Hypertoric varieties, W-Hilbert schemes & Coulomb branches

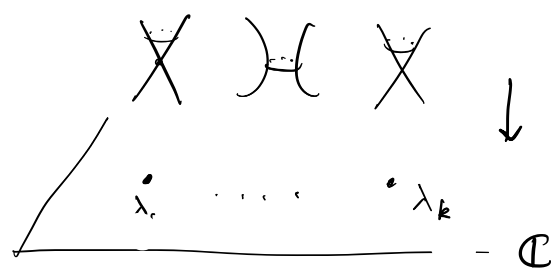
joint w/ R. Bielawski

Aim: study class of non-compact hyperkähler spaces generalising to higher dimensions the geometry of D_k ALF spaces

§. Motivation: ALF spaces

A_{k-1} surfaces: $\quad xy = \prod_{i=1}^{k} (z - \lambda_i)$

$A_{k-1}(\lambda_1, \dots, \lambda_k)$



hyperkähler "upgrade" (Gibbons–Hawking Ansatz):

$\lambda_i \in \mathbb{C} \rightsquigarrow \underline{\lambda}_i \in \mathbb{R}^3$

$$S^1 \hookrightarrow M^4 \downarrow \mathbb{R}^3 \setminus \{\underline{\lambda}_1, \dots, \underline{\lambda}_k\}$$

$\ell \in (0, +\infty]$: $\quad \ell = \infty$ ALE metric asymptotic to $\mathbb{C}^2/\mathbb{Z}_k$

$\ell \in (0, +\infty)$ ALF metric

$$g_{TN_k} \approx g_{\mathbb{R}^3} + \ell^2 \theta_k^2$$

θ_k connection on S^1-bdle deg $= k$ on $\mathbb{R}^3 \setminus B_R$

D_k surfaces:

$D_k(\lambda_1,\dots,\lambda_k)$
$$x^2 - zy^2 = \frac{1}{z}\left(\prod_{i=1}^{k}(z-\lambda_i^2) - (-1)^k\prod_{i=1}^{k}\lambda_i^2\right) + 2\left(\prod_{i=1}^{k}\lambda_i\right)y$$

or

$D_{k+1}^{(0)}(\pm\lambda_1,\dots,\pm\lambda_k)$
$$x^2 - zy^2 = \prod_{i=1}^{k}(z-\lambda_i^2)$$

ALF hyperkähler metrics via gluing (Sen 1997, Schroers-Singer 2020, F. 2019):

(i) Atiyah-Hitchin metric

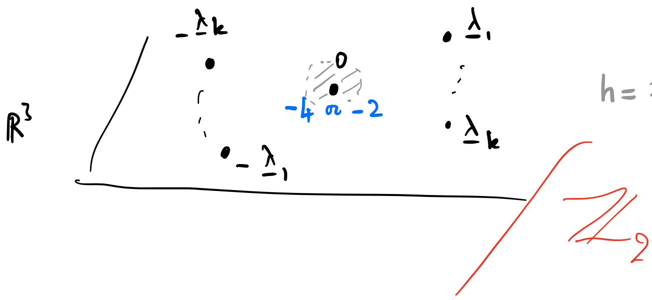
$$g_{AH} \approx g_{\mathbb{R}^3} + \ell^2\,\theta_{-4}^2 \big/ \mathbb{Z}_2 \qquad \text{on } D_0 \text{ surface } \; x^2 - zy^2 = y$$

or

$$g_{\widetilde{AH}} \approx g_{\mathbb{R}^3} + \ell^2\,\theta_{-2}^2 \big/ \mathbb{Z}_2 \qquad \text{on } D_1^{(0)} \text{ surface } \; x^2 - zy^2 = 1$$

(ii) $\mathbb{Z}_2$-inv. A_{2k-1} metric $\quad \ell \ll 1 \quad$ & $\quad \pm\lambda_1,\dots,\pm\lambda_k$

(iii) "glue in" AH or $\widetilde{AH}$ and perturb:



$$h = 1 + \ell\left(-\frac{4}{2|x|} + \sum_{i=1}^{k}\frac{1}{2|x-\lambda_i|} + \frac{1}{2|x+\lambda_i|}\right)$$

Rmk: Dancer 1993, Cherkis-Kapustin 1999, Cherkis-Hitchin 2004 ...
produce families of D_k metrics via twistor theory & Nahm's equations

Q: Higher dimensions?

