Unramified Gromov–Witten and Gopakumar–Vafa invariants

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- Wall-crossing in Gromov–Witten theory

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- Bigger picture

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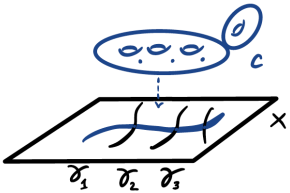
Algebraic geometers want to interpret $\langle \gamma_1 \cdots \gamma_n \rangle^{GW}_{g,\beta}$ as **counts** of curves of genus g in a class β passing through γ_i.



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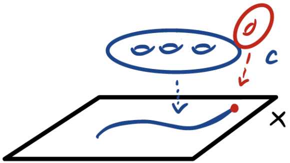

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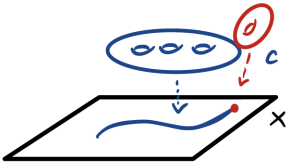

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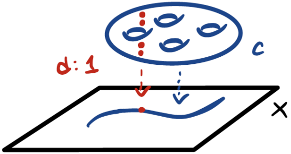

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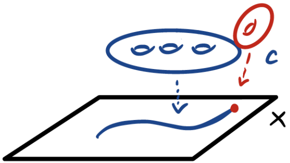

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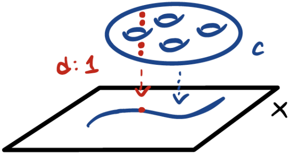

This leads to **overcounts** in GW invariants, i.e. lower-genus counts contribute to higher-genus ones. Moreover, they are **rational** numbers.

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In this case, there are no **multiple covers**. When $\beta \neq d\beta'$ and $c_1(X) \cdot \beta = 0$, these formulas coincide.

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All together this solves all (geometric) problems of GW theory. However, a **direct** construction of GV invariants is very much desired.

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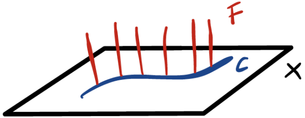

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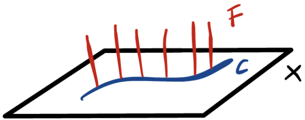

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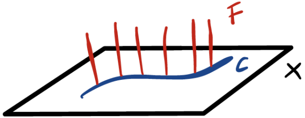

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However, it is not clear how to prove that Maulik–Toda invariants are equal to GV invariants in general. Can be verified in some instances by explicitly computing the invariants. **Very difficult** to compute.

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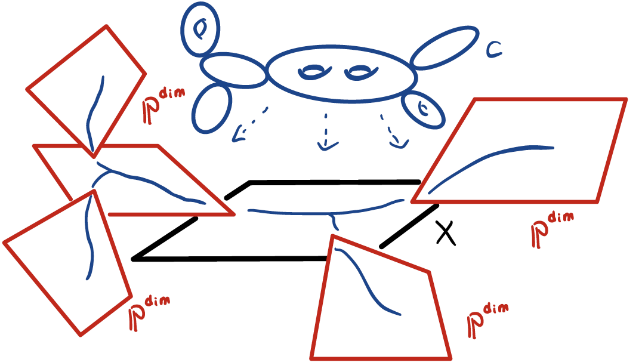

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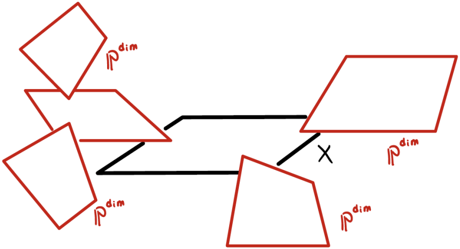

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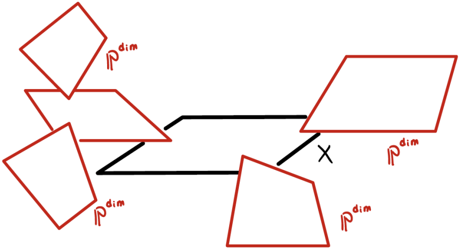

Like in Gromov–Witten theory, we can define spaces of unramified maps to FM degenerations of X,

$$u\overline{M}_{g,n}(X,\beta) = \{f\colon (C, p_1, \ldots, p_n) \to W \mid f \text{ is unramified}\}/\sim .$$

Spaces $u\overline{M}_{g,n}(X,\beta)$ are very similar to $\overline{M}_{g,n}(X,\beta)$. If $\dim(X) = 1$, these are spaces of covers of a curve, known as **Hurwitz** spaces.

