Small complex structure degeneration of Calabi-Yau metrics

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This talk is based on arXiv:1906.03368, which was joint work with Ruobing Zhang (Princeton).

A related baby case was studied in arXiv:1807.09367, which was joint work with H. Hein (Munster), R. Zhang (Princeton), J. Viaclovsky (UC Irvine).

This talk is concerned with Calabi-Yau

This talk is concerned with Calabi-Yau metrics.

Small complex structure degenerations:

Consider a family of hypersurfaces in $\mathbb{CP}^{n+1}(n \geq 2)$ given by

$$X_t = \{t \cdot Q(z) + f_1(z) \cdot f_2(z) = 0\},$$

where Q, f_1, f_2 are generic homogeneous polynomials with

$$d_1 = \deg(f_1) > 0, \quad d_2 = \deg(f_2) > 0$$

and

$$\deg Q = d_1 + d_2 = n + 2.$$

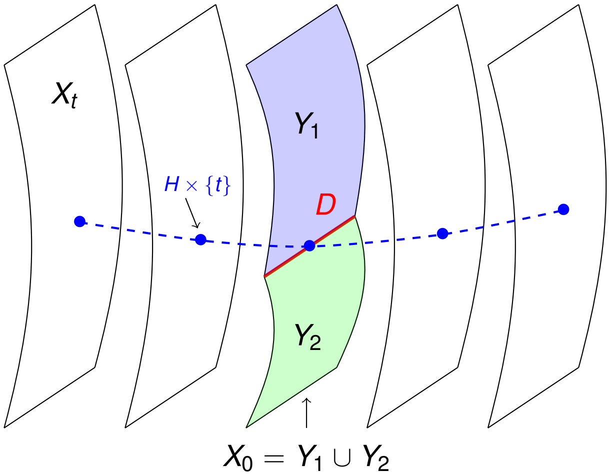
Picture for the algebraic degeneration

$$X_t = \{t \cdot Q(z) + f_1(z) \cdot f_2(z) = 0\}$$

$$Y_1 = \{f_1 = 0\},\ Y_2 = \{f_2 = 0\}$$

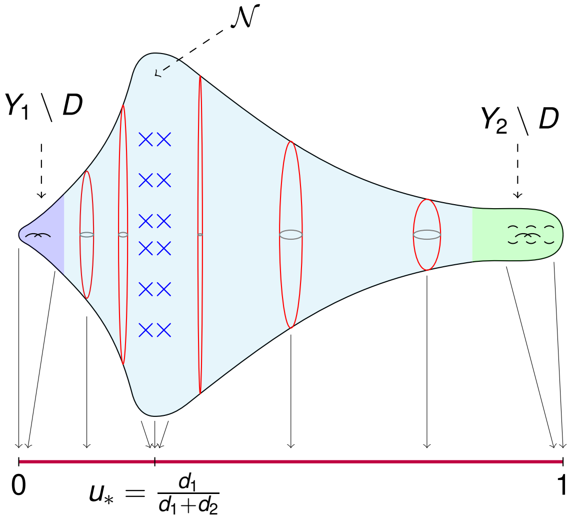
$$D = \{f_1 = f_2 = 0\}$$

$$H = \{Q = f_1 = f_2 = 0\}$$



Yau's proof of the Calabi conjecture yields a unique Calabi-Yau metric ω_t on X_t(for $0 < |t| \ll 1$), normalized with diameter 1, in the cohomology class proportional to the Fubini-Study metric.

Theorem (S.-Zhang 2019): Picture for the Calabi-Yau metrics (X_t, ω_t) as $t \to 0$:



A Calabi-Yau metric is by definition a Riemannian metric with holonomy group contained in $SU(n)$. In particular, they are Ricci-flat.

