Homological link invariants from an A-brane category

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Based on: 2305.13480 with Elise LePage and Miroslav Rapcak

2406.04258 with Ivan Danilenko, Yixuan Li, Vivek Shende and Peng Zhou

and my earlier work: 2207.14104 , 2105.06039 and 2004.14518

The link categorification problem

was introduced in 1998,

by Khovanov.

Khovanov's homology of a link K,

$$\mathcal{H}_K^{*,*} = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{H}_K^{i,j}$$

is a collection of a bigraded vector spaces, each of which is

itself a link invariant.

His link homology theory

$$\mathcal{H}_K^{*,*} = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{H}_K^{i,j}$$

is a categorification of the Jones polynomial,

$$J_K(\mathfrak{q}) = \sum_{i,j\in\mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i,j}$$

which is its graded Euler characteristic.

Khovanov's construction is part of the

categorification program

pioneered by Crane and Frenkel.

A model for categorification

is a Riemannian manifold M whose Euler characteristic

$$\chi(M) = \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathcal{H}^k(M)$$

is categorified by the cohomology groups

$$\mathcal{H}^k(M) = \ker d_k / \mathrm{im}\, d_{k-1}$$

of the de Rham complex

$$C^* = \ldots C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} \ldots .$$

From physics perspective, de Rham cohomology groups

$$\mathcal{H}^k(M) = \ker\, d_k / \mathrm{im}\, d_{k-1}$$

originate from

supersymmetric quantum mechanics with M as a target space,

as the space of exact supersymmetric ground states

of fermion number $\boldsymbol{k}$.

The complex whose cohomology one takes to get

$$\mathcal{H}^k(M) = \ker d_k / \mathrm{im}\ d_{k-1}$$

is the space of all perturbative supersymmetric ground states,

$$C^* = \cdots C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} \cdots$$

equipped with a differential

$$Q = \sum_k d_k$$

obtained by counting instantons.

Khovanov similarly produces link homology groups

$$H_i^j(K) = \frac{\ker \partial_i}{\mathrm{im}\ \partial_{i+1}}$$

by starting with a complex of vector spaces

$$C_\bullet^j(K) = \ldots \to C_i^j(K) \xrightarrow{\partial_i} C_{i+1}^j(K) \to \ldots$$

graded by the fermion number and one additional grading,

and taking cohomology of the differential.

The Jones polynomial is a special case of a

Chern-Simons or quantum group

$$U_{\mathfrak{q}}(\mathfrak{g})$$

link invariant,

where one takes the Lie algebra to be

$$\mathfrak{g} = \mathfrak{su}_2$$

with links colored by its fundamental representation.

Khovanov's construction is explicit, but purely algebraic.

It leaves as a mystery

the physical or geometric origin of his homology groups,

$$\mathcal{H}_K^{*,*} = \bigoplus_{i,j \in \mathbb{Z}} \mathcal{H}_K^{i,j}$$

or what replaces them for a general Lie algebra

$$\mathfrak{g}$$

In 2013 Webster provided an algebraic framework

for categorification of quantum link invariants for arbitrary ordinary Lie algebras

$\mathfrak{g}$

Webster's construction was purely formal,

in contrast to Khovanov's.

The link categorification problem

is to find a general framework for construction of

link homology groups,

categorifying Chern-Simons link invariants,

that works uniformly for all Lie algebras

which explains what link homology groups are,

and why they exist.

The first known polynomial link invariant

predates the Jones polynomial, discovered in 1984.

It was discovered by

Alexander in 1923.

The Alexander polynomial is also a quantum group

$$U_{\mathfrak{q}}(\mathfrak{g})$$

link invariant,

however $\mathfrak{g}$ is not an ordinary Lie algebra,

but a Lie super-algebra

$$\mathfrak{gl}_{1|1}$$

The Alexander polynomial

also has a known categorification,

but of a very different flavor than Khovanov’s.

The categorification of the Alexander polynomial

is based on Floer theory,

which generalizes

supersymmetric quantum mechanics to a theory in one dimension up.

As such, it does come from geometry and physics.

The categorification of the Alexander polynomial

is based on Floer theory, an A-model,

whose target space is a symmetric product of Heegaard surfaces.

The theory, known as Heegard Floer theory

was discovered in 2003

by Ozsvath and Szabo.

The solution to the link categorification problem

is based on a theory that generalizes

Heegaard Floer theory,

from $\mathfrak{gl}_{1|1}$ to all other Lie (super) algebras

$\mathfrak{g}$

Many special features exist in this family of theories,

which will translate to the fact that

problems whose solutions typically only exist formally,

will now be explicitly solvable.

We will have an explicit description

of the corresponding theory,

means of proving it gives homological invariants of links,

and an algorithm for computing them.

The theory originates directly from string theory,

as predicted in the works of Ooguri and Vafa,

and also Gukov, Schwarz and Vafa,

and later Witten, with Gaiotto.

From a mathematical standpoint,

the theory is an application of

homological mirror symmetry

which played a key role

in discovering, formulating and solving the theory.

However, having solved the theory,

homological mirror symmetry is no longer necessary for formulating the answer.

We will obtain invariants of links in

$$\mathbb{R}^2 \times S^1$$

Invariants of links in

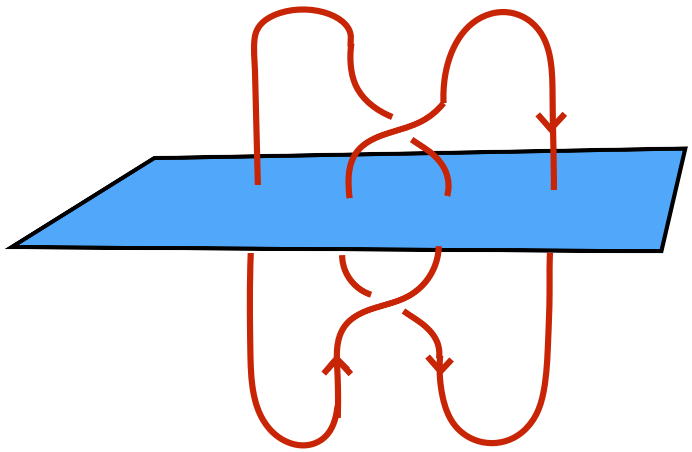
$$\mathbb{R}^3$$

are a special case of these, by taking links not to wrap the S^1

To describe a link in

$$\mathbb{R}^2 \times S^1$$

split the 3-manifold in two



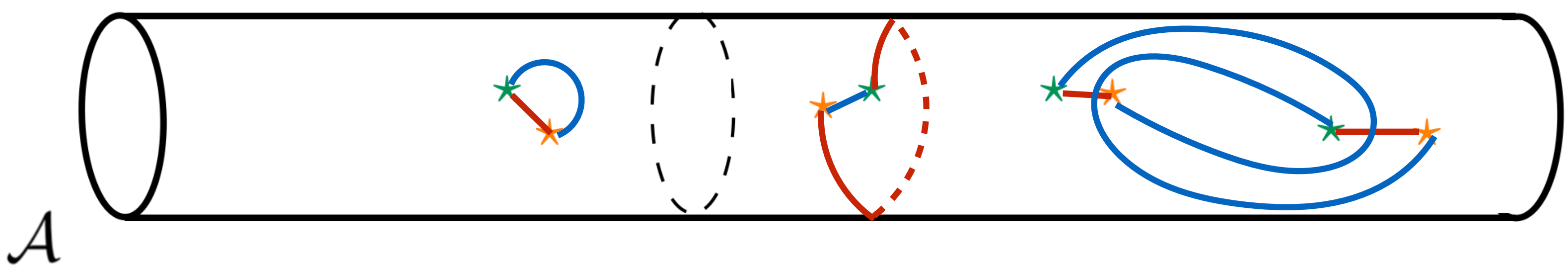
on a Riemann surface

$$\mathcal{A} = \mathbb{R} \times S^1$$

which intersects the link transversely. This can always be arranged.

Such a splitting maps the link into a pair of matchings

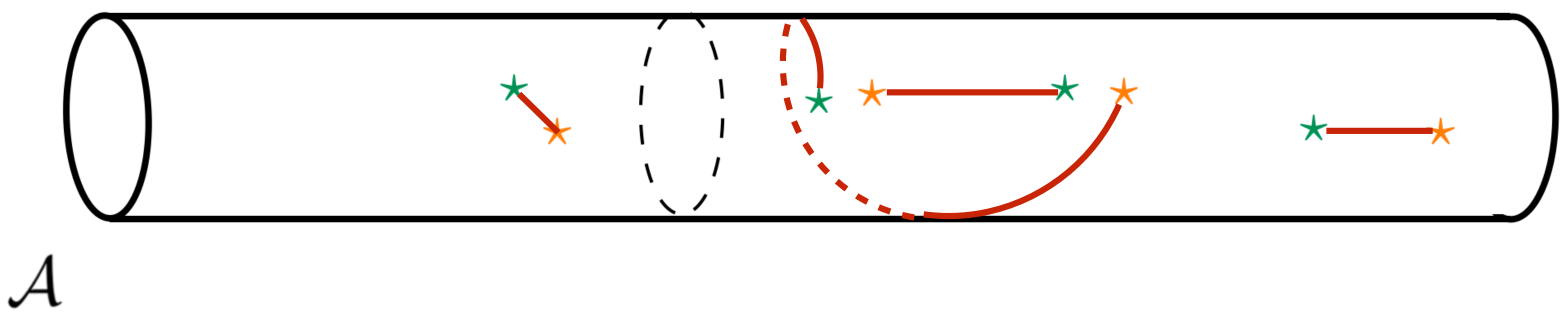
on the Riemann surface



which now has punctures where the two matchings meet.

Each matching is collection of non-intersecting curves on

$$\mathcal{A} = \mathbb{R} \times S^1$$



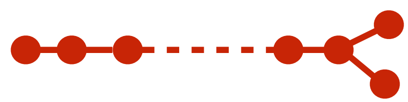
which end on the punctures.

Pick a Lie (super) algebra

$$\mathfrak{g}$$

If $\mathfrak{g}$ is in fact an ordinary Lie algebra

its root system is encoded in the Dynkin diagram

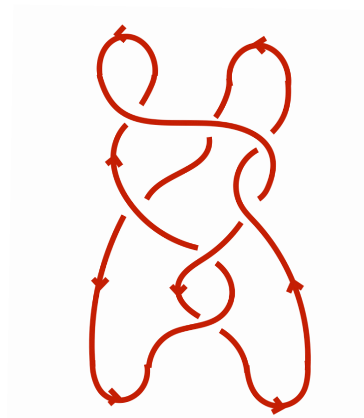


all of whose simple roots are bosonic  .

For a Lie superalgebra

simple roots include fermionic ones  .

We will color the strands of the link



by representations

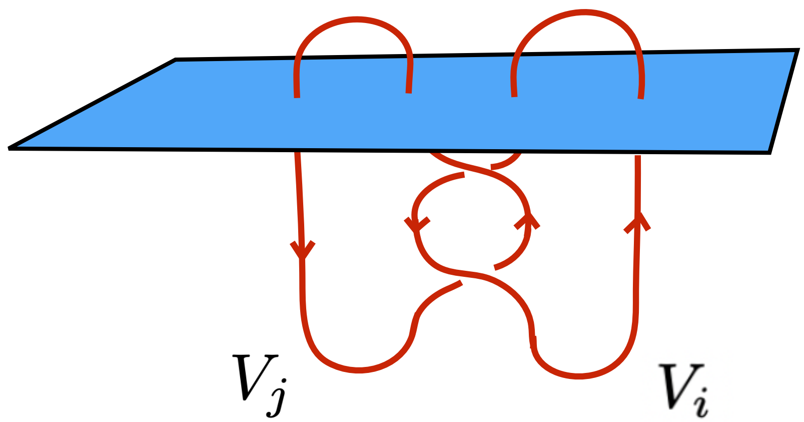
$$V_i$$

of the Lie algebra, which we take to be minuscule.

This condition is technical,

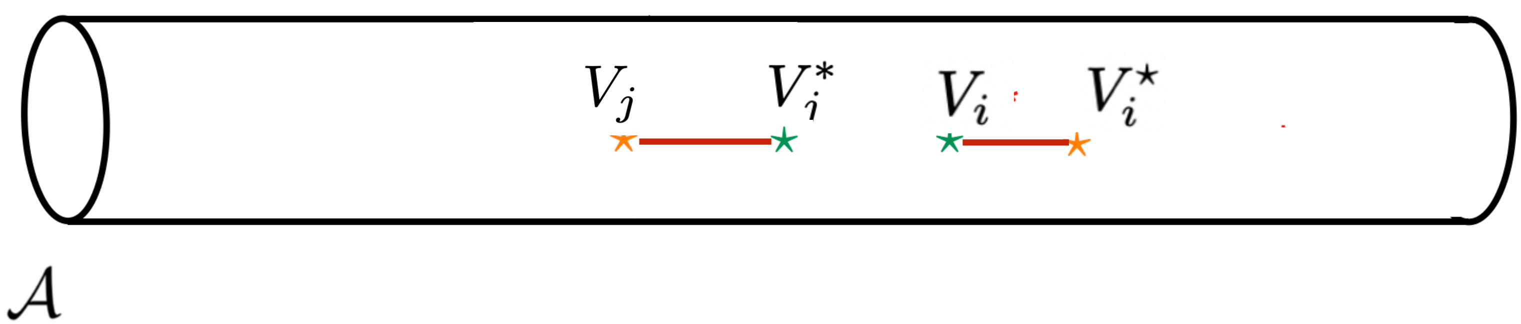
and one may eventually be able to relax it.

Having chosen the representations that color the strands,



the punctures get colored by pairs

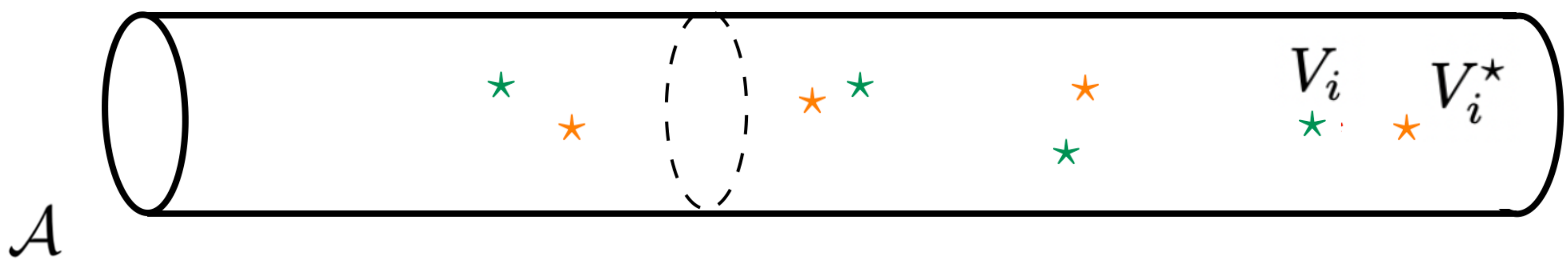
of the representation, and its complex conjugate:



To categorify Chern-Simons link invariants,

we will associate

to the Riemann surface with punctures



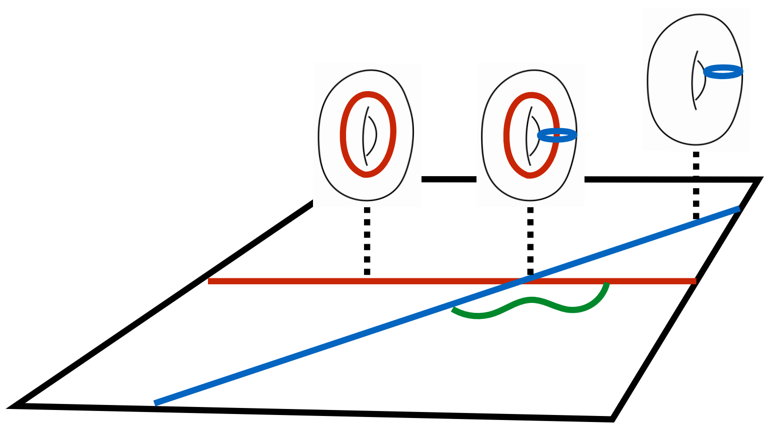
a category,

which depends on the Lie algebra,

and representations coloring the punctures.