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- $H^*(\mathbb{P}(TX)) = H^*(X)[H]/$relations, where H is a hyperplane class. There are more classes to insert.
- $\deg(f^*H) = 2g - 2$. The **genus** parameter becomes a **degree** parameter!

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Theorem (N.'24)

The Conjecture is true. Moreover, the formula above is a **wall-crossing formula**.

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Given a map $f\colon (C, p_1, \dots, p_n) \to W$, to each point $x \in W$, and to each bubble $\mathbb{P}^{\dim} \subset W$, we can associate **weights**,

$$w(x) \in \mathbb{Z}, \quad w(\mathbb{P}^{\dim}) \in \mathbb{Z}.$$

The weight $w(x)$ measures how far the map is from being **unramified** over x, while $w(\mathbb{P}^{\dim})$ measures how **big** the curve is over $\mathbb{P}^{\dim}$.

For example, a contracted component of genus g over x contributes $2g-2$ to $w(x)$, so does a curve of genus g over $\mathbb{P}^{\dim}$ for $w(\mathbb{P}^{\dim})$.

Given $\epsilon \in \mathbb{R}_{>0}$. The map f is **ϵ-unramified**, if

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Each value of ϵ provides a stability condition, such that $\mathbb{R}_{>0}$ is partitioned in **chambers** and **walls**.

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$$\overline{M}^{\epsilon}_{g,n}(X,\beta) = \{f\colon (C, p_1, \ldots, p_n) \to W \mid f \text{ is } \epsilon\text{-unramified}\}.$$

Each value of ϵ provides a stability condition, such that $\mathbb{R}_{>0}$ is partitioned in **chambers** and **walls**.

Spaces $\overline{M}^{\epsilon}_{g,n}(X,\beta)$ are the same for each ϵ in a chamber. As we cross a wall between chambers, the space $\overline{M}^{\epsilon}_{g,n}(X,\beta)$ changes abruptly.

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If $\epsilon \ll 1$, then

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If $\epsilon > 1$, then

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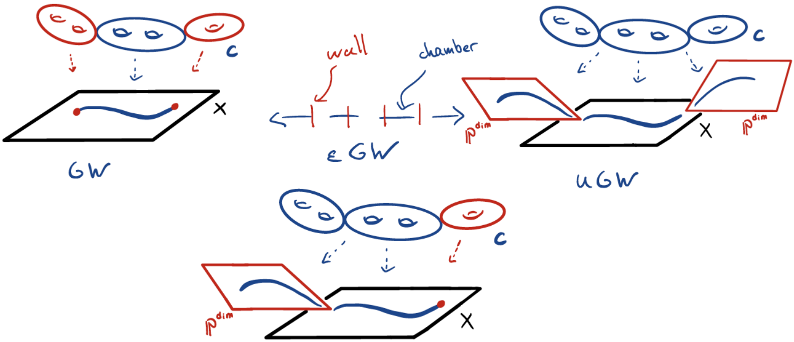
Overall, we get the following picture:



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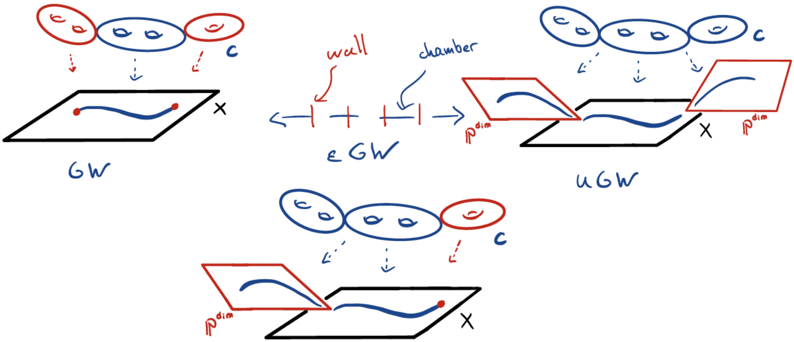

It allows us to compare theories across a single **wall** instead of comparing theories for extremal values of ϵ.

Variation of $\epsilon \implies$ **Wall-crossing formulas**.

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$$\text{Hodge integrals} := \int_{\overline{M}_{g,m}} \prod_i \psi_i^{k_i} \cdot \prod_j \lambda_j^{k_j}.$$

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The contribution from Hodge integrals is **universal**, i.e. depends very little on X.