This is the same as giving a pair (Ω, ω) on a complex manifold X, where Ω is a holomorphic n-form and ω is a Kähler form, with

$$\omega^n = C\Omega \wedge \bar{\Omega}. \qquad (*)$$

Locally $(*)$ takes the form $\det(\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}) = e^{\psi}$. "Non-linear Laplace equation"

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Ω encodes the complex/algebraic data and often can be written down explicitly in terms of the defining equations.

When X is compact Yau's theorem gives a unique ω solving $(*)$ which is co-homologuous to a given Kähler form.

Standard example: $\{F = 0\} \subset \mathbb{CP}^{n+1}$ for F generic homogeneous degree $n + 2$

$\rightsquigarrow$ Calabi-Yau metric ω_F in the cohomology class of the Fubini-Study metric

When F degenerates ω_F can become singular.

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When F degenerates ω_F can become singular.

Example of degenerations: $F_t = t \cdot Q + f_1 \cdots f_k = 0$ for $Q, f_1, \cdots, f_k$ general.

$k = n + 2 \rightsquigarrow$ large complex structure degeneration

$n \geq 2$, $k = 2 \rightsquigarrow$ small complex structure degeneration

Main question:

relate the singularity formation of ω_F to the algebraic degeneration of F.

Picture for the algebraic degeneration

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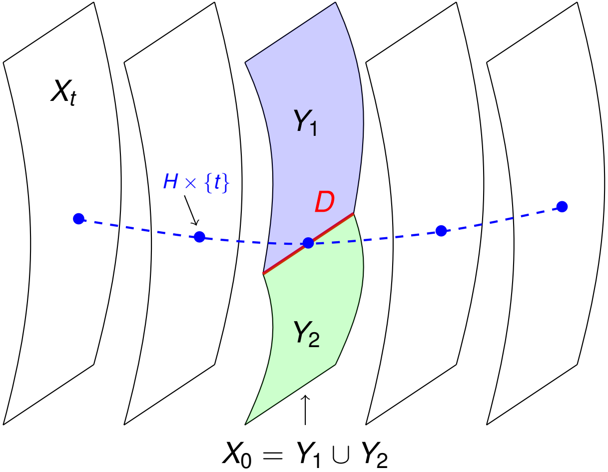


Gromov-Hausdorff convergence:

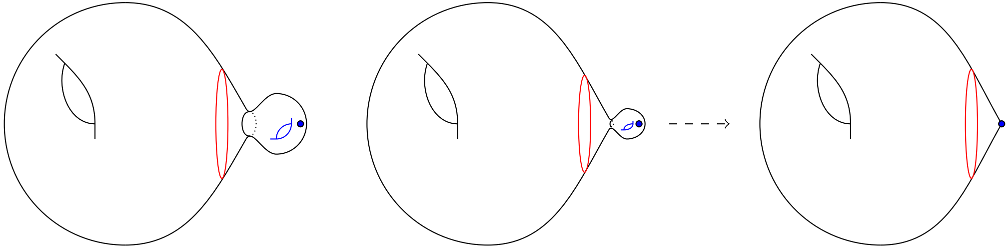
a sequence of compact metric spaces $(M_j, d_j) \xrightarrow{GH} (M_\infty, d_\infty)$ if

there are $\epsilon_j \to 0$ and maps $\phi_j : M_j \to M_\infty$, $\psi_j : M_\infty \to M_j$ which are

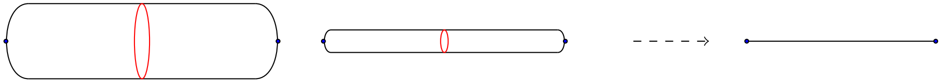
(1). ϵ_j-onto: $M_\infty = B_{\epsilon_j}(\phi_j(M_j))$ and similarly for ψ_j.

(2). ϵ_j-isometric: $|d(\phi_j(x), \phi_j(y)) - d(x, y)| \leq \epsilon_j$ and similarly for ψ_j.