The category will be a category of

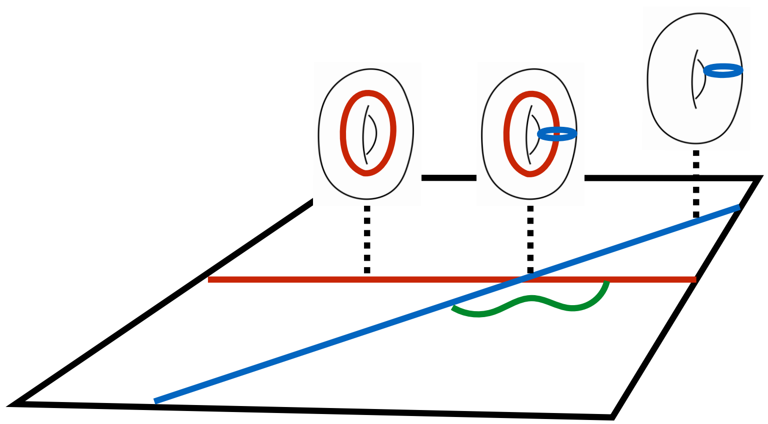
D-branes, preserving A-type supersymmetry



on a Calabi-Yau manifold associated with our data.

The category has objects which are A-branes themselves

and morphisms,



which come from open strings stretching between the branes,

and which one can compose.

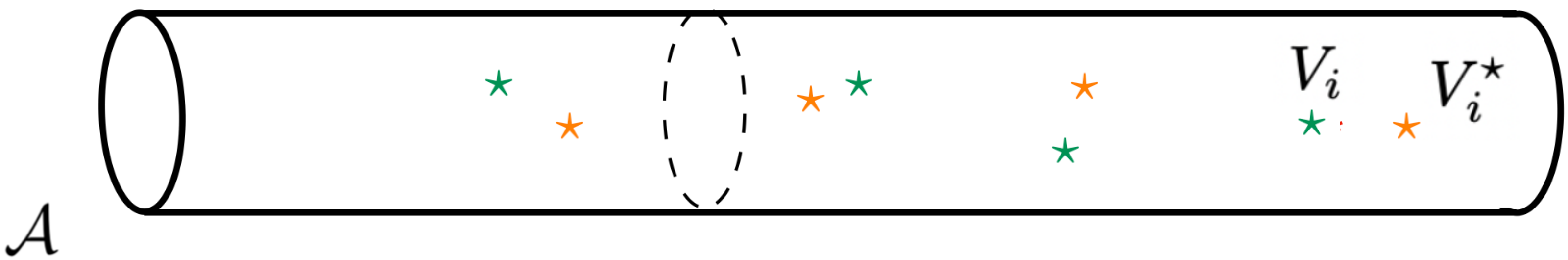
The category will have a

a cohomological $\mathbb{Z}$-grading,

corresponding to the fermion number,

an additional grading that corresponds to

$\mathfrak{q}$



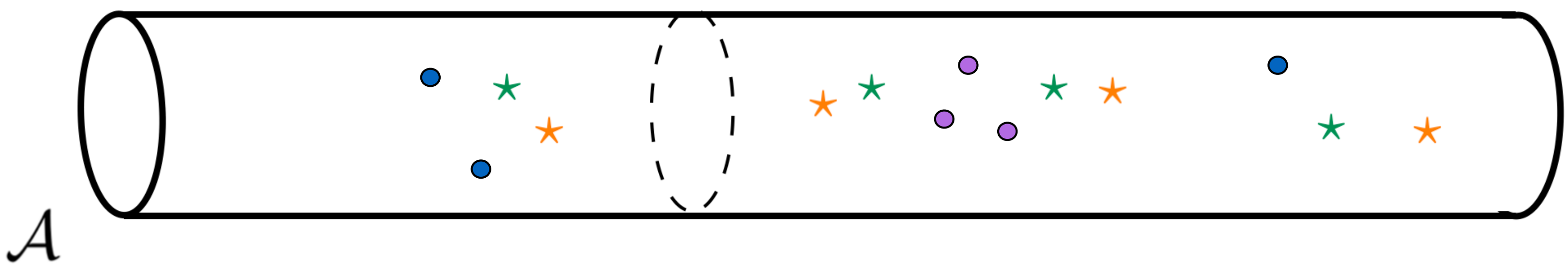
and further gradings associated to holonomies

of Chern-Simons gauge field around the S^1

The target will be based on

$$Y = \prod_{a=1}^{\mathrm{rk}\mathfrak{g}} Sym^{d_a}(\mathcal{A})$$

which is a collection of points



on the punctured Riemann surface $\mathcal{A}$,

colored by simple roots of the Lie algebra

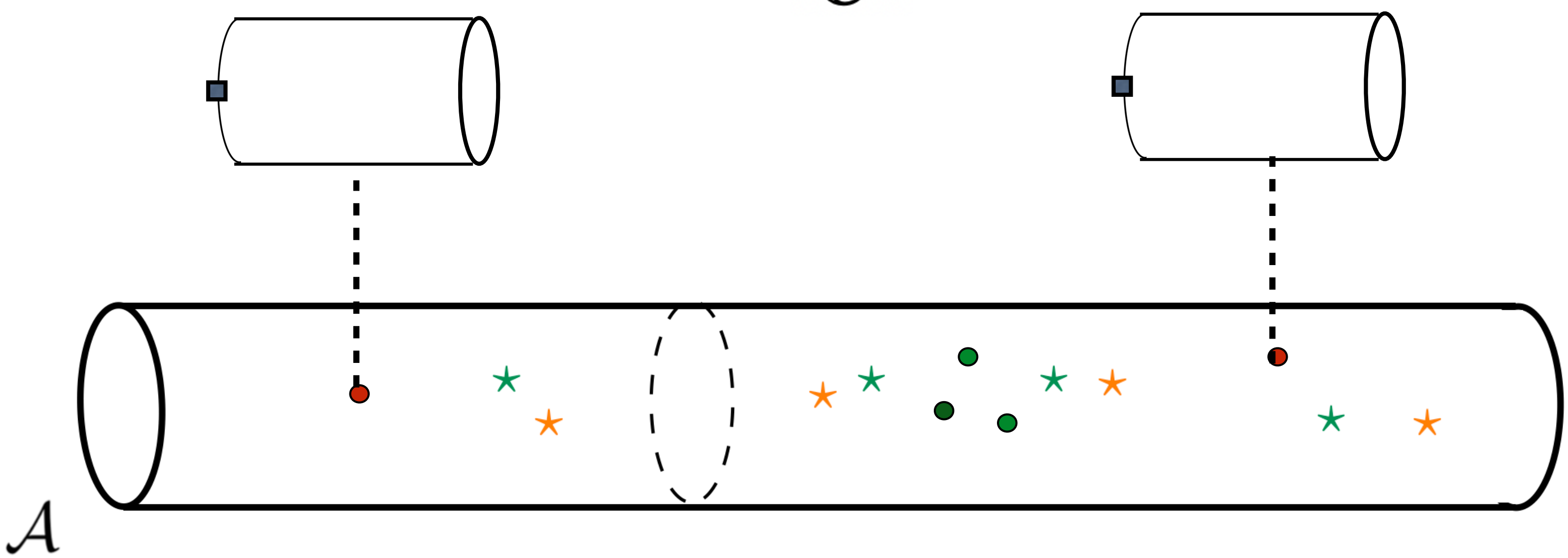
$\mathfrak{g}$

but otherwise identical.

Over every bosonic point in

$$Y = \prod_{a=1}^{\mathrm{rk}\mathfrak{g}} Sym^{d_a}(\mathcal{A})$$

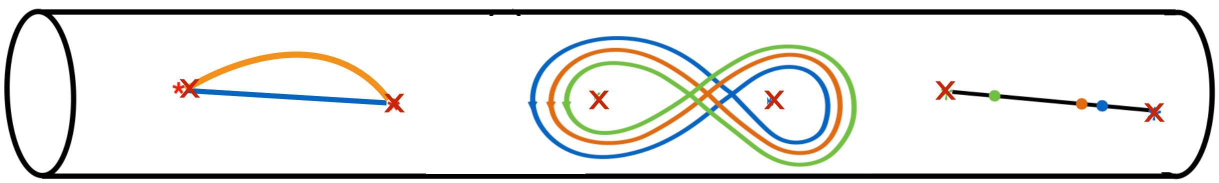
there is a $\mathbb{C}^{\times}$ fiber



The A-branes on

$$Y$$

are all supported either on products of one dimensional curves

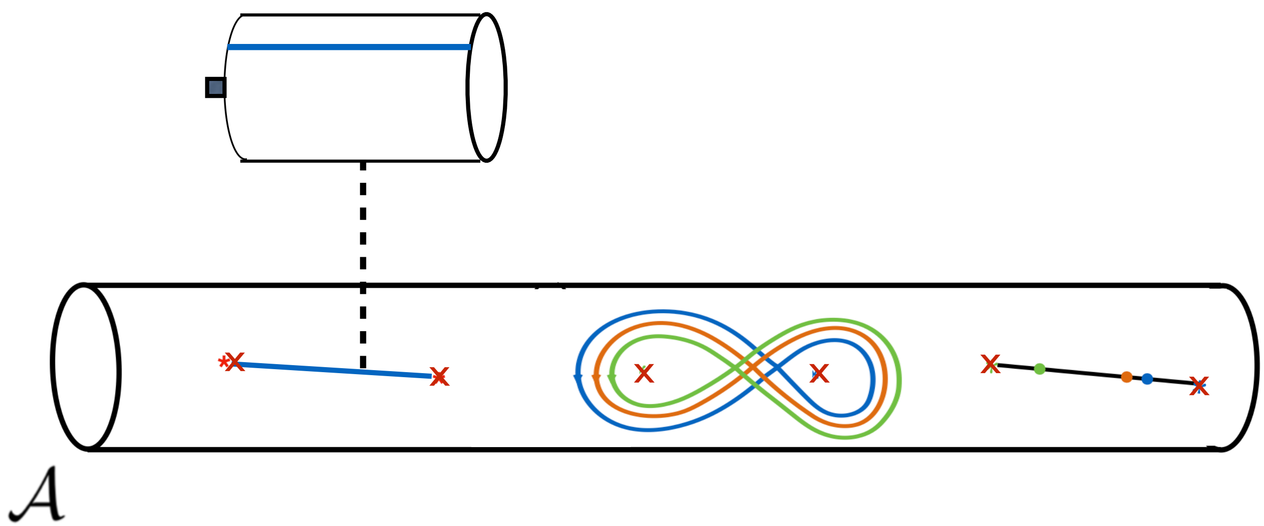


$\mathcal{A}$

on the Riemann surface,

colored by simple roots, or on generalized intervals.

To fully specify the brane



one also picks a brane in the fiber.

The target

is equipped with a top holomorphic form

$$\Omega$$

and a superpotential

$$W$$

The category of A-branes

$$\mathscr{D}_Y$$

is a variant of the Fukaya-Seidel category,

whose objects are graded Lagrangians on the target.

Grading, which turns a Lagrangian L into an A-brane,

an object of the category

$$\mathscr{D}_Y$$

is a choice of a lift of the phase of

$$\Omega e^{-W}|_L$$

to a single-valued function on L .

The cohomological $\mathbb{Z}$ - grading of the A-brane is

the choice of a lift of the phase of the holomorphic volume form

$$\Omega|_L$$

to a single-valued function on L

Change of orientation of L shifts the grading by

$$L \longrightarrow L[1]$$

The category is that of equivariant A-branes

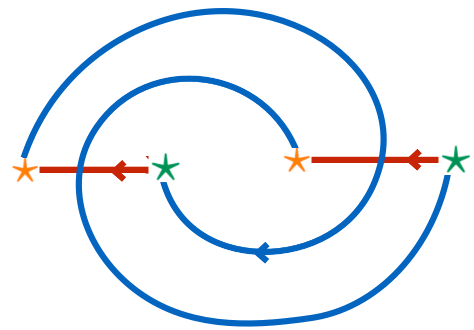
as multi-valuedness of the super potential

$$W|_L$$

introduces additional, equivariant, gradings

coming from a choice of its lift to a single-valued function on L .

To the pair of red and blue matchings,

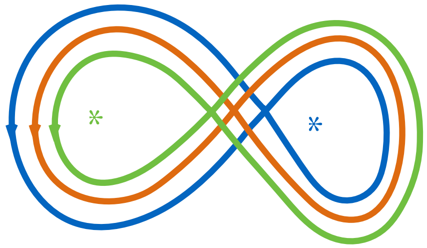


we will associate a pair of A-branes

$$I_{\mathcal{U}}, \ \ \mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y .$$

by replacing the red matchings by interval type branes,

and the blue matchings by braided figure eight type branes.

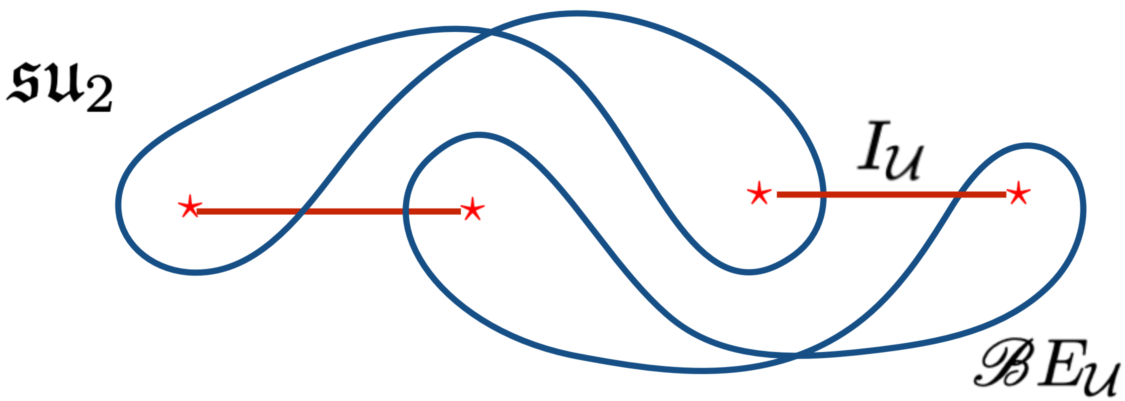


The graded space of morphisms

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

between the two branes

$$I_{\mathcal{U}}, \ \ \mathscr{B}E_{\mathcal{U}} \ \in \mathscr{D}_Y.$$

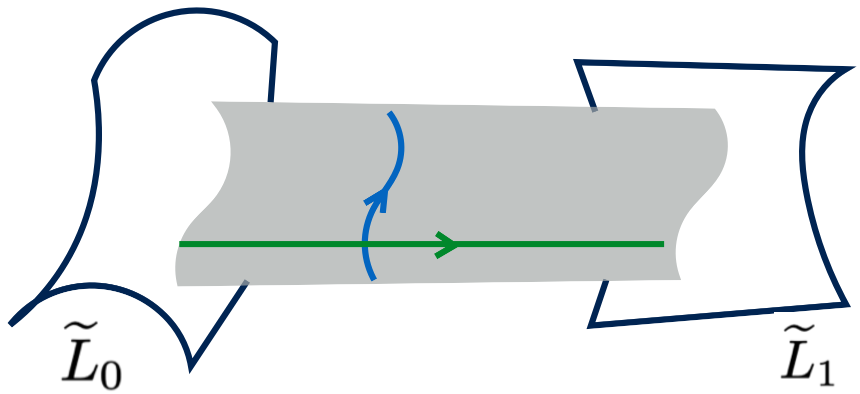


will turn out be an invariant of the link.