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Theorem (N.'24)

In all dimensions, we have

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$$= \sum_{\overrightarrow{(g,n)}} \Big\langle \prod_{j\in N_0} \psi_j^{k_j}\gamma_j \ \Big| \ \prod_{i=1}^{i=k} I_{(g_i,N_i)}\Big(-\Psi_i, \prod_{j\in N_i} \psi_j^{k_j}\gamma_j \Big) \Big\rangle_{g_0,\beta}^{uGW} \Big/ k!,$$

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- invariants on the right from the bar are **relative** invariants, i.e. invariants associated to markings on the **target**;

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Plugging Hodge integrals into the wall-crossing formula, we obtain

$$\sum_g \langle\gamma_1 \cdots \gamma_n\rangle_{g,\beta}^{GW} \lambda^{2g-2} = \sum_g \langle\gamma_1 \cdots \gamma_n\rangle_{g,\beta}^{uGW} \lambda^{2g-2} \left(\frac{\sin(\lambda/2)}{\lambda/2}\right)^{2g-2+\mathrm{c}_1(X)\cdot\beta}.$$

the exponent “$2g - 2 + \mathrm{c}_1(X) \cdot \beta$” arises due to **dilaton** and **divisor** equations.

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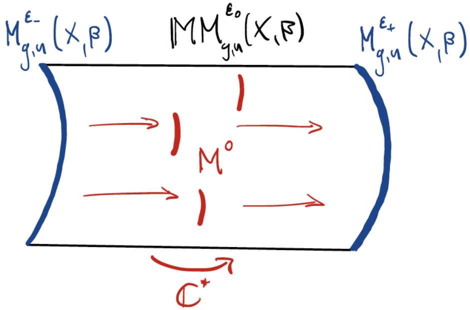

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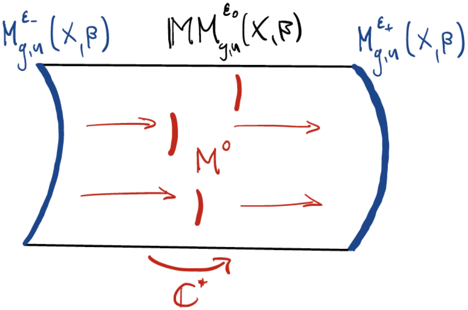

We then apply the equivariant localisation formula and take the equivariant residue, M^0 are responsible for the **wall-crossing** terms.

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Unramified Gromov–Witten theory=Hurwitz theory.

Hurwitz theory counts **covers** of curves with prescribed **ramifications**. For a collection of ramification profiles $\{\eta^1, \dots, \eta^n\}$, $\mathrm{H}_{X,d}(\eta^1, \dots, \eta^n)$ is the associated degree-d Hurwitz number of X.

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The wall-crossing recovers it in a slightly different but equivalent form, expressing $\overline{(k+1)}$ in terms of Hodge integrals, N.–Schimpf'24.

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A similar phenomenon holds for Gromov–Witten theories of $S \times C$ and of an orbifold symmetric product $Sym_d(S)$, N'22.

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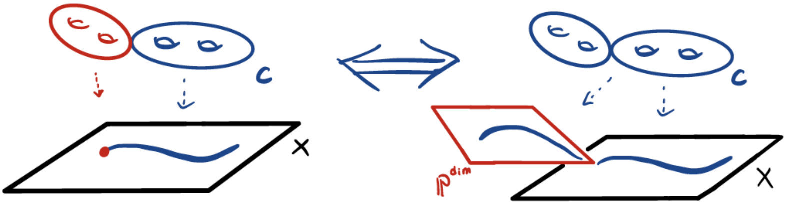

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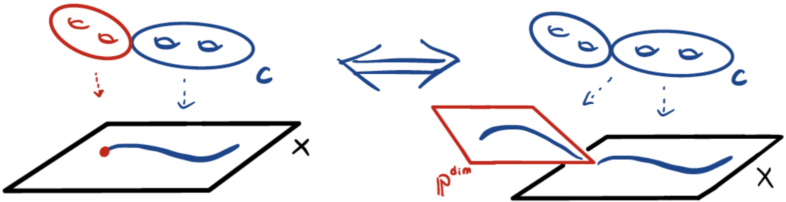

Instead of a map from a curve, there could be a **sheaf**.

Thank you