Schur Quantization of Complex Chern-Simons Theory

String Math 2024

Plan of the Talk

- Quantization and SUSY SQFT
- Schur quantization of character varieties
- Quantization of Complex Chern-Simons theory
- A surprising SYK connection

Quantization

- Classical to Quantum
  - Phase space P, Poisson *-algebra Ob_{cl} of (some) functions on P
  - Family of *-Algebras Ob_h with unitary representations H_h

$$\hat{a}\,\hat{b} = ab + i\hbar\{a, b\} + \cdots$$

- Two aspects: deform the algebra and represent it
- Not canonical! May not be possible.

Extra desiderata

- Preserve symmetries
- Quantize in families.
- States/initial conditions:
  - Lagrangian L in P, functions on L as module M_{cl} for Ob_{cl}
  - State or collection of states, form module M_h for Ob_h in H

Deformation vs representation

- Suppose Ob_h and M_h are given.
  - *-structure $a^* = \tau(a)$
- Images |m> of M_h to H must satisfy

$$\langle \tau(a) m | m' \rangle = \langle m | a m' \rangle$$

- Dual to tensor product $\tau(M_\hbar) \otimes_{A_\hbar} M_\hbar$
- Linear space of tensor products. Positive? It is a bootstrap.

Familiar examples

- Harmonic oscillator: $[a,a^*]=1$, $a|0\rangle=0$ fixes all. $a^*|0\rangle$ does not work
- su(2) representation theory
- Less familiar: unitary irreps of SL(2,R), SL(2,C), other Lie algebras, finite W-algebras, quantum groups, etc.
- Positivity is the hard step! Continuity arguments, but also miracles.

Positive traces and Doubles

- Special case: Algebra A wt positive twisted trace Tr:

$$\mathrm{Tr}\, ab = \mathrm{Tr}\, \rho^2(b) a \qquad\qquad \mathrm{Tr}\, \rho(a) a > 0$$

- Inner product $\langle a|b\rangle \equiv \mathrm{Tr}\, \rho(a) b$

- H= completion of A with inner product

- Action of Ob = A x Aop $a\widetilde{b}|c\rangle = |acb\rangle$

$$|a\rangle = a|1\rangle = \widetilde{a}|1\rangle \qquad\qquad \widetilde{a} = \rho(a)^*$$

Complex phase space

- Complex symplectic P_C as a real phase space
  - A quantizes holomorphic functions
  - A_{op} quantizes anti-holomorphic functions
- Double of A is a quantization of P_C !
  - Which twist? E.g. Weyl algebra has no trace, but X->-X, P->-P works
- We will find a trick: hyperkahler rotation. Will work also for P_R in P_C

Complex Chern-Simons theory I

- Chern-Simons with complex gauge group, e.g. SL(2,C)
  - Do not confuse with analytic continuation. $\int D\mathcal{A}D\bar{\mathcal{A}} \neq \oint D\mathcal{A}$

$$\frac{is}{2}\left(S_{CS}(\mathcal{A}) - S_{CS}(\bar{\mathcal{A}})\right)$$

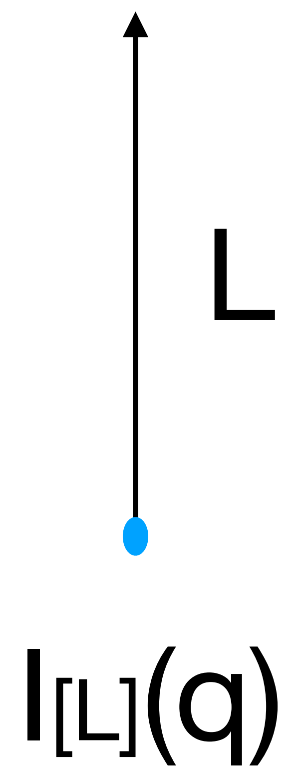
- Does it define a 3d TFT?
- Application to 3d gravity (Lorenzian de Sitter)?