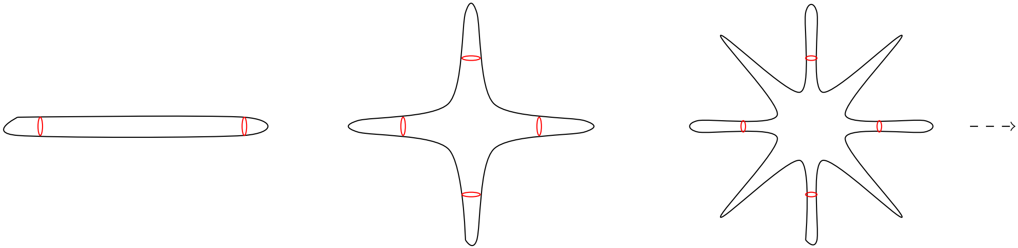
Example



Example



Non-Example:



no GH limit

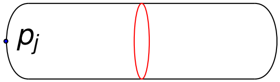
In our case the metrics (X_t, ω_t) are Ricci-flat. This gives pre-compactness in the GH topology.

Bishop-Gromov inequality: $\mathrm{Vol}(B(p, R))/\omega_n R^{2n}$ is decreasing in R.

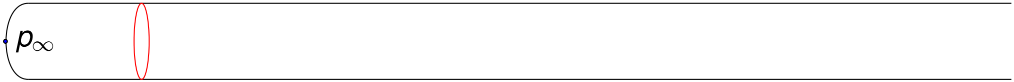
This allows a uniform approximation of (X_t, ω_t) by finite metric spaces in all scales.

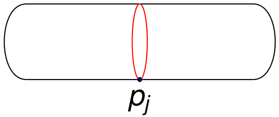
Calabi-Yau metrics admit natural rescalings. The GH limits are sensitive to the scale.

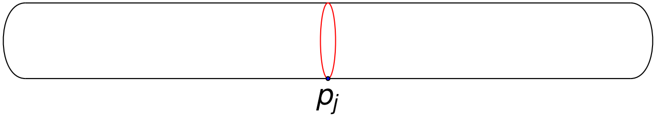
In general for complete metric spaces we say $(M_j, d_j, p_j) \xrightarrow{GH} (M_\infty, d_\infty, p_\infty)$ if for all $R > 0$, $B(p_j, R) \xrightarrow{GH} B(p_\infty, R)$.

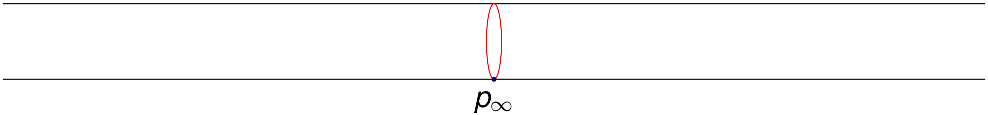

p_j


p_j


p_∞


p_j


p_j

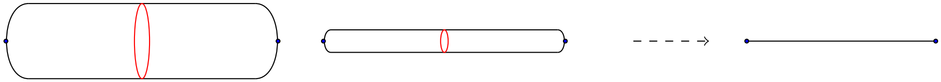

p_∞

The holomorphic form Ω_t yields a probability measure μ_t on X_t, with density given by $C_t \Omega_t \wedge \bar{\Omega}_t$.

Under the GH convergence μ_t also converges to *renormalized limit measures* μ_∞ on the limit.

The limit metric measure space is an RCD space with $Ric \geq 0$ (Cheeger-Colding theory)

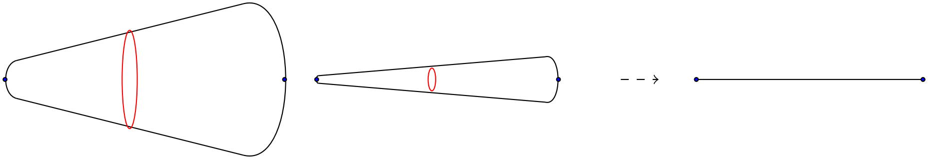
Example (2)



The GH limit $g = dr^2$.