Spaces of morphisms between a pair of A-branes

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \mathrm{Ker}\, Q/\mathrm{Im}\, Q.$$

are defined by Floer theory,



which is modeled after Morse theory approach

to supersymmetric quantum mechanics,

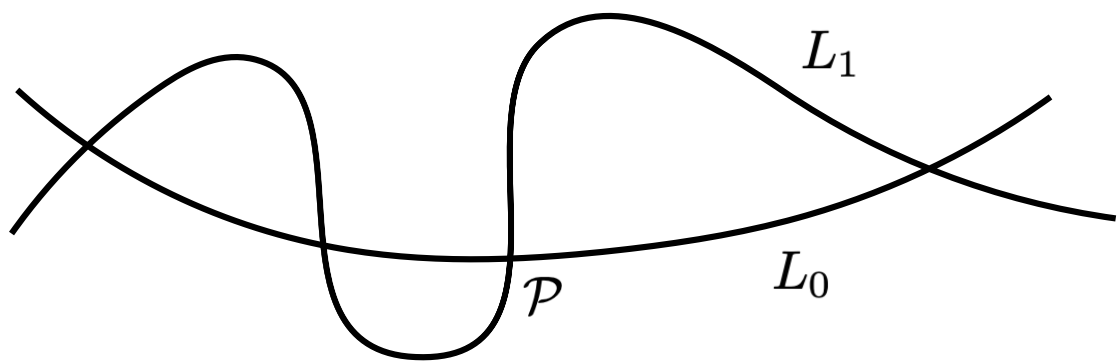
generalized to a theory in one dimension up.

The role of the Morse complex from the beginning of the talk

is taken by the Floer complex, which is a vector space

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

is spanned by the intersection points of the two Lagrangians,



The action of the Floer differential

$$\delta_F$$

which turns the space spanned by intersection points

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

into a complex

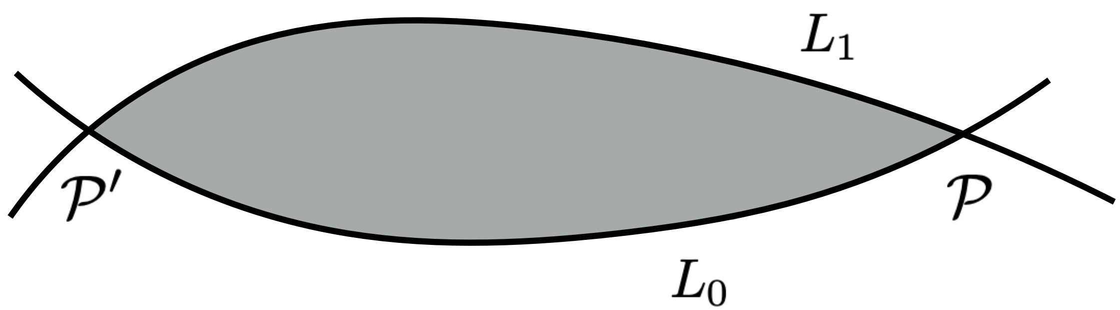
$$\delta_F : CF^{*,*}(L_0, L_1) \to CF^{*+1,*}(L_0, L_1)$$

is generated by instantons.

In Floer theory,

the coefficient of $\mathcal{P}'$ in $\delta_F \mathcal{P}$

is obtained by counting holomorphic maps from a strip to the target,



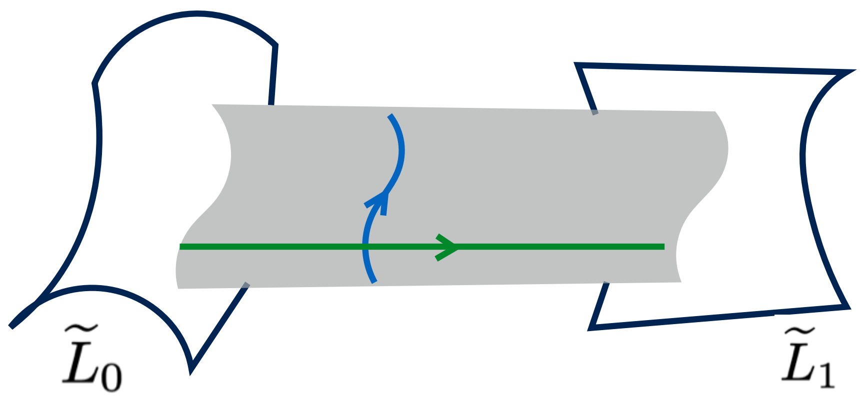
interpolating from $\mathcal{P}$ to $\mathcal{P}'$ of cohomological degree one

and equivariant degree zero.

The cohomology of the resulting complex

$$Hom^{*,*}_{\mathscr{D}_Y}(L_0, L_1) = \mathrm{Ker}\,\delta_F / \mathrm{Im}\,\delta_F$$

is the space of morphisms between the branes in $\mathscr{D}_Y$.



The Euler characteristic of

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

is the equivariant intersection number:

$$\chi(E, \mathscr{B}I) = \sum_{\mathcal{P} \in E \cap \mathscr{B}I} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J(\mathcal{P})}$$

which is the count of intersection points,

$$\mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}$$

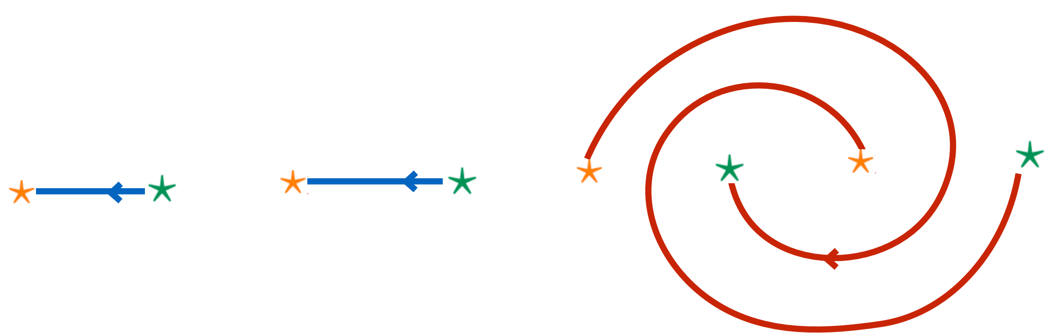
keeping track of gradings.

The fact that the Euler characteristic of

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

is the quantum $U_q(\mathfrak{g})$ link invariant

is guaranteed to hold by construction.



This follows from Picard-Lefshetz theory, and fact that

the equivariant central charge function

$$\mathcal{Z} : \mathscr{D}_Y \to \mathbb{C}$$

which is given by

$$\mathcal{Z}[L] = \int_L \Omega\, e^{-W},$$

is a close cousin of the conformal block of

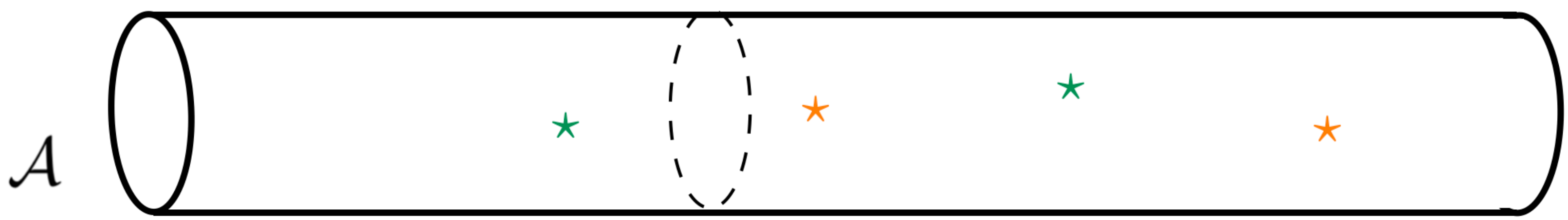
$$\widehat{\mathfrak{g}}_k$$

the affine Lie algebra associated to $\mathfrak{g}$.

The conformal blocks of

$$\widehat{\mathfrak{g}}_k$$

on the Riemann surface we started with



have integral formulation as period integrals:

$$\mathcal{V}_\alpha[L] = \int_L \Phi_\alpha \, \Omega \, e^{-W}$$

of Feigin and E.Frenkel, and Schechtman and Varchenko, which differ from

$$\mathcal{Z}[L] = \int_L \Omega \, e^{-W},$$

by insertions which do not affect monodromy of the integral.

In what follows, I will describe the category,

$$\mathscr{D}_Y$$

and how to solve the theory exactly.

We will learn how to compute the homology theory

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

for any link,

and why the resulting vector spaces are themselves invariants of links.

Rather than computing the action of the differential

by counting holomorphic curves,

for which there is no general algorithm,

we will explain how to sum the instantons up.

One solves all the disk counting problems at once,

by making homological mirror symmetry

$$\mathscr{D}_X \cong \mathscr{D}_Y$$

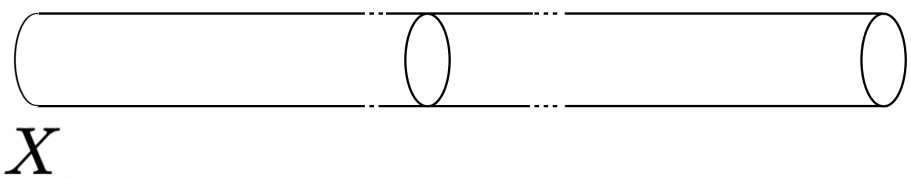
manifest.

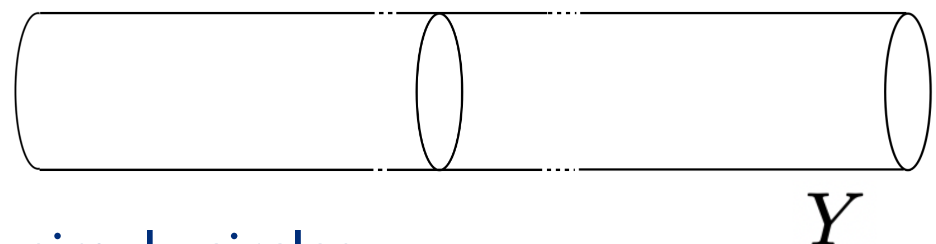
The fact mirror symmetry can sum up curve counts is its basic property.

The very simplest example of homological mirror symmetry is when

X and Y

are taken to be a pair of infinite cylinders,



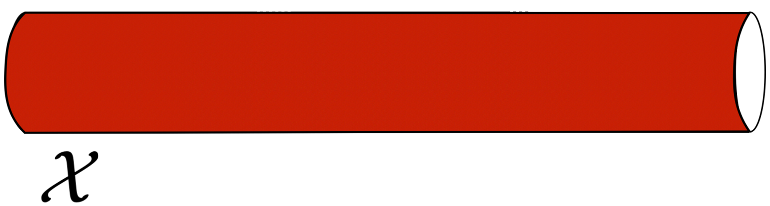


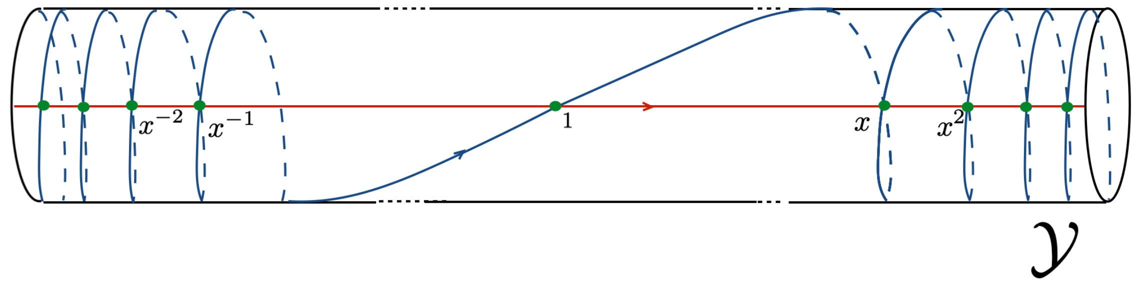
their torus fibers being simply circles,

the common base a real line.

Categories of branes on the two sides turn out to be

each generated by a single brane:

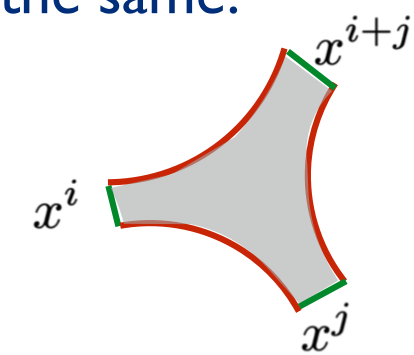




While the branes look different,

their algebras of open strings are the same.

$$\mathscr{A} = \mathbb{C}[x, x^{-1}]$$

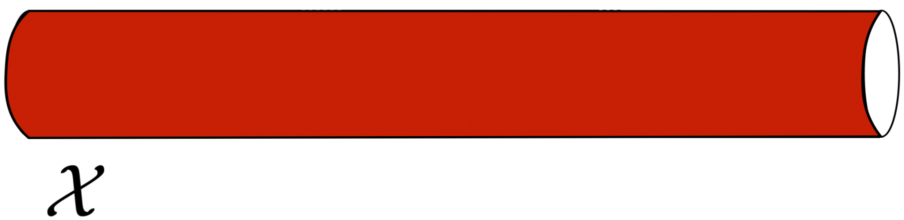


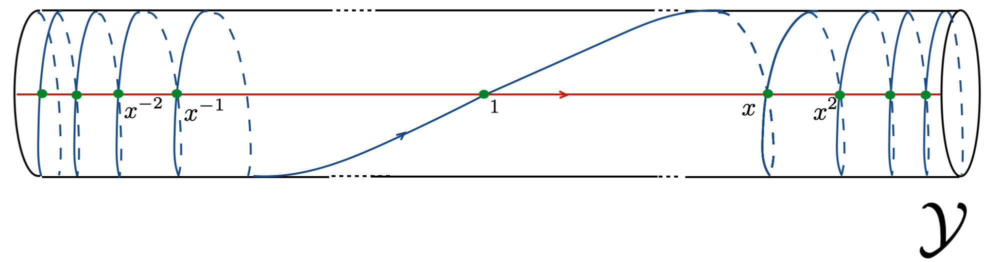
The algebra is simply the algebra of functions on a complex cylinder

The fact that the algebras of open strings on both sides

are the same, equal to

$$A = \mathbb{C}[x, x^{-1}]$$





turns out to mean that the entire categories of branes are equivalent,

$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_Y$$

both being equivalent to a derived category

$$\mathscr{D}_A$$

whose objects are (complexes of) modules of A

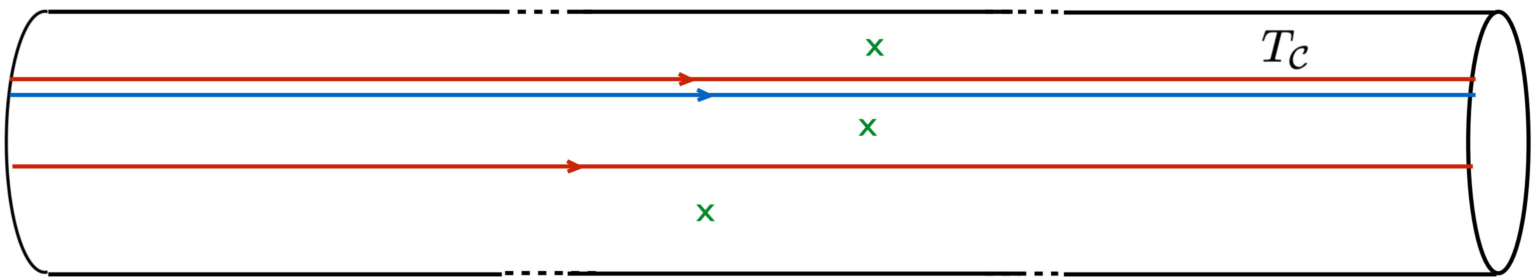
The complex, in turn, describes a geometric brane, in terms of the generators.