§. Hypertoric varieties

complex n-torus $T = \mathfrak{t}/\chi_*(T) \simeq (\mathbb{C}^*)^n$

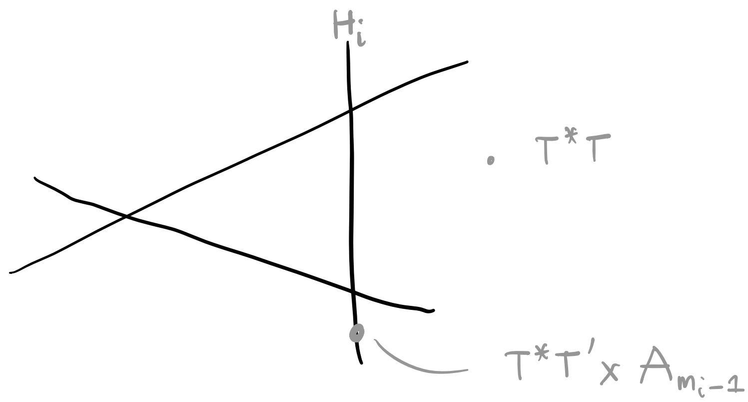
collection $(\alpha_i, m_i, \lambda_i)_{i=1,\dots,d} \in \chi_*(T)_{prim} \times \mathbb{Z}_{\geq 1} \times \mathbb{C}$

$\rightsquigarrow$ affine symplectic $X = (\mathbb{C}^{2d} \times T^*T) /\!/_{\{\lambda_i\}} (\mathbb{C}^*)^d$

- hyperkähler "upgrade" via hyperkähler quotient
  $\lambda_i \rightsquigarrow \underline{\lambda}_i \in \mathbb{R}^3$
  flat metric $L \in Sym^2_{>0}(\mathfrak{t})$ on T

- moment map $\mu : X \longrightarrow \mathfrak{t}^*$

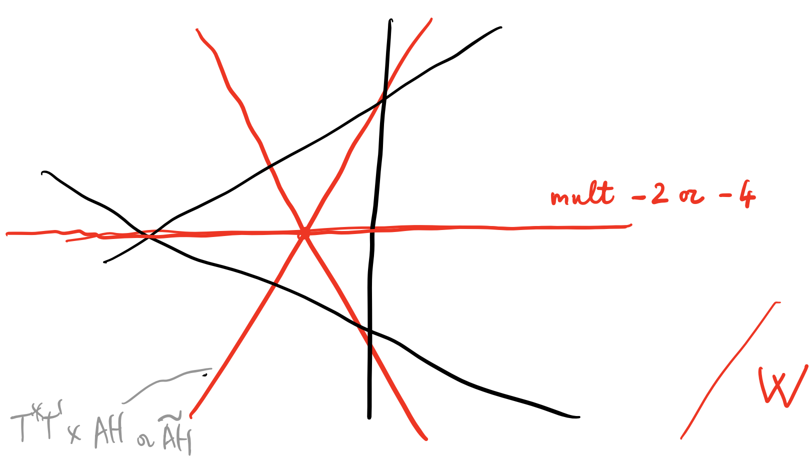
  hyperplane arrangement $H_i = \{ \langle \alpha_i, z \rangle = \lambda_i \} \subset \mathfrak{t}^*$ w/ mult m_i



Rmk: ALF A_{k-1} surfaces = hypertoric metrics in $\dim_{\mathbb{H}} = n = 1$

Now suppose W = Weyl gp $\circlearrowright \mathfrak{t}, T$ in std way

and $W \circlearrowright X$ so that $\mu : X \to \mathfrak{t}^*$ W-equivariant



$\underline{Q}$: Is there a hyperkähler metric "filling in" this asymptotic geometry?

Known examples: classical gauge theoretic moduli spaces, e.g. monopoles on $\mathbb{R}^3$...

Examples from physics: Coulomb branches of 3d quantum gauge theories w/ $\mathcal{N}=4$ supersymmetries

- G cplx reductive Lie gp w/ max torus T & Weyl gp W

Conversely: $W \curvearrowright T \rightsquigarrow G = G_{T,W}$

ambiguity $T_{SO(2n+1)} \underset{W\text{-equiv}}{\cong} T_{Sp(n)}$, take $G_{T,W}$ w/ only $Sp(n)$ factors

- V cplx $G^\vee_{T,W}$-rep $\underset{\text{weights}}{\rightsquigarrow}$ W-inv. collection $\{(\alpha_i, m_i, \lambda_i = 0)\}$