The renormalized limit measure $d\mu_\infty = dr$.

Example (2)



The GH limit $g = dr^2$.

The renormalized limit measure $d\mu_\infty = rdr$.

Back to our degeneration family of Calabi-Yau metrics.

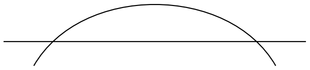
We can talk about GH limits of (X_t, ω_t). In our setting it is known that the volume is *collapsing* as $t \to 0$ (V.Toasatti).

We can also talk about the rescaled GH limits of $(X_t, \lambda_t \omega_t, p_t)$ for $p_t \in X_t$, $\lambda_t \to \infty$. We see more refined structure in smaller scales.

Toy example: the case $n = 1$. This is different from the case $n > 1$!!!

The intersection D is not connected.

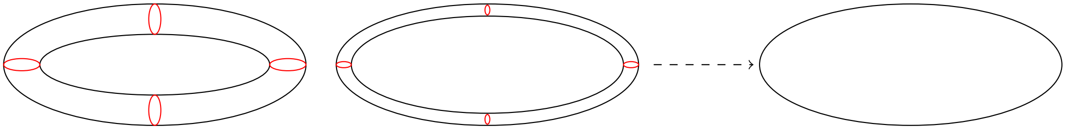
Algebraic picture of X_0:



ω_t is the flat metric on an elliptic curve.

As $t \to 0$, the only possible GH limit is a circle S^1 of unit diameter. This is topologically the same as the dual intersection complex of the degeneration.

The collapsing is along a smooth S^1 fibration.

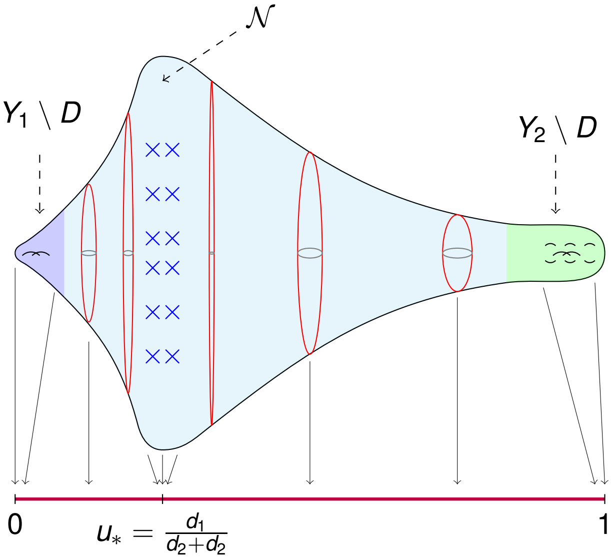


The renormalized limit measure is the standard volume measure.

The rescaled GH limits are flat cylinders $S^1 \times \mathbb{R}$. This can be identified with the components of the smooth locus of X_0.

(X_t, ω_t) can be recovered from gluing of flat cylinders.

$n \geq 2$ case is different



Theorem (precise statement):

(1) GH limit of (X_t, ω_t) is the flat interval $[0, 1]$. Topologically it agrees with the dual intersection complex of X_0

(2) For $|t| \ll 1$ the geometry of ω_t is approximated by 3 building blocks, including 2 complete Calabi-Yau metrics constructed by Tian-Yau 1990 and 1 neck region $\mathcal{N}$.

(3) Multi-scale collapsing: away from $\{0, u_*, 1\}$, the collapsing is along a smooth fiber bundle, where each fiber is an S^1 bundle over D. The circles collapse in a faster rate than that of the base D.

(4) The singular fiber at u_* is an S^1 fibration pinched along H. Suitable rescaled Gromov-Hausdorff limits are given by $\mathbb{C}^{n-2}\times$ Taub-NUT.