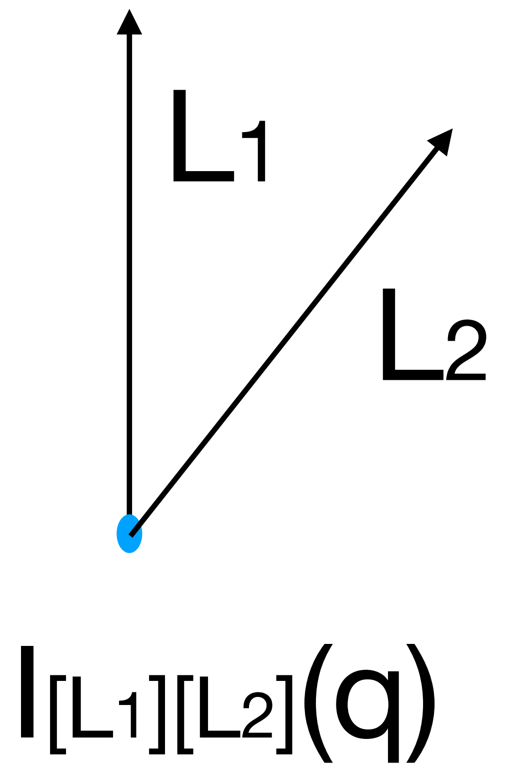
Complex Chern-Simons theory II

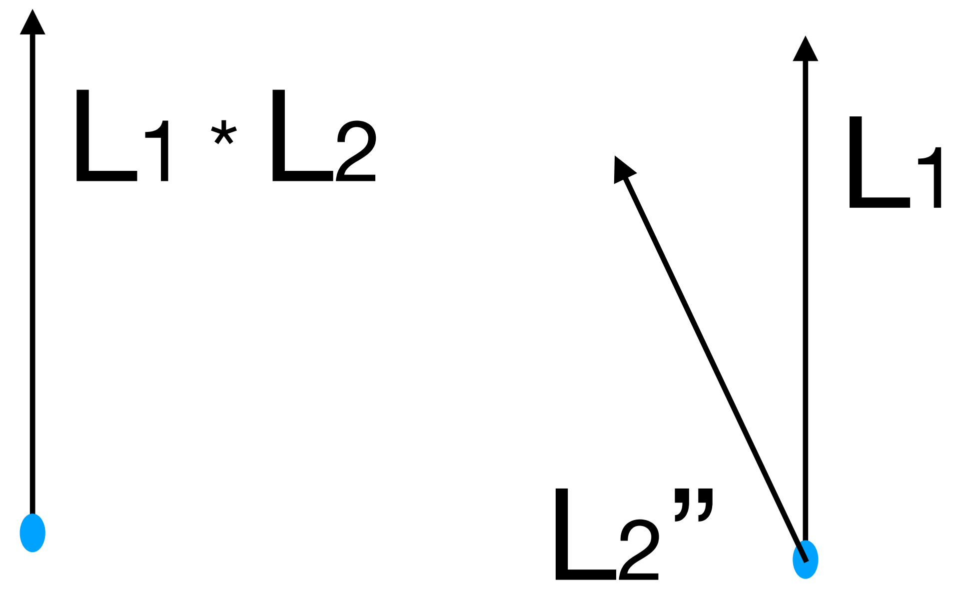
- Phase space = space P_C of complex flat connections as a real manifold
  - A = Skein algebra.
  - Wilson lines $W_{R,\gamma} \equiv \mathrm{Tr}_R \, \mathrm{Pexp} \int_\gamma \mathcal{A}$ with skein relations (real q)
- Positive trace on A => quantization with states labelled by skeins
  - Does it exist? Is it unique?
  - Covariant under mapping class group? States from 3-manifolds?

Positive Schur trace

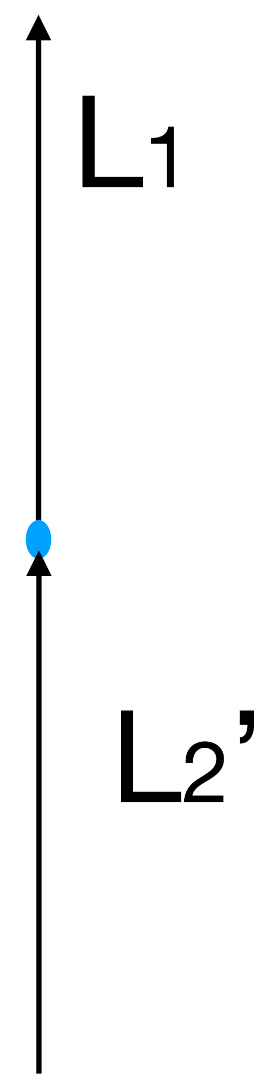
- Schur index: defined for any 4d N=2 SQFT $I(q) = \mathrm{Tr}_{\mathrm{Op}}(-1)^{2R_3} q^{2J_3 - 2R_3}$
  - "Counts" 1/4-BPS local operators / operators in HT twisted theory
  - HT twist allows a definition as partition function on $S^3 \times_q S^1$
  - Conj: the HT $S^3 \times_q S^1$ lifts to a reflection-positive rigid SUGRA bkgr.
- Generalize to operators at endpoint of line defect or junction of many
  $I_a(q)$ $I_{a_1 \cdots a_k}(q)$
  - Depends only on equivariant K-theory $K_{\mathbb{C}^*}(\mathrm{Lines})$ as an algebra



