Calabi-Yau period motives in QFT, String theory and general relativity

String Math 2024

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Based on work with

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[1]=arXiv:1912.06201v2, [2]=arXiv:2008.10574v1,
[3]=arXiv:2108.05310, in JHEP
[4]=arXiv:2209.05291 in PRL and [5]=arXiv: 2212.09550 in JHEP,
[6]= arXiv:2310.08625 in JHEP, [7]= arXiv:2402.19034 in JHEP,
[8]= arXiv:2401.07899 Phys. Rev. D

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III. Post Minkowskian (PM) Worldline Quantum Field Theory approximation to General Relativity to predict the gravitational wave forms in black hole scattering/mergers detected by LIGO,....

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varieties

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
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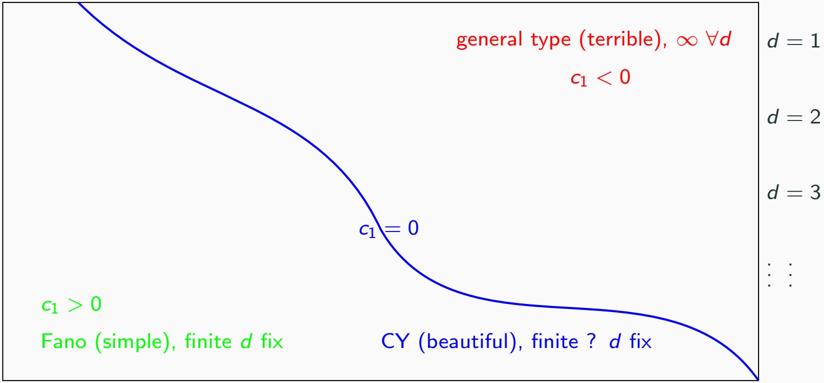
Kodaira map of algebraic varieties



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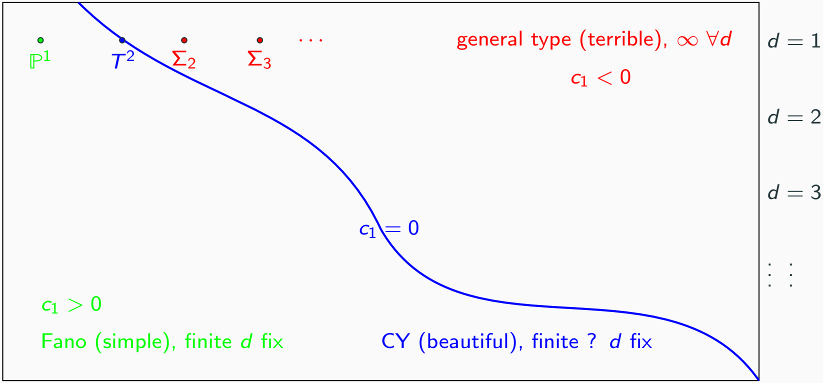
$$\begin{array}{ccccc} l=0 & l=1 & l=2 & l=3 & \cdots \\ g=0 & g=1 & g=2 & g=3 & \cdots \end{array}$$



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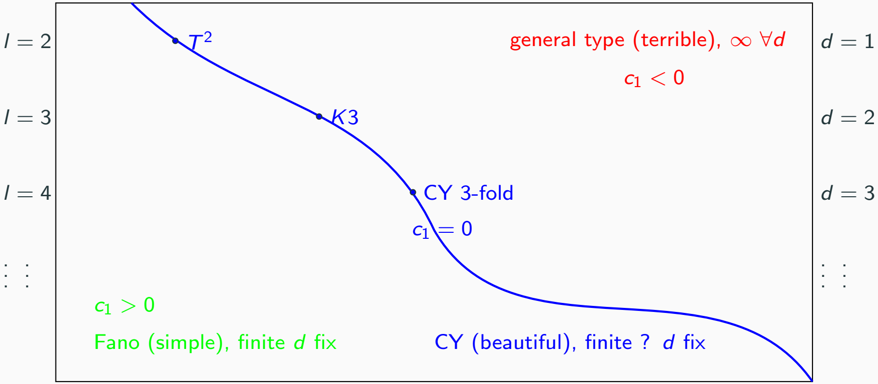

Dictionary Feynman graphs/amplitudes and geometry

Perturbative QFT	Geometry X	Differential eq.	Arithmetic Geometry
maximal cut Feynman integral	Period integral $\underline{\Pi}$ (ϵ-deformed)	Homogeneous Gauss Manin $(d - A(z))\underline{\Pi} = 0$	Motive defined by l-adic coh $H^k_{et}(\overline{X}, \mathbb{Q}_l)$
	⥀ Monodromy group $\in \Gamma(\mathbb{Z})$; irreducible ?		⥀ Galois group $\mathrm{Gal}(\overline{K}/K)$ irreducible ?
actual Feynman integral	Chain integral ϵ-deformed	Inhomogeneous Gauss Manin connection $(d - A(z))\underline{\Pi} = B(z)$	Extended motive

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The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{\jmath}}(g) = 0$,
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Remark: CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with Ω given by $\frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.