$\rightsquigarrow$ "strongly" W-invariant hypertoric X

Conversely: strongly W-inv. hypertoric X $\rightsquigarrow$ element of rep. ring of $G^\vee_{T,W}$ & "masses" $\{\lambda_i\}$

$\rightsquigarrow$ Coulomb branch $\mathcal{M}_C(G^\vee, V \oplus V^*)$

(w/ gauge coupling constant L and masses $\{\lambda_i\}$)

conjectural hyperkähler space w/ expected asymptotic geometry

Braverman–Finkelberg–Nakajima 2019: definition of $\mathcal{M}_C(G^\vee, V \oplus V^*)$

as affine symplectic (Bellamy 2023) variety

via Borel–Moore cohomology of a certain moduli stack

$\underline{Q}$: What about general strongly W-invariant hypertoric X?

What about the hyperkähler metric?

What about a more explicit geometric realisation?

§. Transverse Hilbert schemes

(inspired by Atiyah–Hitchin 1985, Cherkis–Hitchin 2004)

W–inv. hypertoric X w/ W–equivariant moment map $\mu : X \longrightarrow \mathfrak{h}^*$

$$\begin{array}{ccccccc}
\mathrm{Hilb}^W_\mu(X) & \subseteq & \mathrm{Hilb}^W(X) & \subseteq & W\text{-}\mathrm{Hilb}(X) & \subseteq & \mathrm{Hilb}^{[|W|]}(X) \\
 & & \text{closure of } (X/W)_{reg} & & H^0(S;\mathcal{O}_S) \simeq \text{reg } W\text{-rep} & & \text{0-dim'l subschemes } S \subset X \text{ w/ } H^0(S;\mathcal{O}_S) \simeq \mathbb{C}^{|W|}
\end{array}$$

$$\mathrm{Hilb}^W_\mu(X) \xrightarrow{\bar\mu} \mathrm{Hilb}^W(\mathfrak{h}^*) = W\text{-}\mathrm{Hilb}(\mathfrak{h}^*) = \mathfrak{h}^*/W$$

- $\mathrm{Hilb}^W_\mu(X)_{reg}$ has natural holo. symplectic structure
- $\mathrm{Hilb}^W_\mu(X)$ affine variety if X is, regular if X is

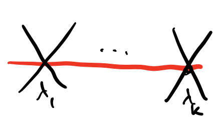
Key examples:

(1) $\mathrm{Hilb}^{\mathbb{Z}_2}_\mu(\mathbb{C}\times\mathbb{C}^*) = D_1^{(0)}$

more generally, $\mathrm{Hilb}^{\mathbb{Z}_2}_\mu\big(A_{2k-1}(\pm\lambda_1,\dots,\pm\lambda_k)\big) = D^{(0)}_{k+1}(\pm\lambda_1,\dots,\pm\lambda_k)$

(2) $\mathrm{Hilb}^{\mathbb{Z}_2}_\mu\big((\mathbb{C}\times\mathbb{C}^*)\times(\mathbb{C}\times\mathbb{C}^*)\big)/\!/\mathbb{C}^* = D_0$

$\mathbb{Z}_2$

more generally, $\mathrm{Hilb}^{\mathbb{Z}_2}_\mu\big($  $\big)/\!/\mathbb{C}^* = D_k(\lambda_1,\dots,\lambda_k)$

(3) $\mathrm{Hilb}^W_\mu(T^*T)$ = universal centraliser of $G_{T,W}$ $\quad \mathcal{Z}_{reg}(G_{T,W})$

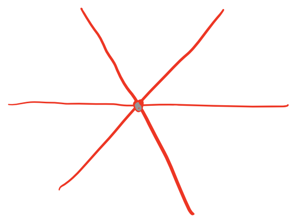
$$\{(S,g)\in \mathfrak{g}_{T,W}\times G_{T,W} \mid S\in \mathcal{S}_{reg},\ Ad_g(S)=S\}$$

proof.

$$\begin{tikzcd} \mathrm{Hilb}^W_\mu(T^*T) \arrow[dr] & & \mathcal{Z}_{reg}(G_{T,W}) \arrow[dl] \\ & \mathfrak{h}^*/W \simeq \mathcal{S}_{reg} & \end{tikzcd}$$

isomorphic away from locus mapped to intersection of two or more "coroot hyperplanes"

$\mathfrak{h}^*$



show fibre over $0\in \mathfrak{h}^*/W$ has dim n + $\mathbb{C}^*$-action

$\rightsquigarrow \bar{\mu}: \mathrm{Hilb}^W_\mu(X) \longrightarrow \mathfrak{h}^*/W$ flat $\quad\blacksquare$