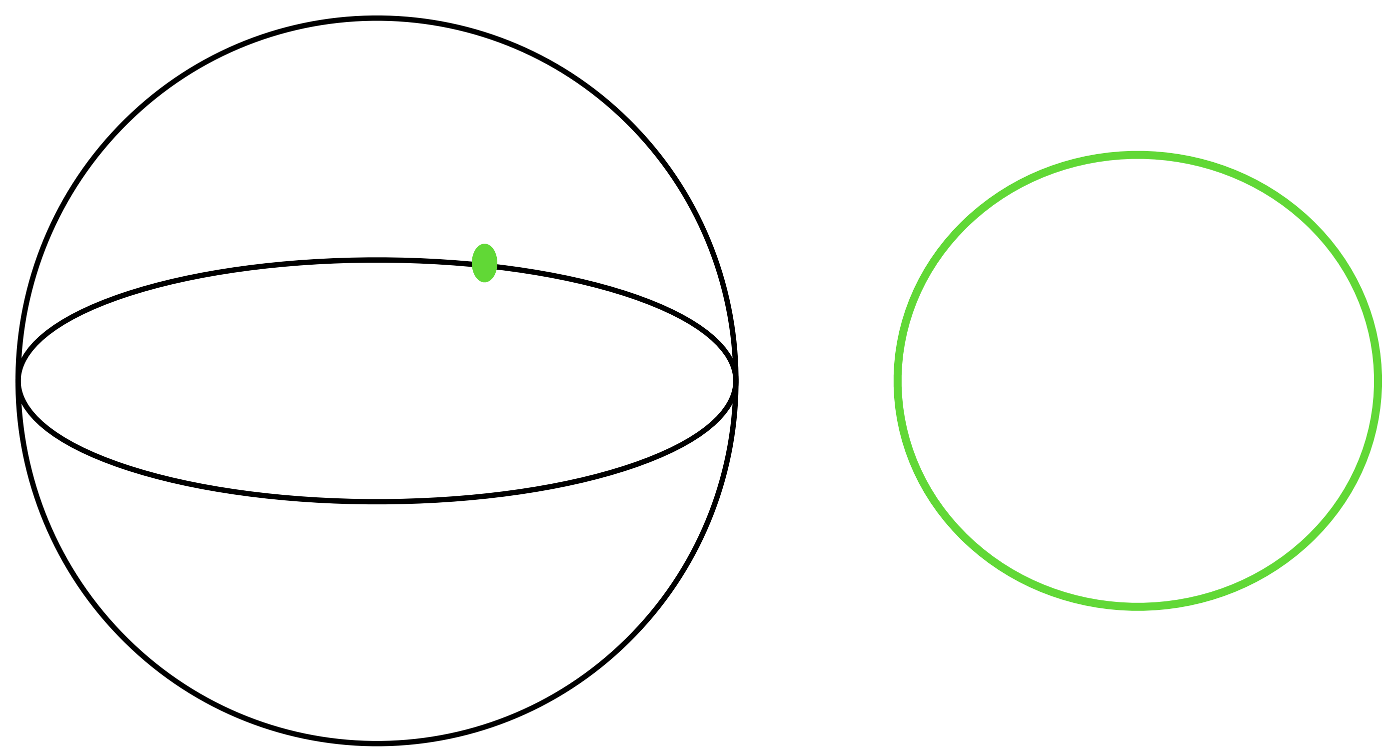


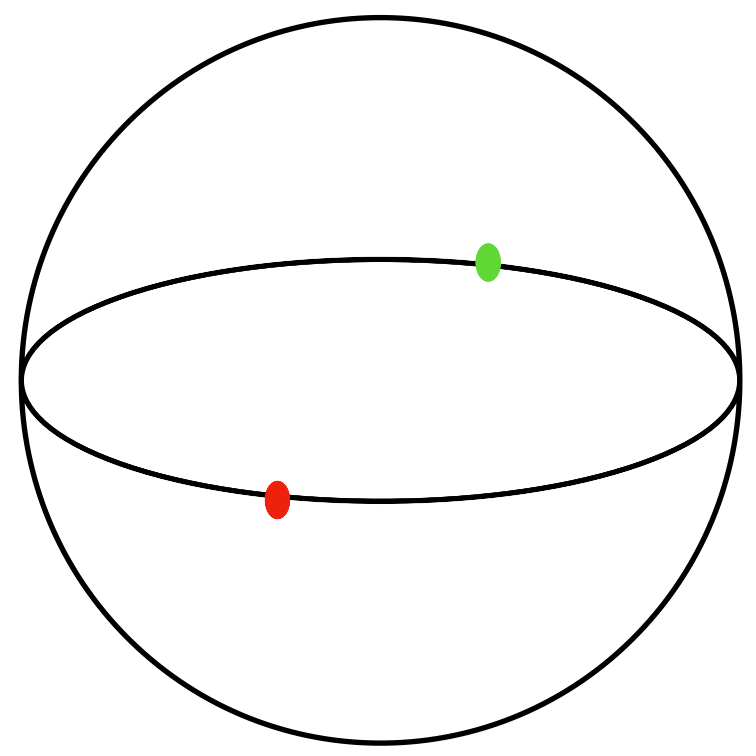


$I_{ab}(q) = I_{(ab)}(q) = I_{\rho^2(b)a}$



$I_{\rho(a)b}(q) = I_{a \to b}(q)$





Conj: positive definite

0<q^2<1

Schur quantization

- Schur indices give a positive twisted trace on algebra $K_{\mathbb{C}^*}(\mathrm{Lines})$
  - Quantization of K-theoretic Coulomb branch
- Class S: 6d SCFT for g on Riemann surface C
  - Quantization of space of complex g flat connections on C
  - E.g. quantization of complex CS theory!
  - Mapping class group from dualities, states from boundary conditions, etc.

Spherical vector

- Duality chain: |1> is created by a topological boundary condition
  - Connection unitary at the boundary. $\mathrm{Pexp} \oint A = \mathrm{Pexp} \oint \bar{A}$
- |a> created by decorated topological b.c.
- Turaev-Viro like, novel unitary structure?
- Relation to (novel?) complex quantum group.

Comparison to other quantizations

- Cluster technology, 3d-3d correspondence
  - Agrees and generalizes via IR formulae for Schur indices
- Holomorphic quantization $\mathcal{H} = L^2(\mathrm{Bun}_G, |K|^{1+is})$
  - |a> from G_C/G WZW model with a Verlinde lines