Construction of Calabi-Yau n-folds from Fano varities

The $c_1(TM) = 0$ condition is topological easy to control in algebraic geometry. Let $\mathbb{P}_k$ a Fano variety, $K_{\mathbb{P}_k} = c_1(\mathbb{P}) > 0$:

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2) $\mathbb{P} = \mathbb{P}_{\Delta_{n+1}}$ with Δ_{n+1} a reflexive lattice polytop. Then the section of $\mathcal{O}(K_{\mathbb{P}})$ defines a Calabi-Yau n-fold hypersurface. $\# = 16, 4319, 473800776, \mathcal{O}(10^{22}), \ldots$ for $n = 1, 2, 3, 4, \ldots$.

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3) Branched covers over $\mathbb{P}_n$: A d-fold cover branched at $\frac{d}{d-1} K_{\mathbb{P}_n}$ is a CY n-variety: $\mathbb{P}_n = (\mathbb{P}^1)^n$ and $d = 2, 3$ are relevant for 2d n-loop fishnets .

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4) Complete intersections: $W_k=0$, $k=1,\ldots,r$ in $\mathbb{P}=\otimes_{l=1}^m \mathbb{P}_l^{n_l}$ define a CY $(\sum n_l-r)$-fold if $\sum_{k=1}^r d_{kl}=n_l+1$, $\forall l=1,\ldots,m$, with d_{kl} degrees of the k'th polynomial in the l'th factor: 2d n-1 loop bananas.

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5) $\mathbb{P}_k$ Grassmanian, Flag, etc etc

Periods on Calabi-Yau n-folds

Periods integrals

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z})$$

define a non-degenerate pairing between between (middle) homology and (middle) cohomology well defined by the theorem of Stokes:

$$\Pi : H_n(M_n, \mathbb{K}) \times H^n(M_n, \mathbb{C}) \to \mathbb{C}\,.$$

It is possible and natural to have $\mathbb{K}$ to be $\mathbb{Z}$. There is an intersection pairing

$$\Sigma : H_n(M_n, \mathbb{K}) \times H_n(M_n, \mathbb{K}) \to \mathbb{K},$$

that can be made in particular integral. If n is odd Σ is antisymmetric and can be made symplectic. If n is even Σ is a symmetric on the even self dual lattice $H_n(M_n, \mathbb{K})$. E.g. for $K3$ $b_2 = 22$ and $\sigma = b_2^+ - b_2^- = \frac{1}{3}\int_{M_2} c_1^2 - 2c_2 = -16$ hence b_2 has signature $(3, 19)$ and is $E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}$.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

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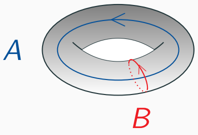

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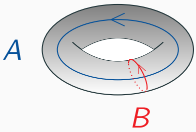

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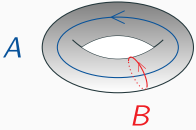

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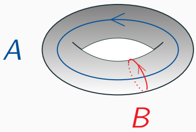

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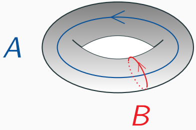

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$$\mathcal{L} \int_\Gamma \Omega = \left[(1-z)\partial_z^2 + (1-2z)\partial_z - \frac{1}{4} \right] \int_\Gamma \Omega = 0 \, .$$

Period geometry on CY n-fold

The main constraints which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \qquad (1)$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{\jmath}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{\jmath}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$.

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$$\int_{M_n} \Omega \wedge \underline{\partial}^k_{I_k} \Omega = \begin{cases} 0 & \text{if } k < n \\ C_{I_n}(z) \in \mathbb{Q}[z] & \text{if } k = n \; . \end{cases} \tag{2}$$

Here $\underline{\partial}^k_{I_k} \Omega = \partial_{z_{I_1}} \ldots \partial_{z_{I_k}} \Omega \in F^{n-k} := \bigoplus_{p=0}^k H^{n-p,p}(W)$ are arbitrary combinations of derivatives w.r.t. to the z_i, $i = 1, \ldots, r$.

