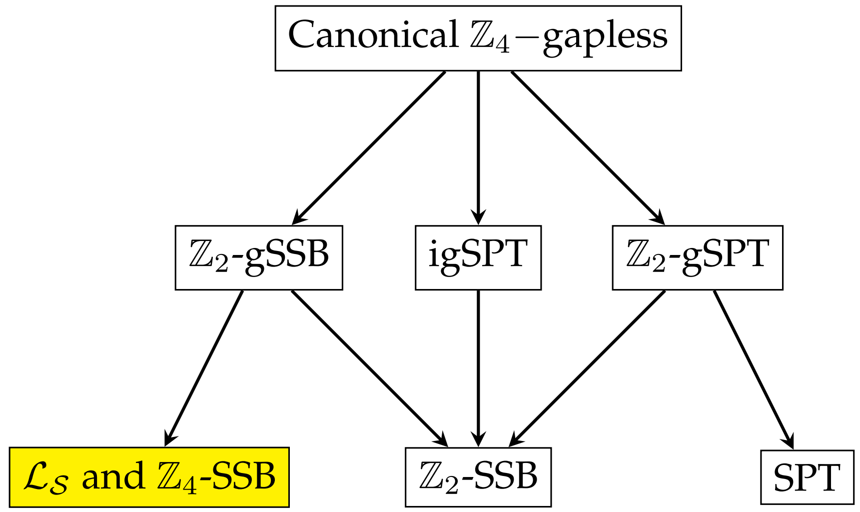
A Roadmap of Phases with Symmetry $\mathcal{S}$

- Construct the SymTFT and Drinfeld center $\mathcal{Z}(\mathcal{S})$
- Determine all condensable algebras and the associated reduced topological orders $\mathcal{Z}(\mathcal{S}')$
- In particular: $\mathcal{L}_1$ and $\mathcal{L}_2$ are Lagrangians, that give rise to gapped phases, then the gapless phase between these is given by $\mathcal{A}_{12} = \mathcal{L}_1 \cap \mathcal{L}_2$.

More generally, there is a partial order on condensable algebras of $\mathcal{Z}(\mathcal{S})$: and thus... a Hasse diagram.