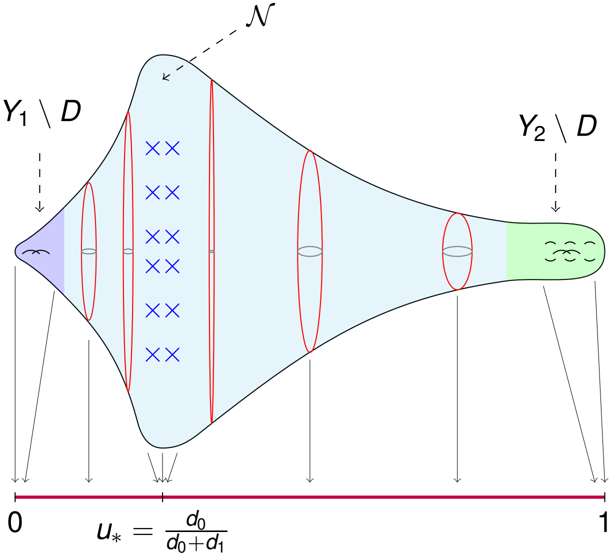
The renormalized limit measure is given by

$$d\mu_\infty = C(\frac{u}{d_1})^{\frac{n-1}{n+1}}$$

when $u \in [0, \frac{d_1}{d_1+d_2}]$;

$$d\mu_\infty = C(\frac{1-u}{d_2})^{\frac{n-1}{n+1}}$$

when $u \in [\frac{d_1}{d_1+d_2}, 1]$.


$\mathcal{N}$
$Y_1 \setminus D$
$Y_2 \setminus D$
0
$u_* = \frac{d_0}{d_0+d_1}$
1

Tian-Yau metrics:

Y^n: Fano manifold, D: smooth anti-canonical divisor with normal bundle L.
$\rightsquigarrow$ D is Calabi-Yau with (Ω_D, ω_D).

A neighborhood of D is approximated by a neighborhood of the zero section $\mathbf{0}$ in L. The latter is equipped with a hermitian metric h with curvature ω_D.

On $L \setminus \{\mathbf{0}\}$ there is a natural holomorphic n-form $\Omega = p^*\Omega_D \wedge \frac{d\zeta}{\zeta}$, where $p : L \to D$ and ζ is the linear coordinate on the fibers. It is invariant under the S^1 action.

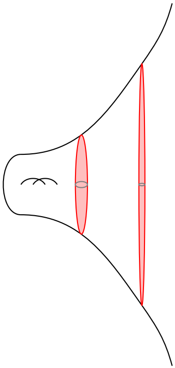
Calabi ansatz gives an S^1 invariant Calabi-Yau metric

$$\omega = \sqrt{-1}\partial\bar{\partial}(-\log|s|_h^2)^{\frac{n+1}{n}}$$

on the subset $|s|_h < 1$.

Using the Calabi model at infinity and extending Yau's proof to the non-compact setting, Tian-Yau constructed a complete Calabi-Yau metric on $Y \setminus D$, asymptotic to ω.

It is not clear that the Tian-Yau metric is canonical or unique on $Y \setminus D$.



Construction of the neck region.

Recall $D = \{f_1(z) = f_2(z) = 0\} \subset X_0$.

Near D, for $|t| \ll 1$, the degenerating family X_t is approximated by the subvariety $\mathcal{N}_t$ in the total space of the bundle $L_1 \oplus L_2$ over D cut-out by the equation

$$s_1 \otimes s_2 + t \cdot q(x) = 0,$$

where L_j is the normal bundle of D in Y_j, $x \in D$, $(s_1, s_2) \in (L_1 \oplus L_2)|_x$ and $q = Q|_D$.

The projection map $\pi_t : \mathcal{N}_t \to D$ is a $\mathbb{C}^*$ fibration, singular over $H = \{Q = 0\} \subset D$.

There is a natural $\mathbb{C}^*$ invariant holomorphic n-form on $\mathcal{N}_t$.

It is natural to ask for S^1 invariant Calabi-Yau metric on $\mathcal{N}_t$ (necessarily incomplete).