__Rmk:__ universal centraliser of $G=SO(2n+1)$ realised as $\mathrm{Hilb}^W_\mu\left(T^*T_{Spin^c(2n+1)}\right)/\!/\mathbb{C}^*$

(4) $X=\mathbb{C}^{2n} \curvearrowleft \Sigma_n = W$

$$\mathrm{Hilb}^{\Sigma_n}_\mu(X) = \{(S,A,B)\in \mathfrak{gl}_n(\mathbb{C})^{\oplus 3} \mid S\in \mathcal{S}_{reg},\ [A,S]=0=[B,S],\ AB=S\}$$

__Thm__ (F. – Bielawski)

Let X be a strongly W-invariant affine hypertoric variety.

(1) $\mathrm{Hilb}^W_\mu(X)$ is a normal affine symplectic variety w/ flat

$$\mu_*: \mathrm{Hilb}^W_\mu(X) \longrightarrow \mathfrak{h}^*/W$$

(2) If G is a cplx reductive Lie gp & V is a $G^\vee$-rep, then $\exists\ X=X(G,V)$ as above such that $\mathcal{M}_C(G^\vee, V\oplus V^*)$ as defined by BFN is isomorphic to $\mathrm{Hilb}^W_\mu(X)/\!/(\mathbb{C}^*)^N$ where N is the number of $SO(2n+1)$ factors of G.

proof.

Reduction to the case of semisimple rank = 1 case, i.e. Key examples 1 & 2, provided we know $\mathrm{Hilb}^W_\mu(X)$ normal & $\mu_*: \mathrm{Hilb}^W_\mu(X) \longrightarrow \mathfrak{h}^*/W$ flat.

Three ingredients:

(1) (BFN) Find $\mathcal{U} = \mathrm{Hilb}^W_\mu(X) \setminus F \xhookrightarrow{j} \mathrm{Hilb}^W_\mu(X)$

w/ F closed codim ≥ 2

$\mathcal{U}$ normal

$\mathcal{O}_{\mathrm{Hilb}^W_\mu(X)} \xrightarrow{\sim} j_* \mathcal{O}_{\mathcal{U}}$

Natural candidate : $F = \mu_*^{-1}$ (intersection of 2 or more "flats")

F codim $\geq 2 \Leftarrow$ fibres of μ_* equidimensional (✓ if local models all product of key examples) e.g. X smooth

(2)

$$\begin{tikzcd} \mathrm{Hilb}^W_\mu(X)^\sigma \arrow[r] & \mathrm{Hilb}^W_\mu(X) \arrow[r] \arrow[dr, "\mu_*"'] & X/W \arrow[d, "\mu\ \text{flat}"] \\ & & \mathfrak{h}^*/W \end{tikzcd}$$

All 3 spaces isomorphic away from μ_*^{-1} (union of flats)

$$\begin{array}{ccc} \mathrm{Hilb}^W_\mu(X)^\sigma & \to & \mathrm{Hilb}^W_\mu(X) \\ \cup\!\! | & & \cup\!\! | \\ \mathcal{U} & = & \mathcal{U} \end{array}$$

weighted affine blow-up $\Longrightarrow \mathcal{O}_{\mathrm{Hilb}^W_\mu(X)} \xrightarrow{\sim} j_* \mathcal{O}_{\mathcal{U}}$

(3) Prove $\mathrm{Hilb}^W_\mu(X) \xrightarrow{\mu_*} \mathfrak{h}^*/W$ flat applying Hilb^W_μ to the symplectic quotient construction of $X = (\mathbb{C}^{2d} \times T^*T)/\!\!/(\mathbb{C}^*)^d$

$\blacksquare$

Remarks:

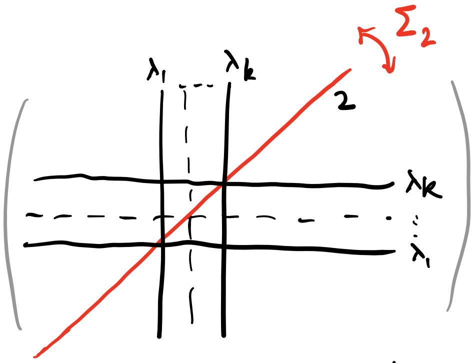
- more $\mathrm{Hilb}^W_\mu(X)$ than $\mathcal{M}_C(G^\vee, V\oplus V^*)$