Hasse diagram for Phases of $\mathbb{Z}_4$

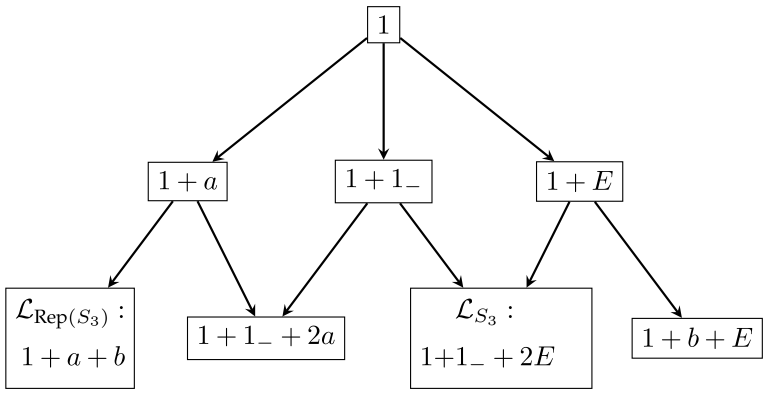


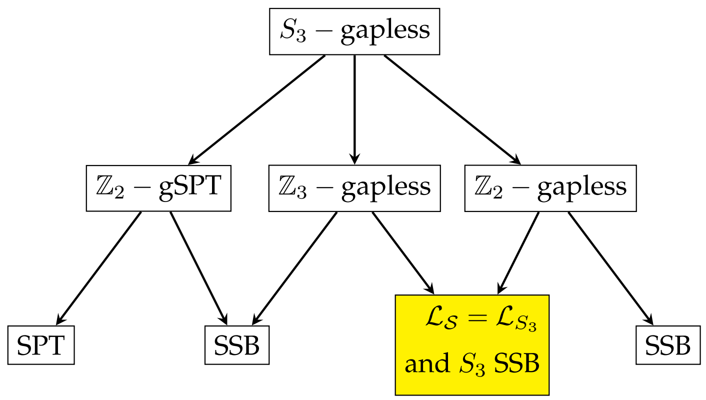


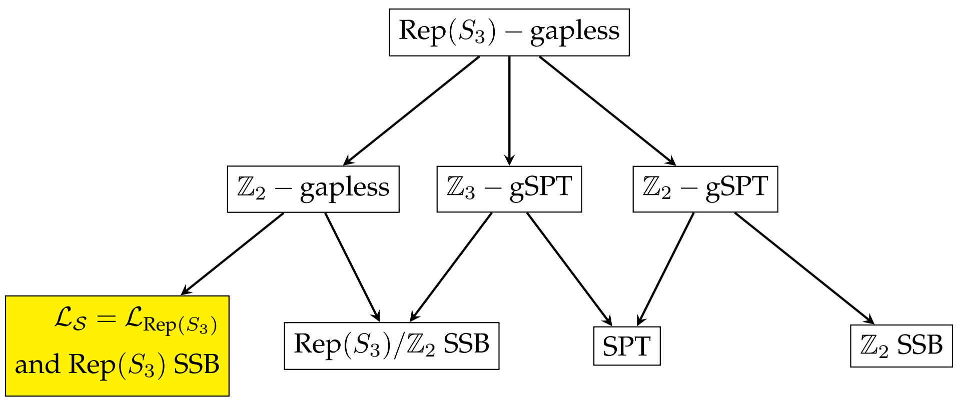
- gSPT (gapless SPT): $\mathcal{A} \cap \mathcal{L}_\mathcal{S} = 1$
- igSPT (intrinsically gapless SPT): gSPT that cannot be deformed to an SPT
- gSSB (gapless SSB): $\mathcal{A} \cap \mathcal{L}_\mathcal{S} \supsetneq 1$
- igSSB (intrinsically gapless SSB): gSSB with n universes, that cannot be deformed to an SSB with n vacua

For $\mathbb{Z}_4$: igSPT was found in [Wen, Potter].
First non-invertible igSPT: Rep(D_{8n}) [Bhardwaj, Pajer, SSN, Warman].