  - $$I(q) = \int_{\mathrm{Bun}} |Z_{\mathrm{WZW}}|^2$$

Real quantization

- Add a boundary condition: half-index $\mathbb{I}_{m\, a_1 \cdots a_k\, m'} = \mathbb{I}_{(m\, a_1 \cdots a_n)(a_{n+1} \cdots a_k\, m')}$
- Algebra, left and right module of boundary lines $\tau(a)\tau(b) = \tau(ba)$
- Some kind of time-reversal symmetry needed $\tau(m)\tau(a) = \tau(am)$
  - Conjectural positivity $\mathbb{I}_{\tau(m)m} > 0$
  - Promote module to Hilbert space

Boundaries and cross-caps

- Complex CS theory admits boundaries and cross-caps
- Reality condition on holonomies
- Challenge: find boundary condition in class S
- Real Schur provides a quantization!

A strange example

- Double-Scaled SYK model

$$H = i^{p/2} \sum J_{i_1 \dots i_p} \psi_{i_1} \dots \psi_{i_p}$$

$$\langle (J_{i_1 \dots i_p})^2 \rangle = \mathcal{J}^2 \binom{N}{p}^{-1}.$$

$$N \to \infty \quad p \to \infty \qquad \lambda \equiv p^2/N = \text{fixed} \qquad \mathfrak{q} = e^{-2\lambda}$$

- Correlation functions identical to Neumann half-indices in SW theory

- SL(2,C) CS theory on a disk with one irregular puncture and SU(2) boundary!

- Coincidence or first of many examples?