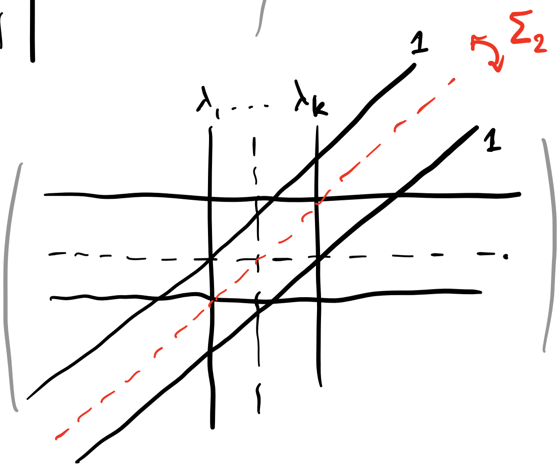
  b/c $X \longleftrightarrow$ element rep. ring of $G^\vee$

also: same $\mathrm{Hilb}^W_\mu(X)$ realised in many different ways as $\mathcal{M}_C(G^\vee, V\oplus V^*)$

- all deformations/resolutions of $\mathrm{Hilb}^W_\mu(X)$ from those of X?

Example: $\mathrm{Hilb}^{[n]}(A_{k-1})$

$\mathrm{Hilb}^{\Sigma_2}_\mu\Big($  $\Big) = \mathrm{Sym}^2(A_{k-1})$

$\rightsquigarrow \mathrm{Hilb}^{\Sigma_2}_\mu\Big($  $\Big)$ deformation of $\mathrm{Hilb}^{[n]}(A_{k-1})$

- $\mathcal{M}_C(G^\vee, \mathbb{V})$ should belong to same class of spaces as $\mathrm{Hilb}^W_\mu(X)$

$\mathbb{V} \rightsquigarrow X_{\mathcal{W}}$ $\quad \mathcal{W} = \{(u, m_u, 0) \mid \pm u \text{ weight of } \mathbb{V}\}$ $\rightsquigarrow$ $1 \hookrightarrow T \to E_X \to W \to 1$ $\quad E \circlearrowright X$

vanishing of $[E] \in H^2(W, T)$ as "anomaly cancellation"

(Braverman–Finkelberg et al., Teleman)

- could attempt to produce hyperkähler metrics on $\mathrm{Hilb}^W_\mu(X)$ by complex Monge–Ampère methods (Hein, Y. Li, Kottke–Rochon)

§. Twistor space & Nahm's equations

$\mu: X \to \mathfrak{t}^*$ W-inv. hypertoric w/ hk metric determined by
$L \in \mathrm{Sym}^2(\mathfrak{t})$
$\{\Delta_i\}_i \subset \mathbb{R}^3$

Hitchin–Karlhede–Lindström–Roček 1987: hk metric encoded in twistor space

$p: Z \to \mathbb{CP}^1$ 2n+1-dim'l cplx + fibrewise symplectic $\omega \in H^0(Z, \Lambda^2 T_F^* \otimes \mathcal{O}(2))$
+ real structure lifting antipodal map

Examples:

- $Z(\mathbb{C}^{2d}) = \mathcal{O}(1)^{2d} \to \mathbb{CP}^1$
- $Z_L(T^*T)$ = total space princ. T-bdle $\mathcal{L}^L \to \mathcal{O}(2) \otimes \mathfrak{t}^*$ w/ cocycle $\exp\left(-\frac{L(\eta)}{\zeta}\right)$
- X hypertoric ⇝ $Z_L(X)$ obtained by holo. symplectic quotient $/\mathbb{CP}^1$ (singular model for twistor space)
$$Z_L(X) \xrightarrow{\mu} \mathcal{O}(2)\otimes\mathfrak{t}^* \to \mathbb{CP}^1$$
- $$\mathrm{Hilb}^W_\mu(Z_L(X)) \xrightarrow{\bar{\mu}} \bigoplus_{i=1}^n \mathcal{O}(2d_i) \to \mathbb{CP}^1$$
  d_i: deg W-inv. poly on $\mathfrak{t}^*$

HKLR: every component of moduli space of twistor lines of Z carries (pseudo)-hyperkähler metric

twistor lines: real sections of $Z \to \mathbb{CP}^1$ w/ normal bdle $\mathcal{O}(1)^{\oplus 2n}$

Q: Twistor lines of $\mathrm{Hilb}^W_\mu(Z_L(X))$?