Hasse Diagram for $\mathcal{Z}(\text{Rep}(S_3))$

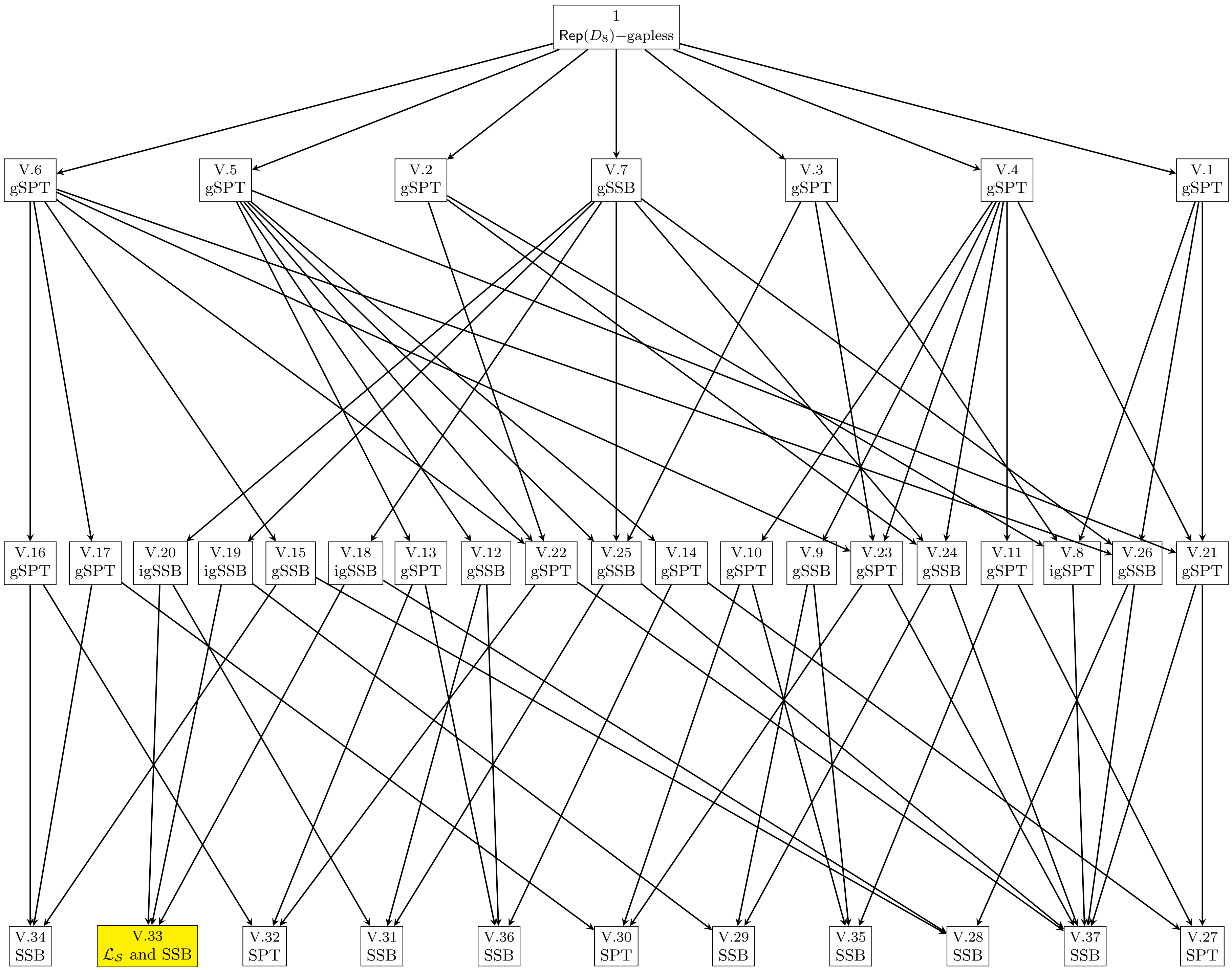






Hasse Diagram for $\mathcal{Z}(\text{Rep}(D_8))$

Dim	Condensable Algebra of $\mathcal{Z}(\mathsf{Rep}(D_8))$ (with label)		Reduced TO $\mathcal{S}'$	Phase for $\mathcal{S} = \mathsf{Rep}(D_8)$	n
1	1	(V.0)	$\mathcal{S}$	$\mathsf{Rep}(D_8)$−gapless	1
2	$1 \oplus e_{RG}$	(V.1)	$\mathbb{Z}_4$	gSPT	1
2	$1 \oplus e_{GB}$	(V.2)	$\mathbb{Z}_4$	gSPT	1
2	$1 \oplus e_{RB}$	(V.3)	$\mathbb{Z}_4$	gSPT	1
2	$1 \oplus e_R$	(V.4)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	gSPT	1
2	$1 \oplus e_G$	(V.5)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	gSPT	1
2	$1 \oplus e_B$	(V.6)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	gSPT	1
2	$1 \oplus e_{RGB}$	(V.7)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_{GB} \oplus e_{RB} \oplus e_{RG}$	(V.8)	$\mathbb{Z}_2^\omega$	igSPT	1
4	$1 \oplus e_R \oplus m_{GB}$	(V.9)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_R \oplus m_G$	(V.10)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_R \oplus m_B$	(V.11)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_G \oplus m_{RB}$	(V.12)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_G \oplus m_R$	(V.13)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_G \oplus m_B$	(V.14)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_B \oplus m_{RG}$	(V.15)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_B \oplus m_R$	(V.16)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_B \oplus m_G$	(V.17)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_{RGB} \oplus m_{RG}$	(V.18)	$\mathbb{Z}_2$	igSSB	3
4	$1 \oplus e_{RGB} \oplus m_{GB}$	(V.19)	$\mathbb{Z}_2$	igSSB	3
4	$1 \oplus e_{RGB} \oplus m_{RB}$	(V.20)	$\mathbb{Z}_2$	igSSB	3
4	$1 \oplus e_G \oplus e_R \oplus e_{RG}$	(V.21)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_B \oplus e_G \oplus e_{GB}$	(V.22)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_B \oplus e_R \oplus e_{RB}$	(V.23)	$\mathbb{Z}_2$	gSPT	1
4	$1 \oplus e_{GB} \oplus e_R \oplus e_{RGB}$	(V.24)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_G \oplus e_{RB} \oplus e_{RGB}$	(V.25)	$\mathbb{Z}_2$	gSSB	2
4	$1 \oplus e_B \oplus e_{RG} \oplus e_{RGB}$	(V.26)	$\mathbb{Z}_2$	gSSB	2
8	$1 \oplus e_G \oplus e_R \oplus e_{RG} \oplus 2m_B$	(V.27)	trivial	SPT	1
8	$1 \oplus e_B \oplus e_{RG} \oplus e_{RGB} \oplus 2m_{RG}$	(V.28)	trivial	SSB	4
8	$1 \oplus e_{GB} \oplus e_R \oplus e_{RGB} \oplus 2m_{GB}$	(V.29)	trivial	SSB	4
8	$1 \oplus e_B \oplus e_R \oplus e_{RB} \oplus 2m_G$	(V.30)	trivial	SPT	1
8	$1 \oplus e_G \oplus e_{RB} \oplus e_{RGB} \oplus 2m_{RB}$	(V.31)	trivial	SSB	4
8	$1 \oplus e_B \oplus e_G \oplus e_{GB} \oplus 2m_R$	(V.32)	trivial	SPT	1
8	$1 \oplus e_{RGB} \oplus m_{GB} \oplus m_{RB} \oplus m_{RG}$	(V.33)	trivial	$\mathcal{L}_\mathcal{S}$ and SSB	5
8	$1 \oplus e_B \oplus m_G \oplus m_R \oplus m_{RG}$	(V.34)	trivial	SSB	2
8	$1 \oplus e_R \oplus m_B \oplus m_G \oplus m_{GB}$	(V.35)	trivial	SSB	2
8	$1 \oplus e_G \oplus m_B \oplus m_R \oplus m_{RB}$	(V.36)	trivial	SSB	2
8	$1 \oplus e_B \oplus e_G \oplus e_{GB} \oplus e_R \oplus e_{RB} \oplus e_{RG} \oplus e_{RGB}$	(V.37)	trivial	SSB	2