Our theories generalize this very simplest example.

From perspective of

Y

the generating set of branes



are all products of real line Lagrangians,

$$T_{\mathcal{C}} = T_{i_1} \times T_{i_2} \times \ldots \times T_{i_d}$$

colored by simple roots.

For ${}^L\mathfrak{g} \neq \mathfrak{gl}_{1|1}$, the branes will get equipped with extra structure,

of a local system.

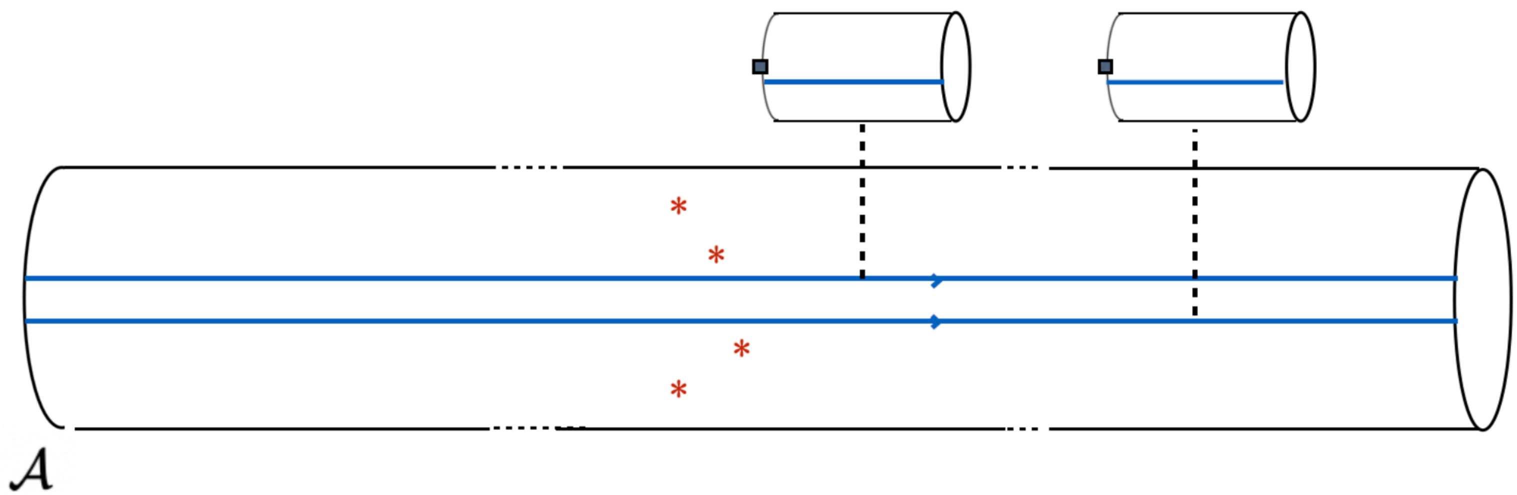
The full $T_{\mathcal{C}}$ -brane, is obtained

by pairing, every real line Lagrangians in the base

colored with a bosonic simple root of

$\mathfrak{g}$

with real-line Lagrangian in the $\mathbb{C}^\times$ fiber over it



The open string algebra

$$A = hom^{*,*}(T,T)$$

where

$$T = \bigoplus_{\mathcal{C}} T_{\mathcal{C}}$$

is computable explicitly.

As a vector space,

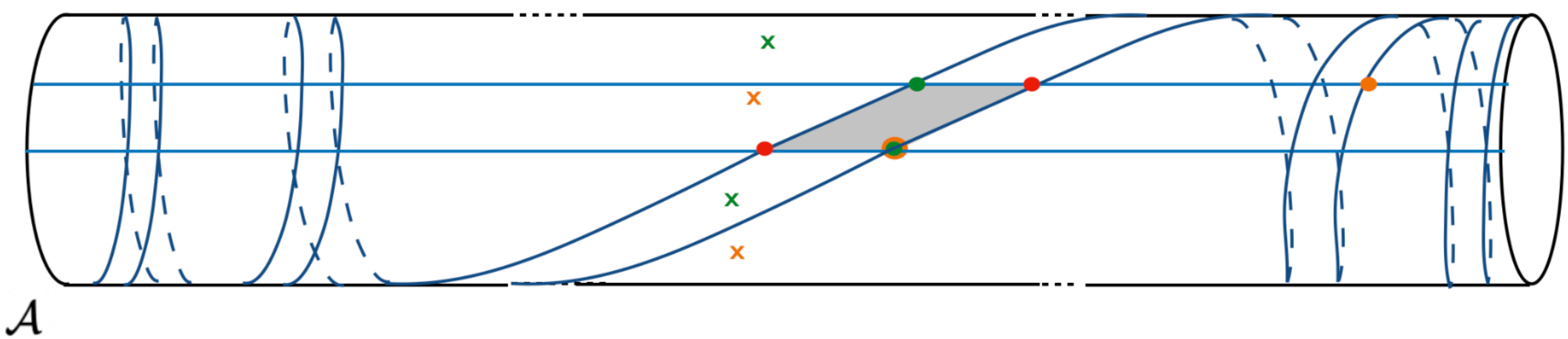
the endomorphism algebra of the T-branes

$$T = \bigoplus_{\mathcal{C}} T_{\mathcal{C}}$$

is generated by their intersection points

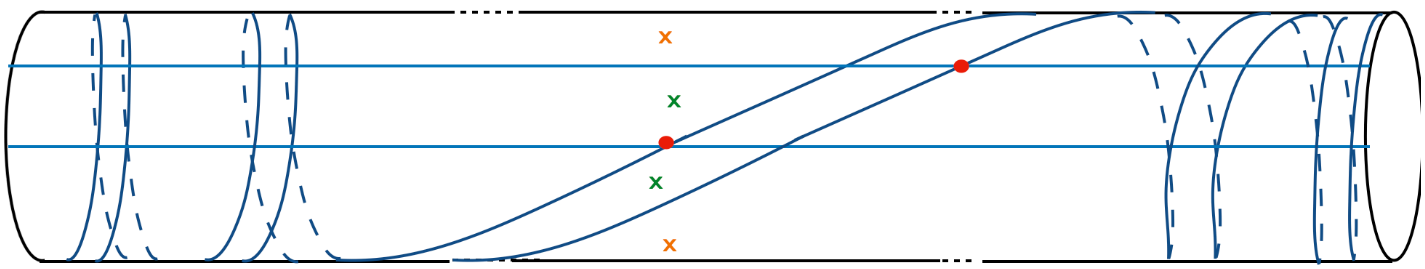
$$A = hom^{*,*}(T, T)$$

defined by the wrapped Fukaya category:

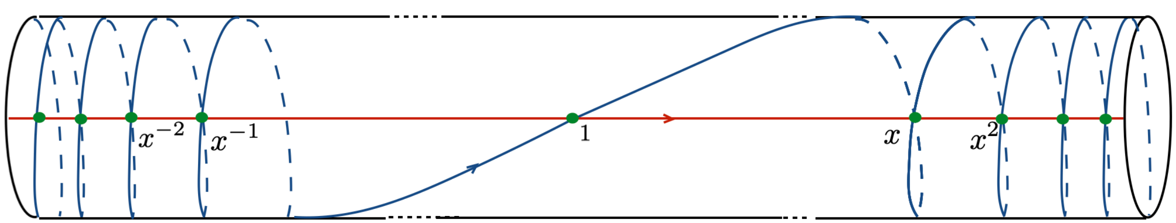


The endomorphism algebra being generated by intersection points

$$\mathcal{P} \in T^{\zeta}_{\mathcal{C}} \cap T_{\mathcal{C}'}$$



is analogous to our model example:

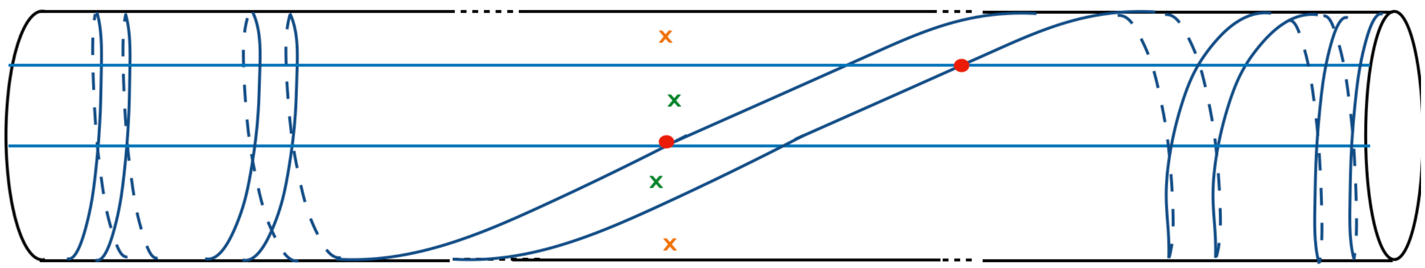


We only need to interpret them as intersections in

$$Y = \prod_{a=1}^{\mathrm{rk}\mathfrak{g}} Sym^{d_a}(\mathcal{A})$$

The intersection points of wrapped branes in the base

$$\mathcal{P} \in T^{\zeta}_{\mathcal{C}} \cap T_{\mathcal{C}'}$$



represent open strings,

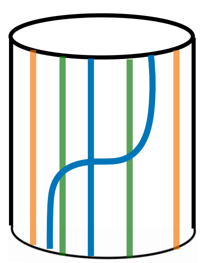
stretching between the pairs of unperturbed branes:

The algebra elements

$$\mathcal{P} \in T_{\mathcal{C}}^{\zeta} \cap T_{\mathcal{C}'}$$

are naturally encoded as

configurations of d blue strings on a cylinder of unit height



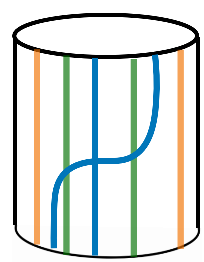
whose vertical direction parameterizes the open string

and whose horizontal direction represents the S^1 in $\mathcal{A}$

The blue strings encode the open string corresponding to

$$\mathcal{P} \in T^{\zeta}_{\mathcal{C}} \cap T_{\mathcal{C}'}$$

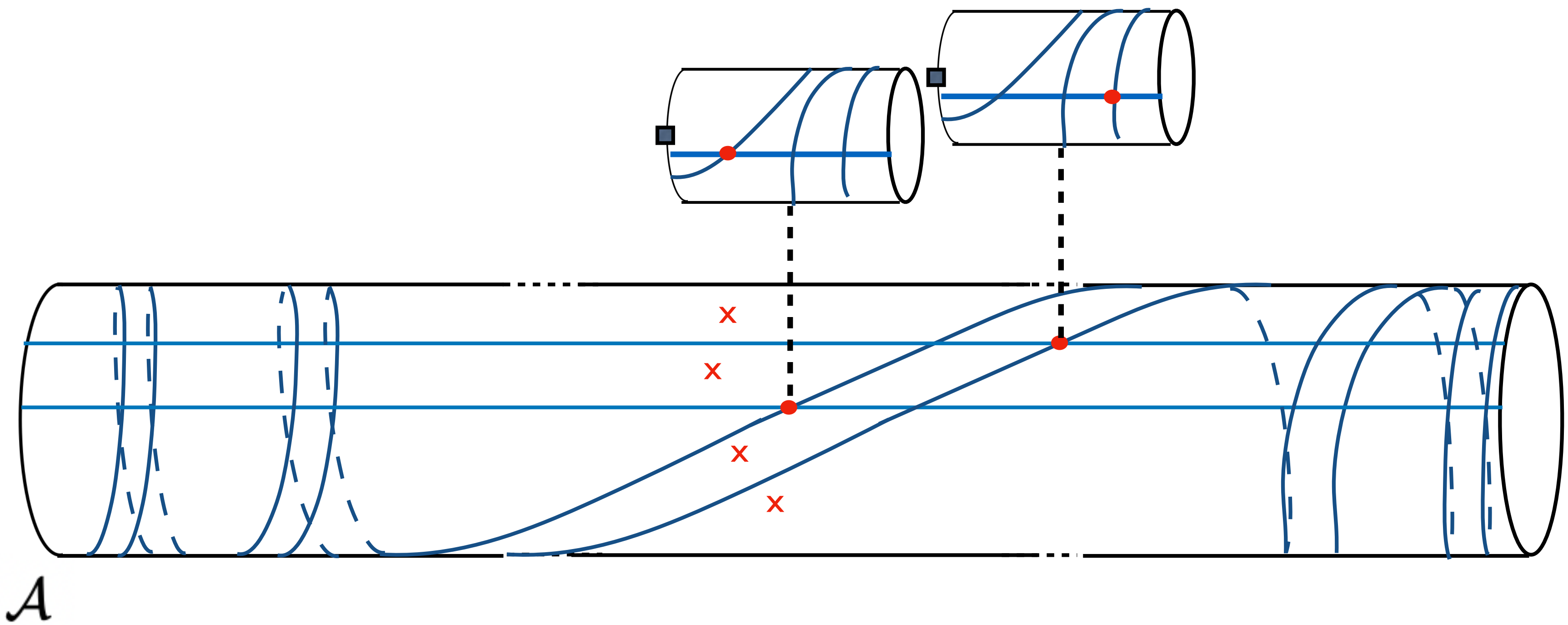
as a path from the position of $T_{\mathcal{C}}$ to $T_{\mathcal{C}'}$ branes on the S^1



The punctures get represented as orange and green strings, frozen in ``time''.

The endomorphism algebra of the branes in the fibers is

a copy of a polynomial algebra,

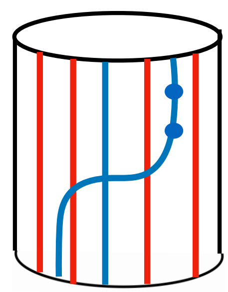


due to the nature of the fiberwise potential.

Multiplying an intersection point

$$\mathcal{P} \in T^{\zeta}_{\mathcal{C}} \cap T_{\mathcal{C}'}$$

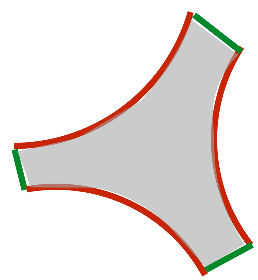
by an element $z_1^{k_1} \cdots z_d^{k_d}$ of the polynomial algebra,



can be represented by post-composing the blue strings

by the corresponding number of dots.

The algebra relations come from the Floer product:



which translates into stacking cylinders,

and rescaling.

To compute the algebra structure, and to describe the resulting category,

we will start not with Y but with

$$Y_0 = Y \backslash \Delta$$

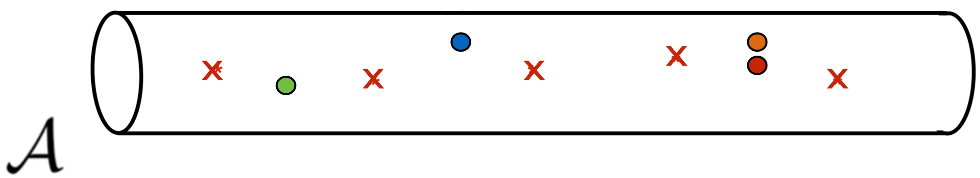
with a divisor Δ deleted.