Step 1. W-inv. $\hat{C} \subset Z_L(X)$ s.t. $\mu|_{\hat{C}} : \hat{C} \xrightarrow{\sim} C \subset \mathcal{O}(2)\otimes\mathfrak{h}^*$

W-inv. flat of deg $|W|$ over $\mathbb{P}^1$

Prop. For generic $C \subset \mathcal{O}(2)\otimes\mathfrak{h}^*$ as above, $\exists$ lift $\hat{C} \in Z_L(X)$ iff a certain W-equiv. princ. T-bdle $P \to C$ is W-equiv. trivial.

Rmk. In general P depends on $[H_i \cap C] = [D_0] + \dots + [D_{m_i}]$

but $X = T^*T \Rightarrow P = \mathcal{L}^L|_C$

Step 2. (Hurtubise, Hitchin, ...)

(C, P) = cameral data of coHiggs bundle $(\mathcal{E}, \Phi)$ on $\mathbb{CP}^1$

W-equiv.

princ. holo. $G = G_{T,W}$ bundle

$\in H^0(\mathbb{CP}^1; \mathcal{O}(2)\otimes \mathrm{ad}\,\mathcal{E})$

$$(C, P \times_C \mathcal{L}^{tL}) \rightsquigarrow (\underbrace{\bar\partial_{\mathcal{E}} + t\Phi \lrcorner \omega_{FS}}_{\cong\, u_t}, \Phi)$$

Assume: $\bar\partial$ on $\mathcal{E}_0 = \mathbb{CP}^1 \times G$

$$\Rightarrow \mathrm{Ad}_{u_t}(\Phi) = (\Phi_2 + i\Phi_3) + 2i\zeta\,\Phi_1 + (\Phi_2 - i\Phi_3)\zeta^2$$

w/ $\quad \frac{d}{dt}\Phi_i + L([\Phi_j, \Phi_k]) = 0 \qquad$ (Nahm)

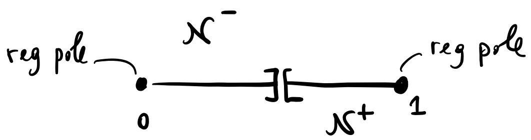
Prop. $\mathcal{L}^L|_C$ trivial $\Leftrightarrow$ sol. to (Nahm) develops <u>regular pole</u> as $t \to 1$

$\dot\Phi_i = \frac{1}{1-t}\, \rho_i$

$(\frac{i}{2}\rho_1, \rho_2 + i\rho_3, \rho_2 - i\rho_3) : \mathfrak{sl}_2(\mathbb{C}) \xrightarrow{\text{principal}} \mathfrak{g}$

Corollary For $X = T^*T$, $\exists$ component of twistor lines of twistor space of $\mathrm{Hilb}^W_\mu(X)$ yielding complete positive definite hyperkähler metric described as natural L^2-metric on moduli space of (Nahm) on $(0,1)$ w/ structure gp $G_{\mathbb{R}}$ and regular poles at the two endpoints

Example: $X = \mathbb{C}^{2n} \circlearrowleft \Sigma_n = W$



$$\left(\mathcal{N}^- \times \left(\mathfrak{gl}_n(\mathbb{R})\otimes\mathbb{H}\right) \times \mathcal{N}^+\right) /\!/\!/ \, U(n)\times U(n)$$

Proposal: For many strongly W-invariant hypertoric X (moduli spaces of monopoles, $\mathrm{Hilb}^{[n]}(A_k \setminus D_k)$ ...)

- $X \subset \mathfrak{h} \oplus \underbrace{V}_{G\text{-rep}} \simeq \Delta\mathfrak{h} \oplus \mathfrak{z}\left(\mathrm{End}(V)^T\right) \subset \mathfrak{g}\oplus\mathfrak{g}\oplus \mathrm{End}(V) \circlearrowleft G\times G$

- $M_G(X) :=$ normalisation of closure of $G\times G(X)$

  has natural twistor space

- Expect: $M_G(X)$ has "complete" hyperkähler metric

$$\Rightarrow \quad \mathrm{Hilb}^W_\mu(X) = \left(\mathcal{N}^- \times M_G(X) \times \mathcal{N}^+\right) /\!/\!/ \, G_{\mathbb{R}} \times G_{\mathbb{R}}$$

- Advantage: $M_G(X) =$ finite quotient of $T^*G' \times M'$

  where M' has algebraic twistor space,

  often realised as hk quotient of $\mathbb{H}^N$