1
Rep(D_8)−gapless
V.6 gSPT
V.5 gSPT
V.2 gSPT
V.7 gSSB
V.3 gSPT
V.4 gSPT
V.1 gSPT
V.16 gSPT
V.17 gSPT
V.20 igSSB
V.19 igSSB
V.15 gSSB
V.18 igSSB
V.13 gSPT
V.12 gSSB
V.22 gSPT
V.25 gSSB
V.14 gSPT
V.10 gSPT
V.9 gSSB
V.23 gSPT
V.24 gSSB
V.11 gSPT
V.8 igSPT
V.26 gSSB
V.21 gSPT
V.34 SSB
V.33 $\mathcal{L}_\mathcal{S}$ and SSB
V.32 SPT
V.31 SSB
V.36 SSB
V.30 SPT
V.29 SSB
V.35 SSB
V.28 SSB
V.37 SSB
V.27 SPT

Classification of Phases

Two key distinctions:

- Gapped versus gapless: energy gap $\Delta > 0$ or $\Delta = 0$
- SPT-ness (gapless or gapped): symmetry gap $\Delta_{\mathcal{S}} > 0$ or $\Delta_{\mathcal{S}} = 0$. The symmetry gap $\Delta_{\mathcal{S}} > 0$ means, that not all charges of $\mathcal{S}$ are realized in the IR phase, i.e. some $\mathcal{S}$-charges are confined. They are realized as excited states, that enter the spectrum at $\Delta_{\mathcal{S}}$.

Note: $\Delta_{\mathcal{S}} \geq \Delta$.

Number of universes/vacua: n, which is 1 for SPTs (gapless or gapped) and $n > 1$ for SSB.

Finally: whether or not an $\mathcal{S}$-symmetric phase can be deformed to another $\mathcal{S}$-symmetric phase may imply the symmetry is protected ("symmetry protected criticality"). This is the distinction between gSPT and igSPT (intrinsic) and gSSB and igSSB.