The deleted divisor

$$\Delta$$

has as its main component the ``diagonals'',

where any two points of the same color come together on $\mathcal{A}$.



along with some additional ones.

We have the equivalence,

$$\mathscr{D}_{A_0} \cong \mathscr{D}_{Y_0}$$

where

$$A_0 = hom^{*,*}_{Y_0}(T,T)$$

is the endomorphism algebra of the same

$$T = \bigoplus_{\mathcal{C}} T_{\mathcal{C}}$$

brane just based on $Y_0 = Y \backslash \Delta$ instead of on Y , The two categories

$\mathscr{D}_{Y_0}$ and $\mathscr{D}_{Y}$

have the same objects,

and the generators of Floer complexes are the same as well,

since both are based on branes that avoid the diagonal.

The theory with

$$\Delta$$

turns out to be very simple.

By deleting the diagonal,

the only maps that contribute are products of

one dimensional maps to $\mathcal{A} \times \mathbb{C}^{\times}$

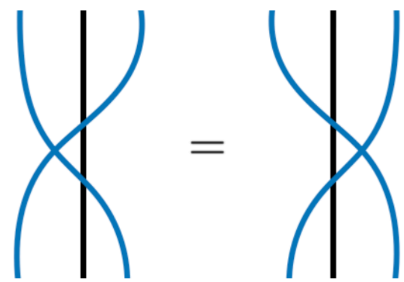
which are easy to count.

The relations in

$$Y_0 = Y \backslash \Delta$$

say that string diagrams must have no excess intersection,

or the algebra product vanishes, as well as



As a result, A_0 is an associative algebra,

graded by cohomological and equivariant degrees,

which is easy to compute explicitly.

The theory we want, on

$$Y$$

is a deformation of the theory on

$$Y_0 = Y \backslash \Delta$$

The deformation parameter is the instanton counting parameter

$$\hbar^{\#}$$

that counts the intersection $\#$ with the diagonal Δ

As vector spaces, the algebras

$$A_0 = hom^{*,*}_{Y_0}(T, T)$$

and

$$A_\hbar = hom^{*,*}_{Y}(T, T)$$

are the same, as T -branes avoid the diagonal Δ

$$Y_0 = Y \backslash \Delta$$

only the algebra structure deforms.

It turns out that understanding the

$\mathfrak{gl}_{1|1}$ and $\mathfrak{su}_2$

theories suffices to solve the theory for general

$\mathfrak{g}$

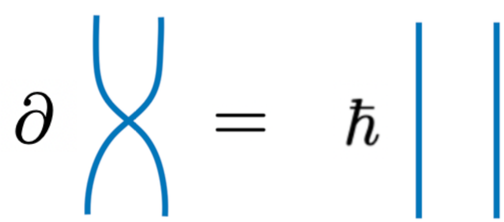
We only need to count two non-trivial disks,

in the entire theory,

one coming from $\mathfrak{gl}_{1|1}$ and (a highly non-trivial) one from $\mathfrak{su}_2$

In the $\mathfrak{gl}_{1|1}$ case, the algebra gains a non-trivial differential ∂

a cohomological degree one operator which acts by

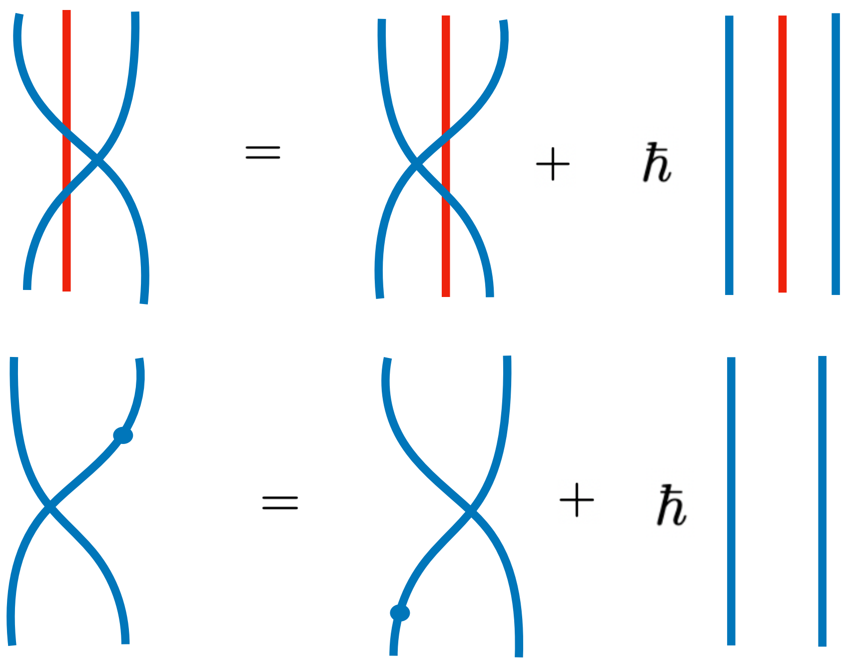


and squares to zero.

For

$\mathfrak{su}_2$

the relations deform:



For any

$$\mathfrak{g}$$

the same strategy of working in the complement of

$$\Delta = \sum_{a=1}^{\mathrm{rk}\mathfrak{g}} \Delta_a$$

the diagonal divisor in

$$Y = \prod_{a=1}^{\mathrm{rk}\mathfrak{g}} Sym^{d_a}(\mathcal{A})$$

and then filling it back in, to solves the theory, For

$$\mathfrak{gl}_{1|1}$$

the algebra of endomorphisms of the T-brane,

$$A_{\mathfrak{gl}_{1|1}} = hom^{*,*}(T,T)$$

is the differential graded, associative algebra of

Lipshitz, Ozsvath and Thurston.

Theorem (A., Danilenko, Li, Zhou, Shende)

For $\mathfrak{g}$ which is ordinary Lie algebra

$$A_{\mathfrak{g}} = hom^{*,*}(T, T)$$

coincides with the algebra

discovered by Khovanov and Lauda, and by Rouquier

and generalized by Webster,

known as KLRW algebras.

The fact that, for $\mathfrak{g}$ which is ordinary Lie algebra

$$A_{\mathfrak{g}} = hom^{*,*}(T,T)$$

coincides with the cylindrical KLRW algebra implies:

Corollary

For $\mathfrak{g}$ which is ordinary Lie algebra

our invariants of links in $\mathbb{R}^3$

coincide with the invariants Webster defined in 2013.

For Lie superalgebras other than

$$\mathfrak{gl}_{n|1}$$

the resulting algebra

$$A_{\mathfrak{g}} = hom^{*,*}(T,T)$$

is new.

We will now apply this to the problem of computing

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

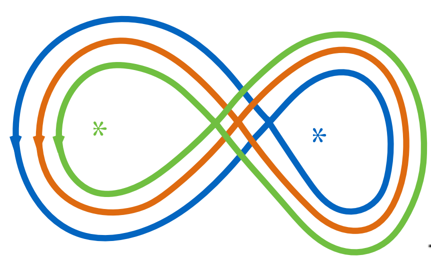
for the cap and cup branes

$$I_{\mathcal{U}}, \quad \mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y.$$

the former of which are simple interval type branes,

$I_{\mathcal{U}}$

the later are braided figure eight type branes:

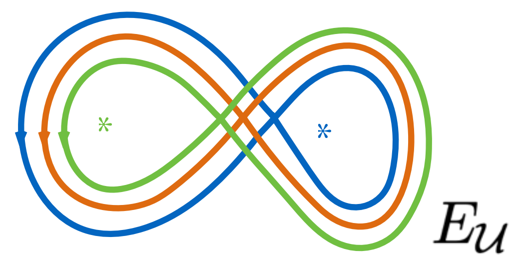


$E_{\mathcal{U}}$

The fact that T -branes generate the category means that,

as any brane in $\mathscr{D}_Y$, the braided cup branes

$$\mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y$$



have a description as a complex ,

$$(\mathscr{B}E(T), \delta)$$

which is a direct sum of $T_{\mathcal{C}}$ branes, with a differential $\delta \in A$

The differential

$$\delta \in A = hom^{*,*}(T,T)$$

is a degree one operator

$$\delta : \mathscr{B}E(T) \to \mathscr{B}E(T)[1]$$

that squares to zero in an appropriate sense,

It is an open string tachyon which gives a prescription for how to glue

$$\mathscr{B}E_{\mathcal{U}} \cong (\mathscr{B}E(T), \delta)$$

the brane of interest, from T-branes.

One of the simple, but key properties of the $T_{\mathcal{C}}$ -branes is that

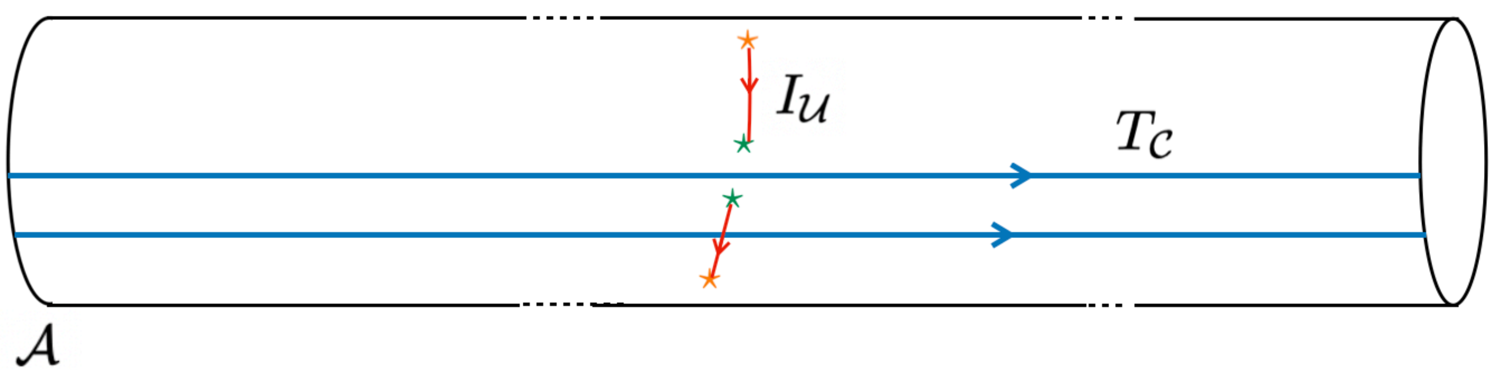
the cap branes

$$I_{\mathcal{U}} = I_1 \times \ldots \times I_d$$

are dual to them,

in the sense that the only non-zero morphism between them is:

$$hom(T_{\mathcal{C}}, I_{\mathcal{U}}) = \mathbb{C}\delta_{\mathcal{C},\mathcal{U}}$$



If we know the complex resolving the braided cup branes in terms of T-branes

$$\mathscr{B}E_{\mathcal{U}} \cong (\mathscr{B}E(T), \delta)$$

by applying the functor,

$$hom^*_Y(-, I_{\mathcal{U}}\{\vec{J}\}) = \bigoplus_{k\in\mathbb{Z}} \mathrm{hom}_Y(-, I_{\mathcal{U}}[k]\{\vec{J}\}),$$

we get a complex of vector spaces whose cohomology

we are after:

$$\mathrm{Hom}_{\mathscr{D}_Y}\left(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}[k]\{\vec{J}\}\right) = H^k\left(\mathrm{hom}^*_Y(\mathscr{B}E_{\mathcal{U}}(T), I_{\mathcal{U}}\{\vec{J}\})\right)$$

The identification of the space of morphisms

$$\mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}[k]\{\vec{J}\})$$

with

$$H^k(\mathrm{hom}^*_Y(\mathscr{B}E_{\mathcal{U}}(T), I_{\mathcal{U}}\{\vec{J}\}))$$

the k-th cohomology group of the complex we found

is a tautological consequence of the equvalence of two descriptions

$$\mathscr{B}E_{\mathcal{U}} \cong (\mathscr{B}E(T), \delta)$$

of a single A-brane.

To find the resolution

$$(\mathscr{B}E(T), \delta)$$

one apriori has to compute which module of the algebra

$$A = hom^{*,*}(T, T)$$

the brane maps to by the functor

$$\mathscr{B}E \in \mathscr{D}_Y \to hom^{*,*}(T, \mathscr{B}E) \in \mathscr{D}_A$$

and then find the resolution of this module.

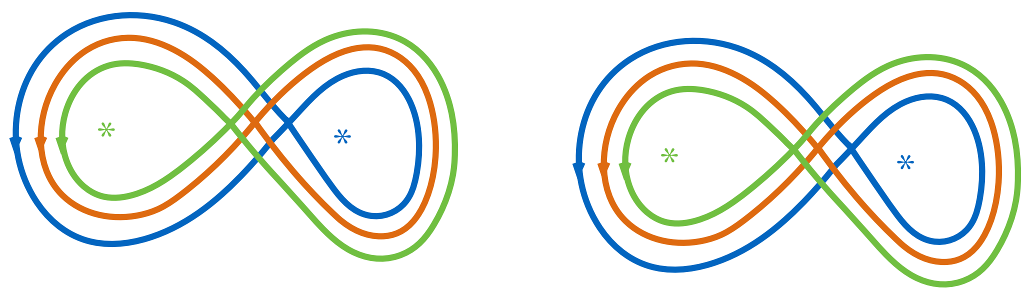
Both of these problems are generally solvable only in principle,

though not in practice.

In our setting, we can solve them both at once.

thanks to the fact the braided cap branes

$$\mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y$$



are products of one dimensional Lagrangians on $\mathcal{A}$

I will describe only the two simplest examples,

when the Lie algebra

$$\mathfrak{g}$$

is either

$$\mathfrak{su}_2 \qquad\qquad \mathfrak{gl}_{1|1}$$

with links colored by the defining representation and its conjugate.

The former theory will categorify the Jones polynomial,

the later the Alexander polynomial.

In both cases, the Lie algebra has a single simple root,

which is fermionic for

$$\mathfrak{gl}_{1|1}$$

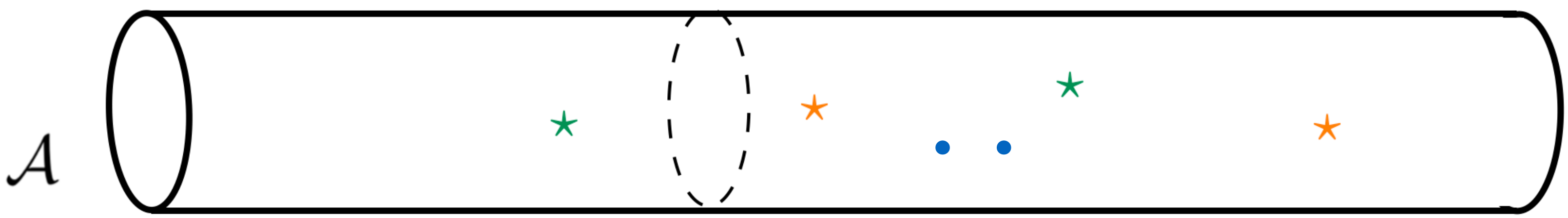
and bosonic for

$$\mathfrak{su}_2$$

Both theories are based on the same

symmetric product of d copies of the Riemann surface,

$$Y = Sym^d(\mathcal{A}),$$



The difference between the two theories is in

the multi-valued potential W and the top holomorphic form Ω .

and the flat vector bundles the branes may be equipped with.

Objects of

$$\mathscr{D}_Y$$

are Lagrangians on $Y = Sym^d(\mathcal{A})$ which are products

$$L = L_1 \times L_2 \times \ldots \times L_d$$

of d one-dimensional curves on $\mathcal{A}$

which we take to be non-intersecting.