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$$\mathcal{L}_k^* = (-1)^{n+1} C(z) \mathcal{L}_k \,,$$

with $\mathcal{L}^* = \sum_{i=0}(-\partial_z)^i a_i(z)$ and $\frac{\partial_z C(z)}{C(z)} = \frac{2}{n+1} a_n(z)$.

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b.) Let $W = (\partial_z^i \Pi_j)$ the Wronskian of $\mathcal{L}$. Then

$$W^{-1} = \Sigma W^T Z^{-1}$$

with $Z^{-1} \in \mathbb{Q}[z]$, e.g. for $n = 3$

$$Z^{-1} = \frac{(2\pi i)^3}{C} \begin{pmatrix} 0 & \frac{C''}{C} - 2\frac{C'}{C} + \frac{a_2}{a_4} & -\frac{C'}{C} & 1 \\ 2\frac{C'}{C} - \frac{C''}{C} - \frac{a_2}{a_4} & 0 & -1 & 0 \\ \frac{C'}{C} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

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c.) The Gauss-Mania connection can be brought into a canonical form

$$\partial_{t^i_*} \begin{pmatrix} \Pi_0 \\ \Pi_j \\ \Pi^j \\ \Pi^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Pi_0 \\ \Pi_k \\ \Pi^k \\ \Pi^0 \end{pmatrix} .$$

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$$\partial_{t^i_*} \begin{pmatrix} \Pi_0 \\ \Pi_j \\ \Pi^j \\ \Pi^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Pi_0 \\ \Pi_k \\ \Pi^k \\ \Pi^0 \end{pmatrix} .$$

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$\Rightarrow$ Special forms of the Gauss Manin connection as c.) imply that the terms in the ϵ expansion can be written as as iterated integrals of $\partial_i^k \Pi_j$ modulo rational functions also for multi parameter integrals.

I. Amplitudes and Yangian integrable symmetries [5,6,7]

Driving question: Which symmetries allow to solve n.t. QFT's.

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Integrable Deformations: Marginal β deformations Leigh, Strassler (95) Maldacena Luni (05). Here most relevant the supersymmetry breaking γ_i, $i = 1, 2, 3$ deformations in the double scaling limit $g \to 0$, $\gamma_3 \to i\infty$ with $\xi^2 = g^2 N_c e^{-i\gamma_3}$ fixed Gürdoğan, Kazakov (16), with Caetano (18) and the bi-scalar model χFT Kazakov, Olivucci (18) leading to holographic dual pairs of integrable fishnet and fishchain theories in D dimensions.

Orginal Fishnet Lagrangians

These bi-"scalar" fishnet theories in D dimensions have a Lagrangian with quartic interaction $V = 4$

$$\mathcal{L}_{\mathrm{quad}}^{\omega D} = N_{\mathrm{c}} \mathrm{tr}[-X(-\partial_\mu \partial^\mu)^\omega \bar{X} - Z(-\partial_\mu \partial^\mu)^{\frac{D}{2}-\omega} \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}] \,.$$

ω determines the propagator power in the Feynman graphs. E.g. $D = 4$, $\omega = 1$ and $D = 2$, $\omega = 1/2$ are conformal choices.

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Most importantly this theory exhibit as symmetry the Yangian extension of the bosonic conformal symmetry.

Hexagonal Fishnets Lagrangian

A generalization with analogous symmetry properties are Fishnet theories with cubic interaction $V = 3$ Kazakov, Olivucci (23) and Lagrangian

$$\mathcal{L}^D_{\mathrm{cub}} = \quad N_{\mathrm{c}}\mathrm{tr}\big[-X(-\partial_\mu\partial^\mu)^{\omega_1}\bar{X} - Y(-\partial_\mu\partial^\mu)^{\omega_2}\bar{Y} - Z(-\partial_\mu\partial^\mu)^{\omega_3}\bar{Z}$$
$$+\xi_1^2\bar{X}YZ + \xi_2^2 X\bar{Y}\bar{Z}\big]\,,$$

with $\sum_{i=1}^{V}\omega_i = D$ at vertex, e.g. $D = 2$ and $\omega_1 = \omega_2 = \omega_3 = 2/3$.