Baby case $n = 2$, then Calabi-Yau metrics are hyperkähler ($SU(2) = Sp(1)$).

S^1 invariant Calabi-Yau metrics $\rightsquigarrow$ Gibbons-Hawking ansatz, the equation becomes the (exactly!) the linear Laplace equation.

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Away from the fixed points, the metric is given by

$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

for $V > 0$ harmonic function on $\Omega \subset \mathbb{R}^3 = \mathbb{C}_{w=x+iy} \oplus \mathbb{R}_z$ and θ is the $U(1)$ connection with curvature $*dV$.

Projection to the w-plane realizes the space as a holomorphic fibration.

The local structure near a fixed point corresponds to V having singularities of the form $V = \frac{1}{2r} + C^\infty$.

$\Omega = \mathbb{R}^3$, $V = \frac{1}{2r} \rightsquigarrow$ flat metric on $\mathbb{C}^2$ with projection map $(z_1, z_2) \mapsto z_1 z_2$

$V = \frac{1}{2r} + 1 \rightsquigarrow$ non-flat Taub-NUT metric on $\mathbb{C}^2$

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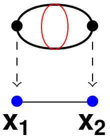
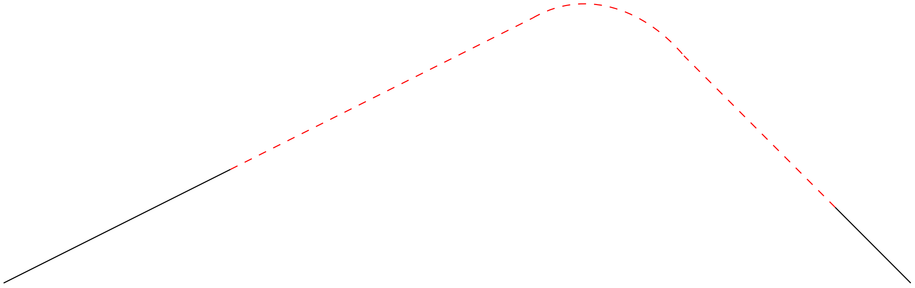
Linear superposition $\rightsquigarrow$ Calabi-Eguchi-Hanson metric



$\Omega = T^2 \times [1, \infty)$, $V = z$ $\quad \rightsquigarrow$ Calabi model space when $n = 2$

We need to interpolate two linear functions.



On $\Omega = T^2 \times (-\infty, \infty)$, take V to be the electric potential of k points

$$\Delta V = 2\pi \sum_{l=1}^{k} \delta_{p_l}$$

Let $v = \int_{T^2 \times \{z\}}$, then $\partial_z^2 v = 2\pi \sum_{l=1}^{k} \delta_{z_l} \implies V \sim -k\pi|z|$ for $|z| \gg 1$.

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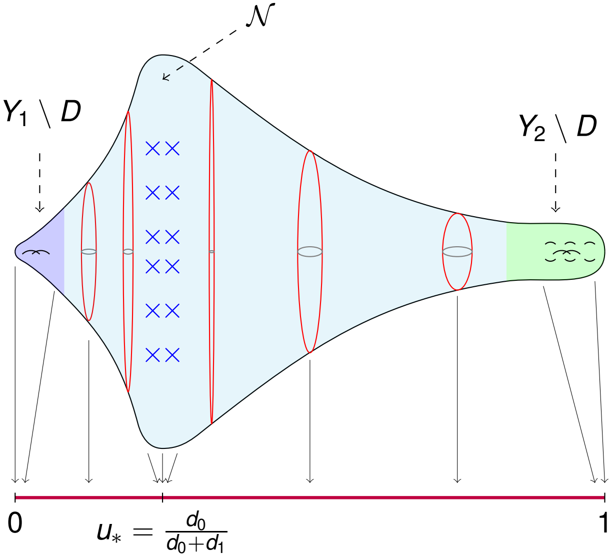
For $T \gg 1$, $m \in \mathbb{Z}$, $V + m\pi z + T$ defines a Calabi-Yau metric with two boundaries, on the conic bundle over T^2 with k singular fibers.