Given a link in

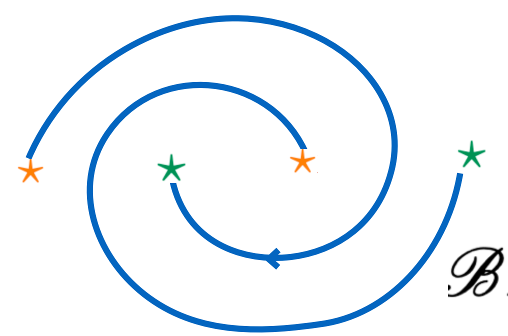
$$\mathcal{A} \times \mathbb{R}$$

we get a pair of such branes,

$$I_{\mathcal{U}}, \quad \mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y .$$

derived from the corresponding the red and blue matchings,

$I_{\mathcal{U}}$



$\mathscr{B}E_{\mathcal{U}}$

In both theories,

the objects which we will associate to the cups,

will be "I-branes"

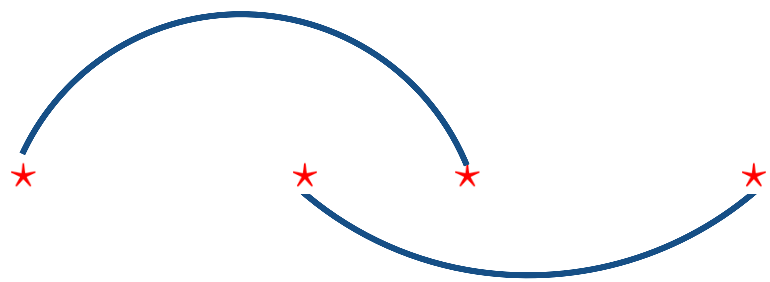
$$I_{\mathcal{U}} = I_1 \times \ldots \times I_d$$

which are the simple products of intervals pictured, The cap branes, which we will denote by:

$$E_{\mathcal{U}} = E_1 \times \ldots \times E_d$$

will be products of closed curves

whose homology class is proportional to the class of



the braided caps.

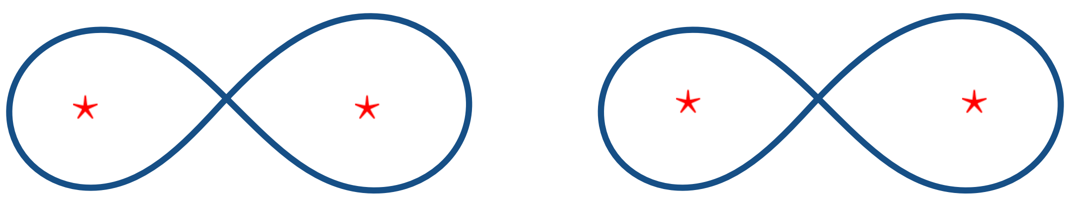
When the Lie algebra is

$$\mathfrak{su}_2$$

the cap branes

$$E_{\mathcal{U}} = E_1 \times \ldots \times E_d$$

are products of figure eights:



When the Lie algebra is

$$\mathfrak{gl}_{1|1}$$

the cap branes

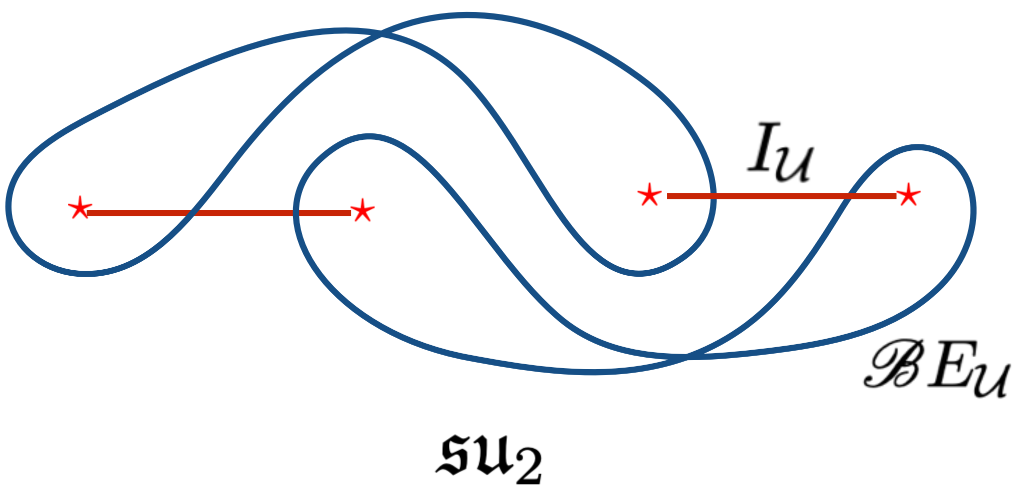
$$E_{\mathcal{U}} = E_1 \times \ldots \times E_d$$

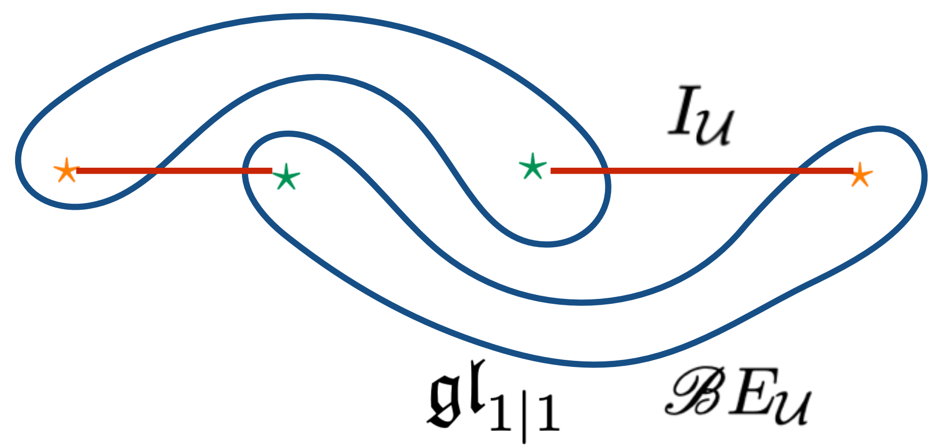
are products of ovals:

The link homologies

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \mathrm{Ker}\,\delta_F/\mathrm{Im}\,\delta_F$$

are the cohomologies of the Floer differential,





acting on the vector space spanned by graded intersection points of the branes.

We can describe the story

for the

$\mathfrak{su}_2$ and $\mathfrak{gl}_{1|1}$

in parallel.

As a warmup, we will start by considering

$$d = 1$$

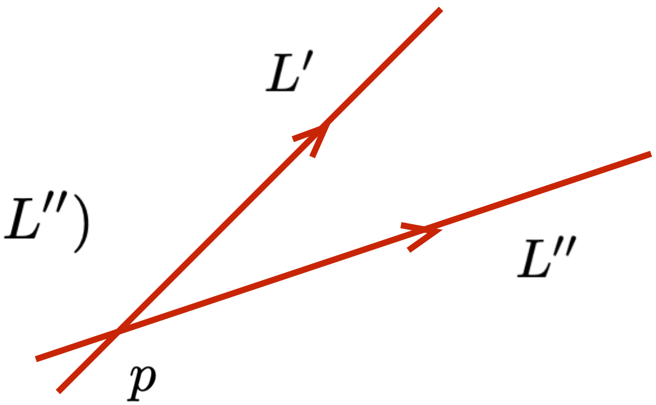
theories, when our target is simply the Riemann surface itself.

$$Y = \mathcal{A}$$

This case is fundamental for all that will follow.

If two one dimensional Lagrangians, intersect over a point

$$p \in Hom_{\mathscr{D}_Y}(L', L'')$$

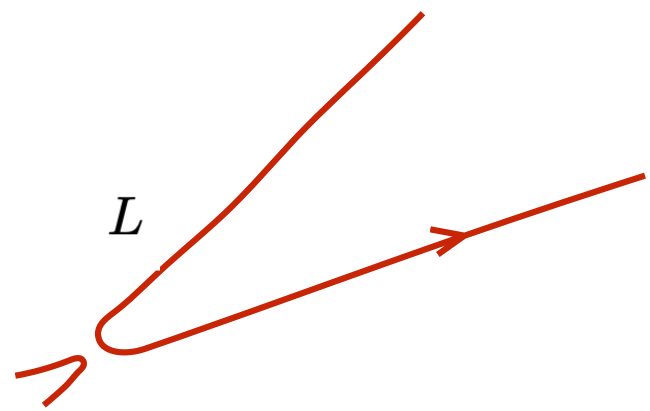


we get a new one dimensional Lagrangian

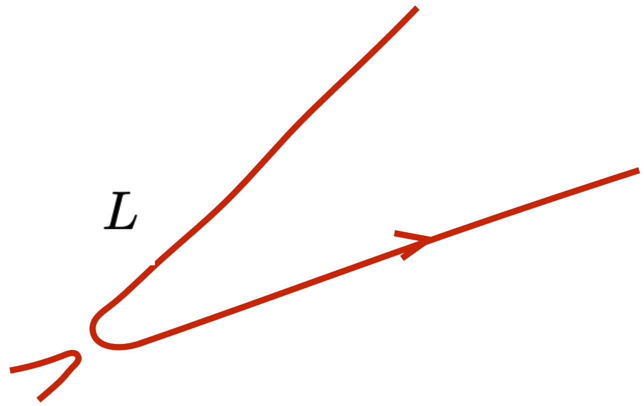
by starting with

$$L'[1] \oplus L''$$

by taking their connected sum at p .



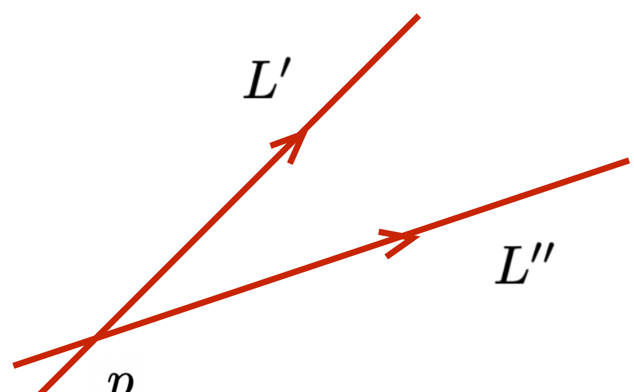
As object of $\mathscr{D}_Y$, the connected sum brane



is equivalent to the complex

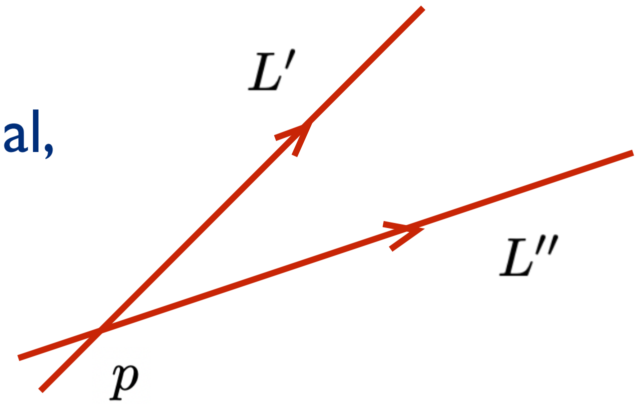
$$L \;\cong\; L' \xrightarrow{p} L''$$

which is known as the cone over



$$p \in Hom_{\mathscr{D}_Y}(L', L'')$$

Thus, if the target is one dimensional,

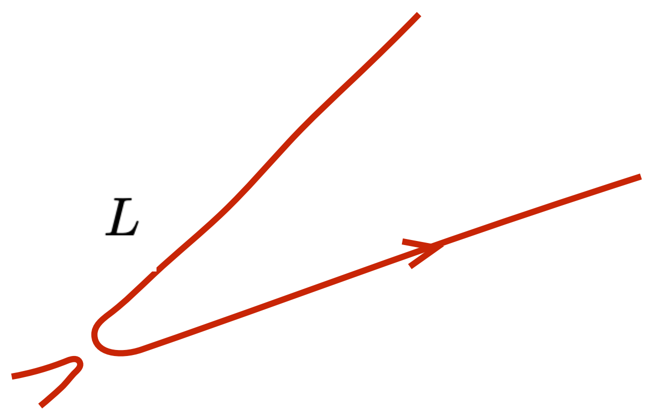


taking cones

$$L \;\cong\; L' \xrightarrow{p} L''$$

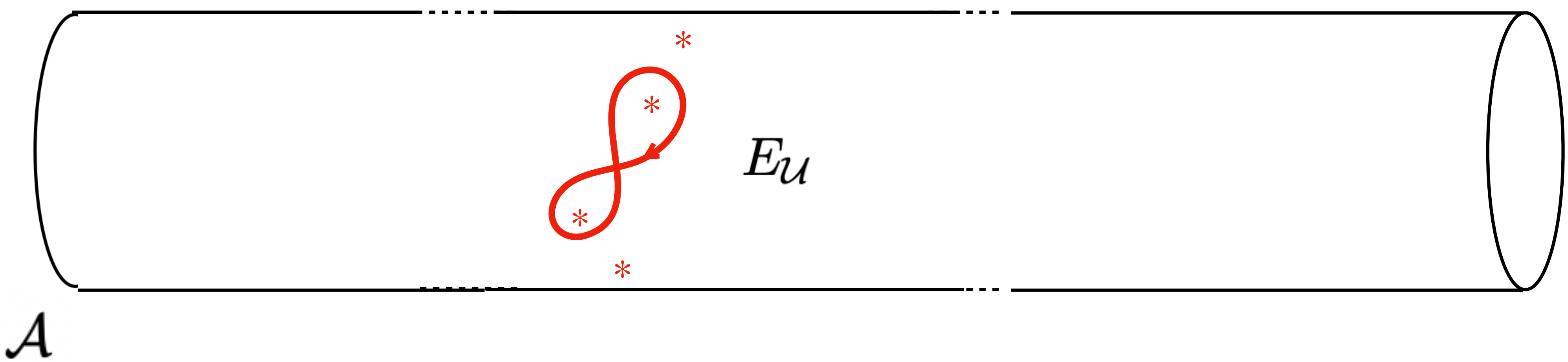
over morphisms in the derived category

has a geometric interpretation of taking connected sums.