Classification of Phases

Phase	Physical characterization	Energy gap Δ Symmetry gap $\Delta_{\mathcal{S}}$	Condition on $\mathcal{A}$ in (1+1)d	n
SPT	Gapped system with energy gap $\Delta > 0$. IR: trivial TQFT. $\mathcal{S}$-charges confined in IR appear at an energy scale (symmetry gap) $\Delta_{\mathcal{S}} \geq \Delta > 0$. Order parameters (OPs) are all of string type (i.e. in twisted-sectors for $\mathcal{S}$).	$\Delta > 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} = \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} = 1$	1
gSPT	Gapless system with $\Delta = 0$ and a unique ground state on circle. Not all charges of $\mathcal{S}$ appear in IR. The confined charges appear at a symmetry gap $\Delta_{\mathcal{S}} > 0$. OPs are all of string type.	$\Delta = 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} \neq \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} = 1$	1
igSPT	A gSPT phase that cannot be deformed to a gapped SPT phase, because it has confined charges not exhibited by any of the gapped SPTs.	$\Delta = 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} \neq \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} = 1$	1
SSB	Gapped system with n degenerate vacua (labeled by i) permuted by $\mathcal{S}$ action. Each vacuum i has energy gap $\Delta^{(i)} > 0$. Going from i to j costs $\Delta^{(ij)} > 0$. Not all charges realized in IR $\implies$ symmetry gap $\Delta_{\mathcal{S}} > 0$. OPs are multiplets with string and non-string type.	$\Delta^{(i)} > 0$ $\Delta^{(ij)} > 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} = \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} \supsetneq 1$	> 1
gSSB	Gapless system with n degenerate gapless universes labeled by i. Each universe has a unique ground state on a circle. Going from i and j costs $\Delta^{(ij)} > 0$. Not all charges realized in IR $\implies$ symmetry gap $\Delta_{\mathcal{S}} > 0$. OPs string and non-string type	$\Delta^{(i)} = 0$ $\Delta^{(ij)} > 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} \neq \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} \supsetneq 1$	> 1
igSSB	A gSSB phase with n universes that cannot be deformed to a gapped SSB phase with n vacua.	$\Delta^{(i)} = 0$ $\Delta^{(ij)} > 0$ $\Delta_{\mathcal{S}} > 0$	$\mathcal{A} \neq \mathcal{L}$ $\mathcal{A} \cap \mathcal{L}_{\mathcal{S}} \supsetneq 1$	> 1

Δ is the energy gap. $\Delta_{\mathcal{S}}$ the symmetry gap: not all $\mathcal{S}$-charges are realized in the IR. The missing/confined charges are realized by excited states. The symmetry gap $\Delta_{\mathcal{S}}$, is the energy of the first excited state carrying one of the confined charges. The symmetry becomes less faithful going downwards.

A Roadmap of Phases with Symmetry $\mathcal{S}$

- Construct the SymTFT and its topological defects.
- Determine all condensable algebras of topological defects.
- In particular: $\mathcal{L}_1$ and $\mathcal{L}_2$ are Lagrangians, that give rise to gapped phases, then the gapless phase between these is given by $\mathcal{A}_{12} = \mathcal{L}_1 \cap \mathcal{L}_2$.
- SymTFT encodes the order parameters and symmetry implementation.

Results in new phases with non-invertible symmetries, e.g. found non-invertible SPTs and igSPTs for $\text{Rep}(D_{8n})$.

Crucially, this is applicable to any fusion category symmetry.

Conclusions and Open Questions

The field of categorical symmetries has seen enormous progress in the last years, in string/high-energy theory, condensed matter and math, with lots of synergies.

In view of the applications to phases of matter, there are many open questions, e.g.:

1. Classification of symmetric phases: 3d and 4d where the full structure of higher fusion categories will need to be tapped in [wip Oxford]

2. gSPT, igSPT, gSSB, igSSB phases in higher dimensions: QFT examples? [Antinucci, Copetti, SSN, wip]
   gSPTs in 4d [Dumitrescu, Hsin]

3. Extension of this framework of SymTFT, gapped, gapless phases to non-semisimple categories, and continuous symmetries.

https://sites.google.com/view/symmetries2024/home

https://www.kitp.ucsb.edu/activities/gensym25