This is the cousin of the Ooguri-Vafa metric when $\Omega = S^1 \times \mathbb{R}^2$, which is important in the work of Gross-Wilson 2000 on the collapsing of Calabi-Yau metrics on elliptic fibered K3 surfaces.



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When $n > 2$, there are generalizations of Gibbons-Hawking ansatz by Matessi and Zharkov, but the equations are still non-linear.

The idea is to consider the adiabatic limit when fibers are small and use the linear equation as an approximation.

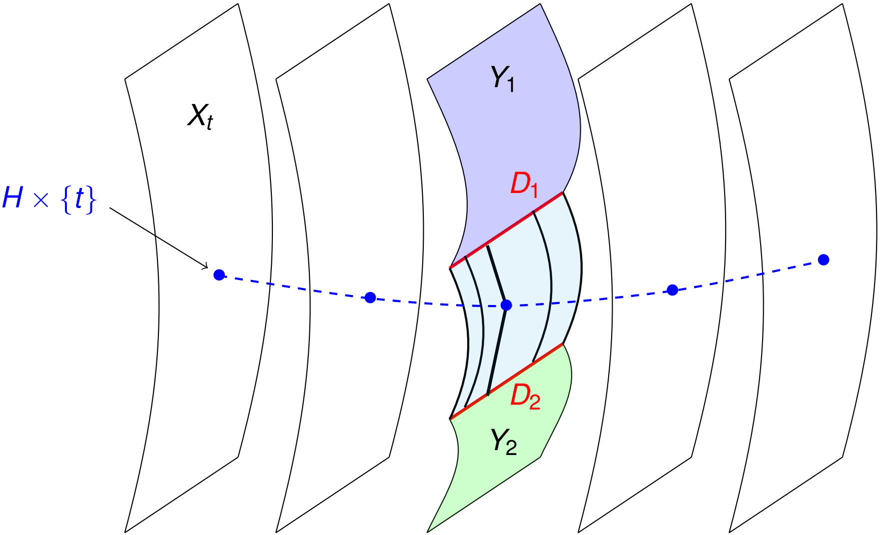
Green currents:

$H \subset D$ smooth divisor, look for a 3-current ψ on $D \times (-\infty, \infty)$ such that

$$\Delta\psi = 2\pi \cdot \delta_P,$$

where $P = H \times \{0\}$.

To perform a gluing construction, needs to extract a new component in terms of algebraic geometry



The result gives a possible inductive description of the geometry of Calabi-Yau metrics for special class of degenerations.

Conjecture:

(1) (generalizing Gross-Wilson/Kontsevich-Soibelman)

Gromov-Hausdorff limits (differential geometry)

$\longleftrightarrow$

essential skeleton (non-Archimedean geometry)

There are recent progress by Yang Li and others which gives information about the geometry in the generic region, for certain classes of degenerations (including examples of large complex structure degenerations).

Conjecture:

(2) Rescaled Gromov-Hausdorff limits (differential geometry)

$\longleftrightarrow$

Algebro-geometric limits (algebraic geometry)

There are progress by Yuji Odaka in the algebro-geometric aspect.

Some related further results:

Hein-S.-Viaclovsky-Zhang 2021: Complete Calabi-Yau metrics asymptotic to the Calabi model are all given by the generalized Tian-Yau-Hein construction.

When $n = 2$, these are ALH* hyperkähler gravitational instantons (complete hyperkähler with $\int |Rm|^2 < \infty$).

S.-Zhang 2021: all hyperkähler gravitational instantons are known to be asymptotic to a model end of type $ALE, ALF, ALG, ALH, ALG^*, ALH^*$.

Each $AL*$ class is also classified (Kronheimer, Minerbe, Chen-Chen, Chen-Viaclovsky-Zhang, Collins-Jacob-Lin, Hein-S.-V. Zhang).