Consider, for example, the cup brane

in the $\mathfrak{su}_2$ theory



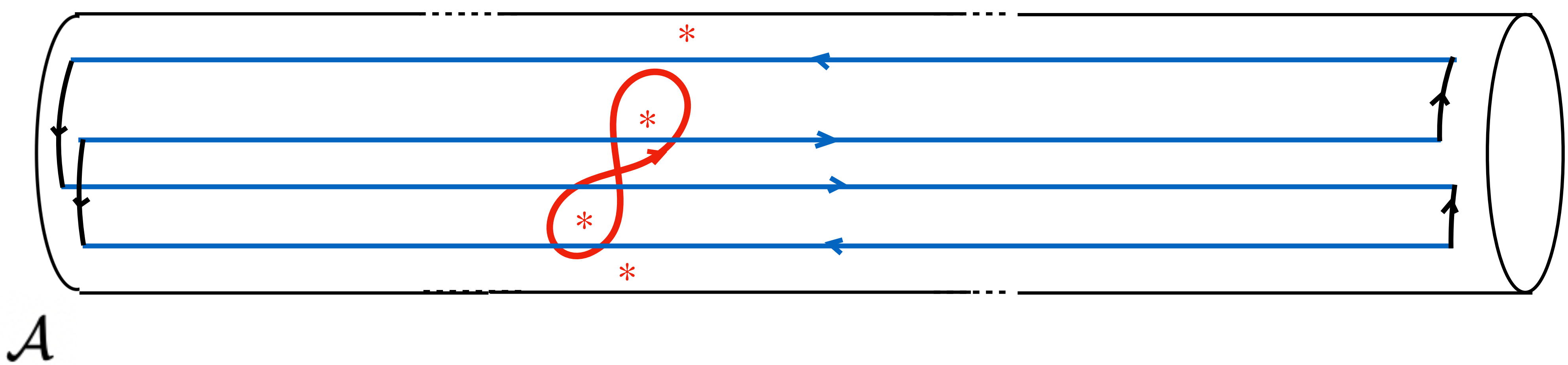
We recover the

$$E_{\mathcal{U}}$$

brane from the direct sum of T- branes

$$E_{\mathcal{U}}(T)$$

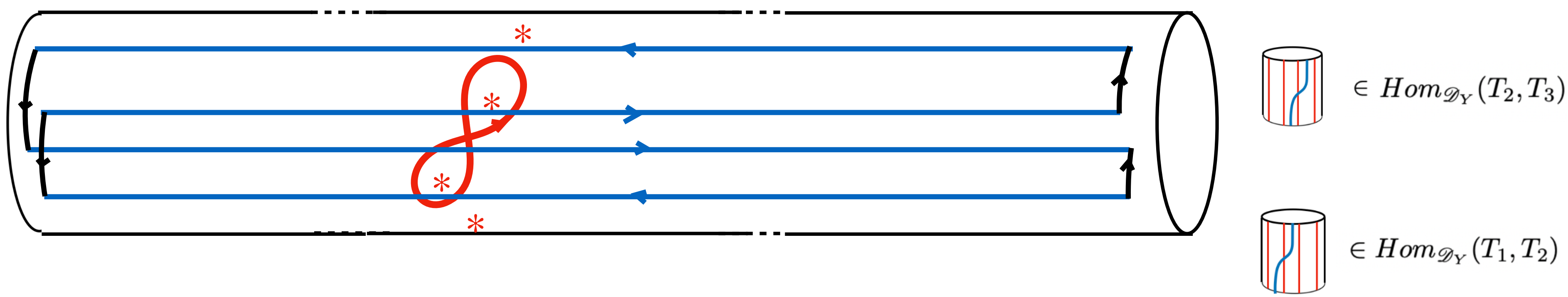
by taking connected sums over intersection points at infinity



Each intersection point at infinity is a specific

element of the algebra

of definite equivariant and cohomological degrees.



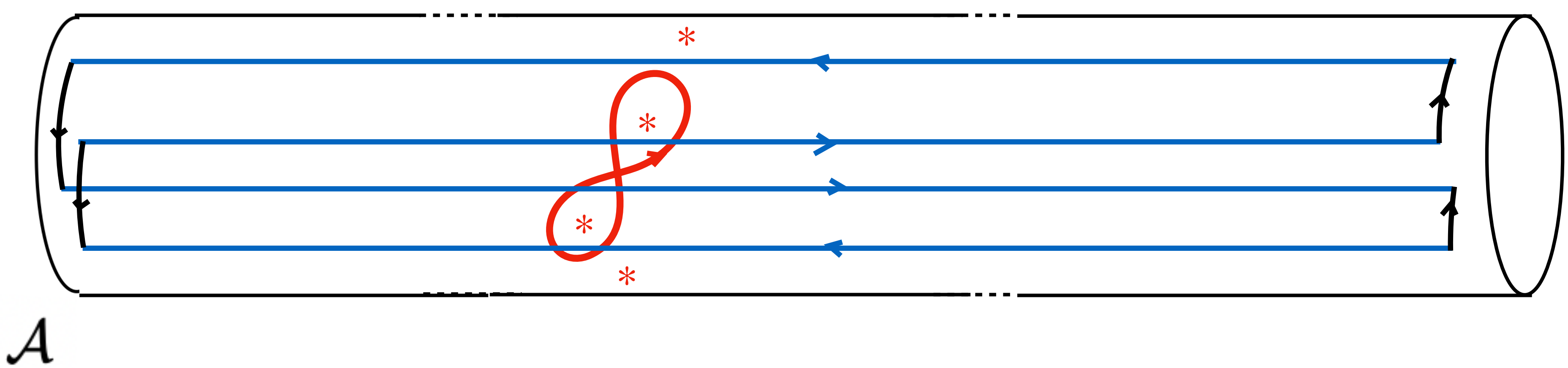
To write the corresponding complex,

we can start on any one T-brane, and record the Hom's we find,

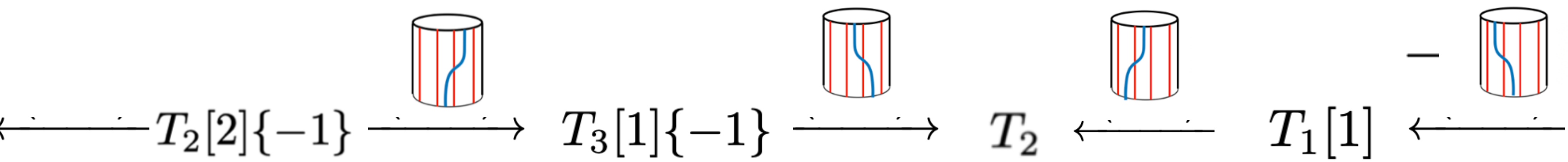
as we go around the brane,

shifting degrees accordingly

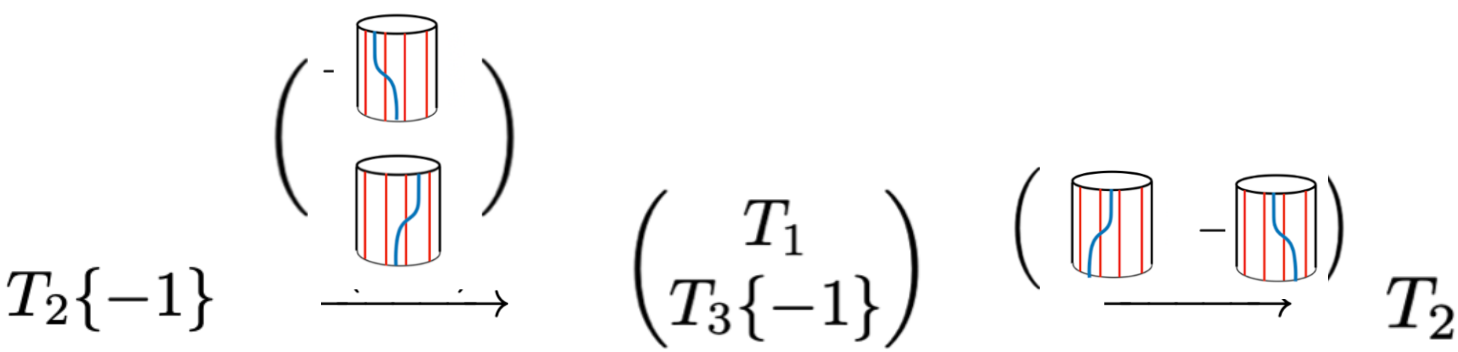
$$\longleftarrow T_2[2]\{-1\} \longrightarrow T_3[1]\{-1\} \longrightarrow T_2 \longleftarrow T_1[1] \longleftarrow$$



Organizing the result by cohomological degree

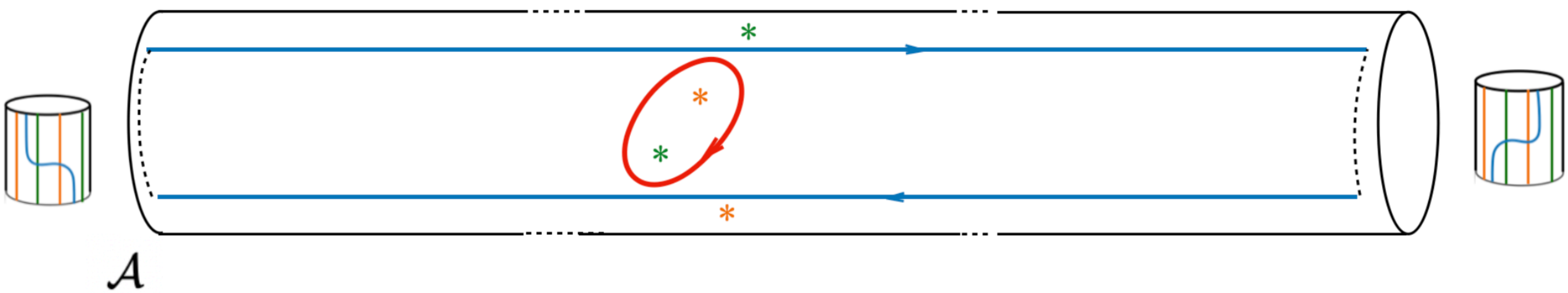


we get the complex resolving the $E_{\mathcal{U}}$ brane



With the differential that squares to zero.

The cup brane in the $\mathfrak{gl}_{1|1}$ theory



is even simpler:

$$E(T) = T_0[1] \oplus T_2$$

with the differential $\delta = \begin{pmatrix} 0 & \cdot \\ \cdot & 0 \end{pmatrix}$, with upper-right entry  and lower-left entry , that squares to zero since

 $= \ 0 \ =$ 

More generally,

branes of interest are products of one dimensional ones,

$$\mathscr{B}E_{\mathcal{U}} = \mathscr{B}E_1 \times \cdots \times \mathscr{B}E_d$$

so, by working in

$$Y_0 = Y \backslash \Delta$$

all the holomorphic maps to Y which are not products of one dimensional maps to copies of $\mathcal{A}$ are removed.

As a result, the complex which describes the brane

$$\mathscr{B}E_{\mathcal{U}} = \mathscr{B}E_1 \times \cdots \times \mathscr{B}E_d$$

as an object of the category

$$\mathscr{D}_{Y_0} \cong \mathscr{D}_{A_0}$$

corresponding to

$$Y_0 = Y \backslash \Delta$$

with the divisor of diagonal removed

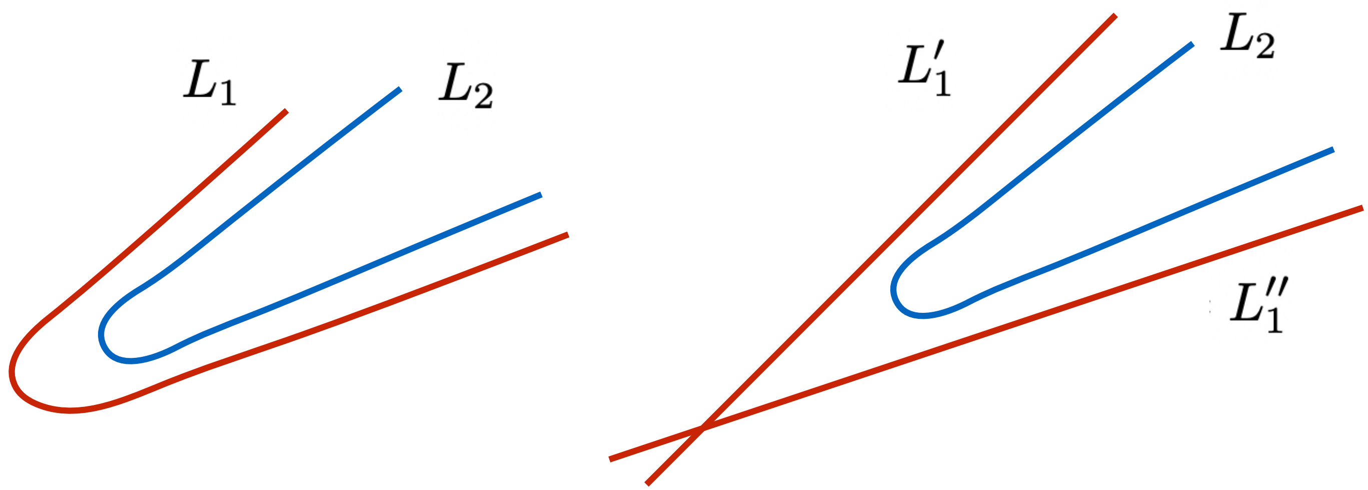
is a product of one dimensional complexes,

which are elementary to find.

Geometrically, each map in the product complex

is a cone over an intersection point of the form

$$\mathcal{P} = (p, id_{L_2}, ..., id_{L_d})$$



Having found the complex resolving the brane in

$$\mathscr{D}_{Y_0} \cong \mathscr{D}_{A_0}$$

we can use the fact that the theory on

$$Y$$

is the $\hbar$ - deformation of the theory on

$$Y_0 = Y \backslash \Delta$$

to find the complexes resolving the brane in

$$\mathscr{D}_Y \cong \mathscr{D}_{A_\hbar}$$

The algorithm can be run, in principle, for arbitrarily complicated branes,

starting with the(twisted) products of one dimensional complexes,

which are elementary to write down

and which by construction describe the brane in

$$\mathscr{D}_{Y_0} \cong \mathscr{D}_{A_0}$$

and then finding the deformation of the resulting differential to that in

$$\mathscr{D}_{Y} \cong \mathscr{D}_{A_\hbar}$$

This generalizes to an explicit algorithm for computing arbitrary

$$U_{\mathfrak{q}}(\mathfrak{g})$$

link homologies.

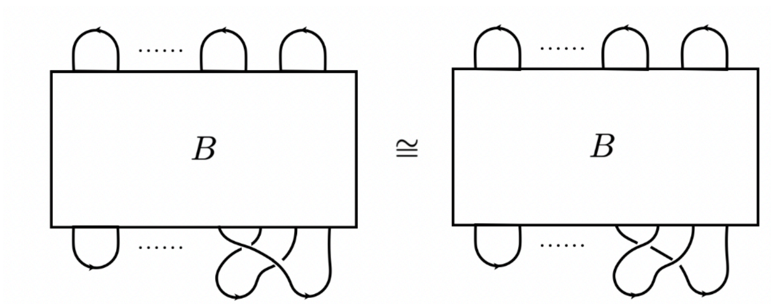
Theorem (A., LePage, Rapcak)

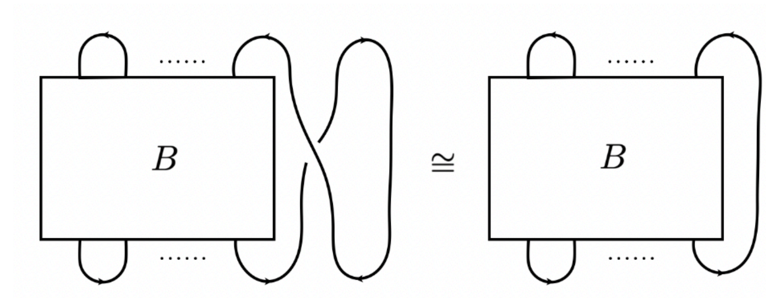
Homology groups $Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$ are invariants of links.

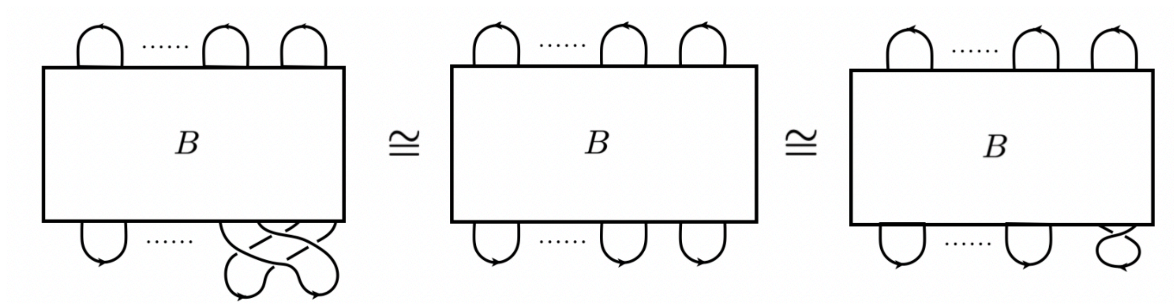
To prove

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

are invariants of a link we need to prove they satisfy Markov moves:

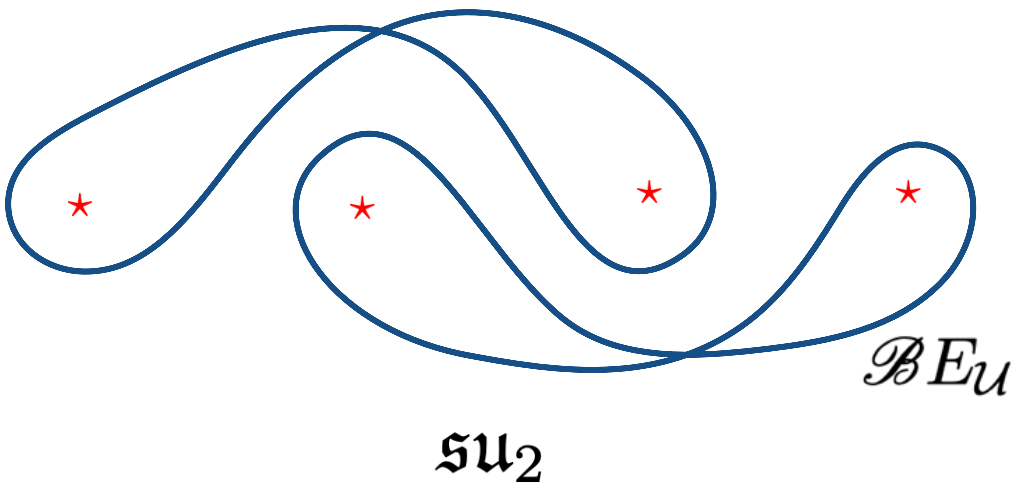


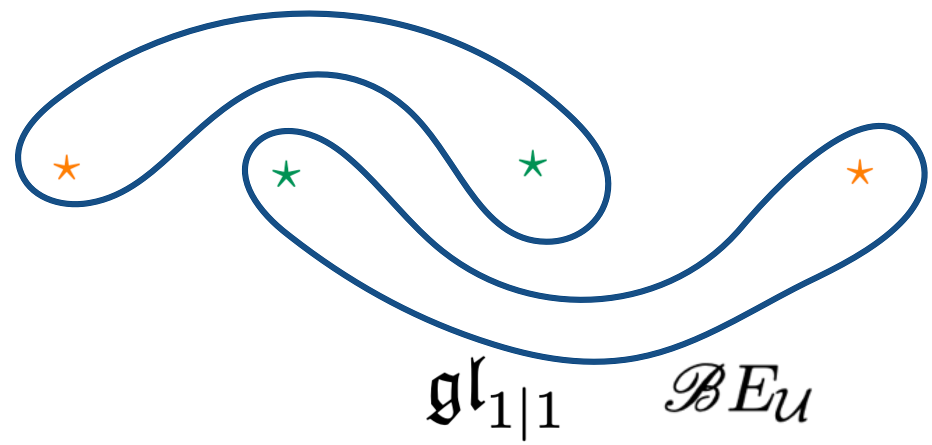




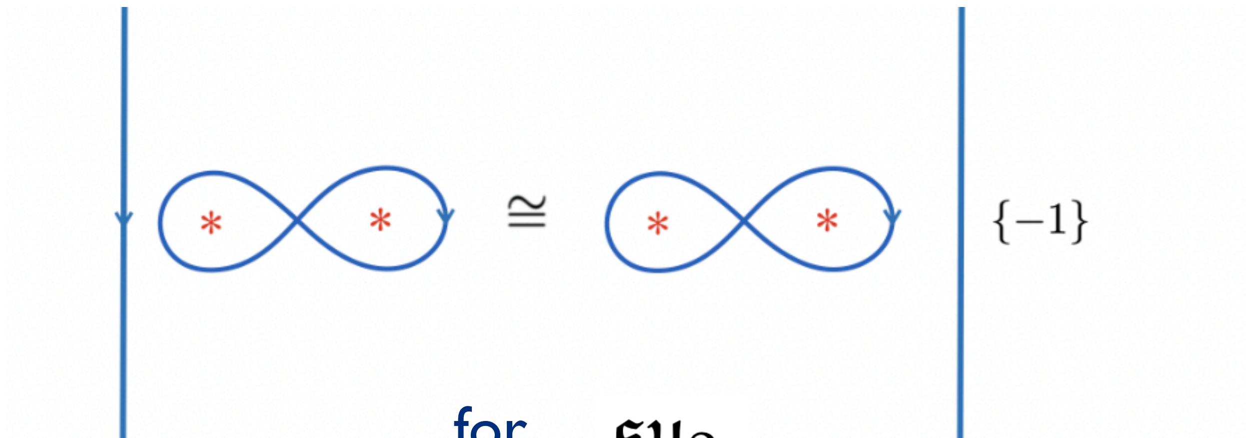
In our case, the moves hold as a consequence of equivalences

satisfied by the $\mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y$ branes

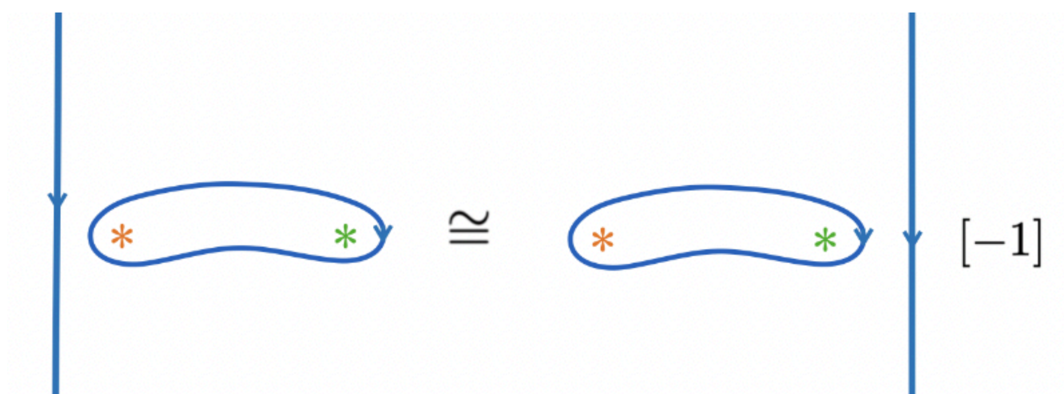




All moves follow, up to degree shifts, from a single statement:



for $\mathfrak{su}_2$

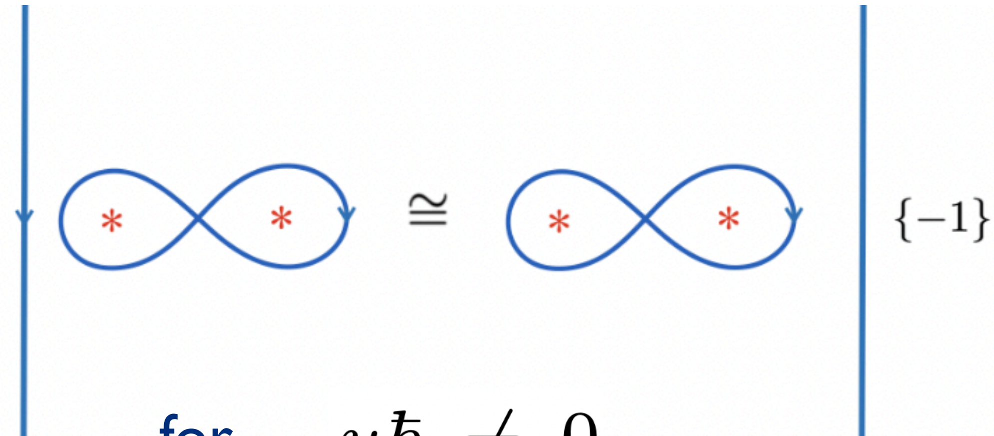


for $\mathfrak{gl}_{1|1}$

Since we know the explicit resolutions of branes on both sides

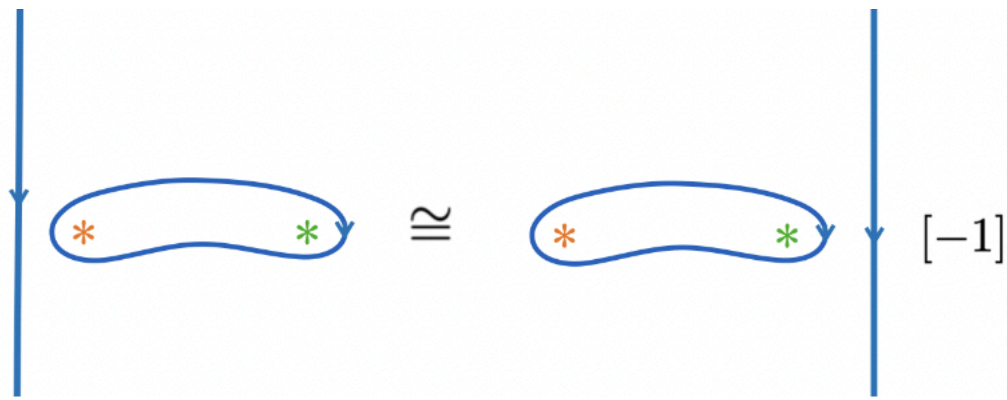
both, equivalences can be proven easily,

$\mathfrak{su}_2$



for $u\hbar \neq 0$

$\mathfrak{gl}_{1|1}$



for $\hbar \neq 0$

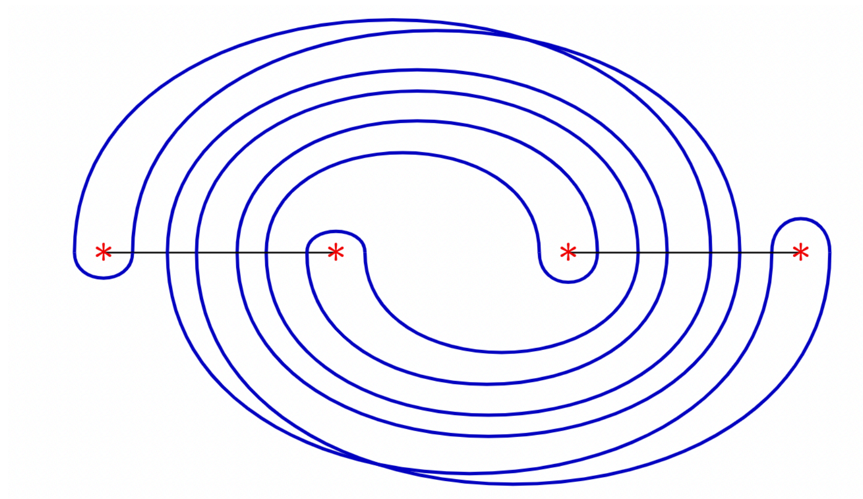
as homotopy equivalences of the underlying complexes

In the $\mathfrak{su}_2$ case,

the theory reproduces Khovanov homology

including torsion.

For example, for the trefoil, we get the following pair of branes.



The theory computes the Floer complex, over $\mathbb{C}$, as:

$$\mathbb{C} \to \emptyset \to \begin{matrix}\mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-1\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\\-1&-1&0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\-1&-1&0&0\\0&0&-1&-1\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}0&0&0&0\\0&0&0&0\\-1&-1&0&0\\1&1&0&0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-4\}\\ \mathbb{C}\{-4\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}1&1&0&0\\0&0&1&1\end{pmatrix}} \begin{matrix}\mathbb{C}\{-4\}\\ \mathbb{C}\{-3\}\end{matrix}$$

The complex is 18-dimensional,

which should be compared to 30 dimensional complex Khovanov defined.

The same complex

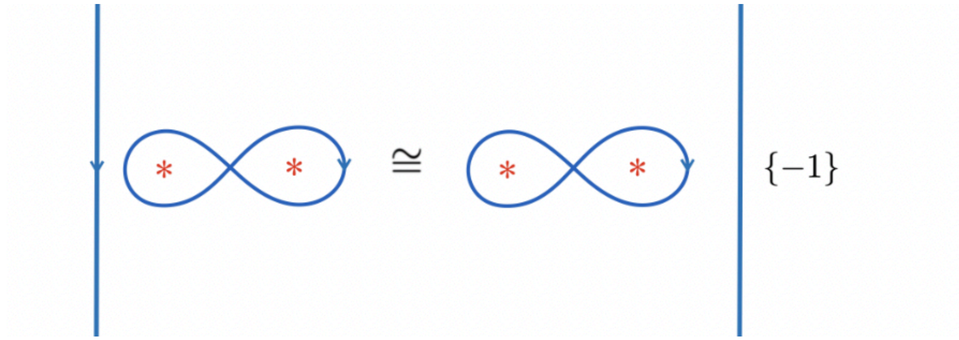
$$\mathbb{C} \to \emptyset \to \begin{matrix}\mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-1\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\\-1&-1&0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\-1&-1&0&0\\0&0&-1&-1\end{pmatrix}} \begin{matrix}\mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-2\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}0&0&0&0\\0&0&0&0\\-1&-1&0&0\\1&1&0&0\end{pmatrix}} \begin{matrix}\mathbb{C}\{-4\}\\ \mathbb{C}\{-4\}\\ \mathbb{C}\{-3\}\\ \mathbb{C}\{-3\}\end{matrix} \xrightarrow{u\hbar\cdot\begin{pmatrix}1&1&0&0\\0&0&1&1\end{pmatrix}} \begin{matrix}\mathbb{C}\{-4\}\\ \mathbb{C}\{-3\}\end{matrix}$$

with $\mathbb{C}$ replaced with $\mathbb{Z}$ reproduces Khovanov homology

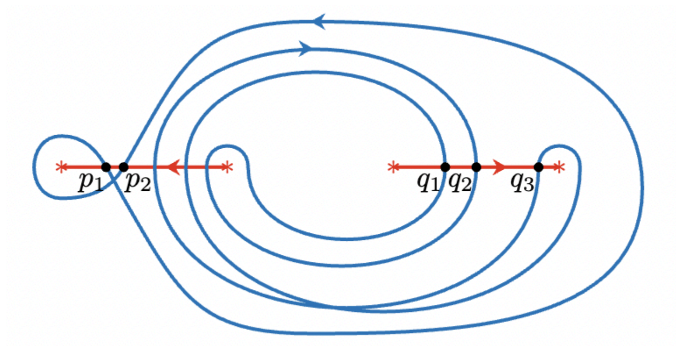
$$\mathbb{Z} \oplus \mathbb{Z}[2]\{-2\} \oplus \mathbb{Z}[2]\{-1\} \oplus \mathbb{Z}_2[4]\{-3\} \oplus \mathbb{Z}[5]\{4\}$$

including torsion.

The swiping move



can be used to simplify the brane to an equivalent one:



Its resolution leads to a complex which is only 6 dimensional

$$\mathbb{C} \to 0 \to \begin{matrix}\mathbb{C}\{-2\}\\ \mathbb{C}\{-1\}\end{matrix} \xrightarrow{0} \mathbb{C}\{-3\} \xrightarrow{-2u^2\hbar^2} \mathbb{C}\{-3\} \xrightarrow{0} \mathbb{C}\{-4\}$$

with homology that is unchanged:

$$\mathbb{Z} \oplus \mathbb{Z}[2]\{-2\} \oplus \mathbb{Z}[2]\{-1\} \oplus \mathbb{Z}_2[4]\{-3\} \oplus \mathbb{Z}[5]\{4\}$$

The theory extends to arbitrary simple Lie algebras

$$\mathfrak{g}$$

as well as Lie superalgebras

We are currently developing a Sage code whose input is

an arbitrary link, a choice of a Lie algebra, and its representations

coloring its strands, and whose output is the

$$U_{\mathfrak{q}}(\mathfrak{g})$$

link homology.