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with $\sum_{i=1}^{V}\omega_i = D$ at vertex, e.g. $D = 2$ and $\omega_1 = \omega_2 = \omega_3 = 2/3$. Scalar field have conformal dimension $\Delta_\phi = (D-2)/2$ and conformal interactions have to have valency $V = 2D/(D-2)$, i.e. $D = 6, 4, 3$ enforce $V = 3, 4, 6$ respectively.

Regular tilings and Calabi-Yau motives

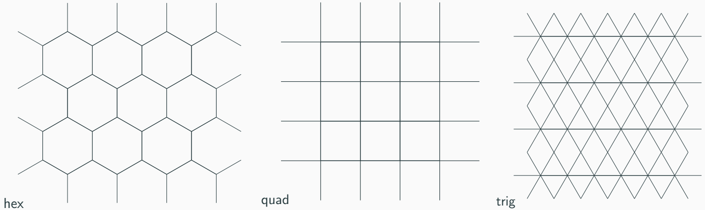

Figure 1: The three regular tilings of the plan with vertices of valence $\nu = 3, 4, 6$ respectively.

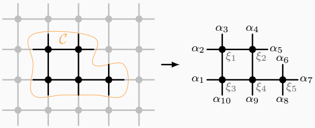

Figure 2: Ten-point five-loop fishnet integral cut out of a square tiling of the plane.

Regular tilings and Calabi-Yau motives

To obtain a graph G consider a convex closed oriented curve $\mathcal{C}$ that cuts edges of the tiling and does not pass to vertices. To each vertex inside the curve $\mathcal{C}$ we associate a $\mathbb{P}^1$ with homogeneous coordinates $[x_i : u_i]$, $i = 1, \ldots, l$ over which we want to integrate with the measure

$$\mathrm{d}\mu_i = u_i \mathrm{d}x_i - x_i \mathrm{d}u_i \ . \tag{3}$$

To the end point of each cut edge outside $\mathcal{C}$ we associate a parameter $a_j \in \mathbb{C}$, $j = 1, \ldots, r$. The graph is constructed by the l vertices with propagators

$$P^I_{ij} = \frac{1}{(x_i - x_j)^{w_{ij}}}, \qquad P^E_{ij} = \frac{1}{(x_i - a_j)^{w_{ij}}} \ . \tag{4}$$

To be conformal in D dimension the weights of propagators incident to each vertex V_i has to fullfill

$$\sum w_{ij} = D \tag{5}$$

Regular tilings and Calabi-Yau motives

We deal mainly with $D = 2$ and choose the propagator weights all equal $w_{ij} = w = 2/\nu(V)$, where $\nu(V)$ is the valence of the vertices, i.e. for the hexagonal tiling we have $w = \frac{2}{3}$, for the quartic tiling $w = \frac{1}{4}$ and for the trigonal tiling $w = \frac{1}{3}$.

To the hexagonal and the quartic lattice we can associate an in general singular l-dimensional Calabi-Yau variety M_l as the $d = 3$ or $d = 2$ fold cover

$$W = \frac{y^d}{d} - P([\underline{x} : \underline{u}]; \underline{a}) = 0 \qquad (6)$$

over the base $B = (\mathbb{P}^1)^l$ branched at

$$P([\underline{x} : \underline{w}]; \underline{a}) = \prod_{ij}(u_j x_i - x_j u_i) \prod_{ij}(x_i - a_j u_i) = 0 \ , \qquad (7)$$

respectively. The orders of the covering automorpishm exchanging the sheets will play a crucial role in the following geometric analysis.

Regular tilings and Calabi-Yau motives

Note that (6) defines a Calabi-Yau manifold, because the canonical class of the base is with H_i the hyperplane class of the i'th $\mathbb{P}^1$ given by

$$K_B = 2\bigoplus_{i=1} H_i, \tag{8}$$

and the Calabi-Yau condition ensuring $K_{M_l} = 0$

$$\frac{d}{d-1} K_B = [P([\underline{x} : \underline{u}]; \underline{a})] = \nu \bigoplus_{i=1} H_i \tag{9}$$

is true with $d = 3, 2$ as $\nu = 3, 4$ for graphs from the hexagonal and the quartic tiling, respectively.

Calabi-Yau period motives and Amplitdudes

Another way of stating this is that the periods over the unique holomorphic (ℓ,0)-form, given by the Griffiths residuum form Ω

$$\Pi_G = \int_C \Omega = \int_\Gamma \frac{1}{2\pi i} \oint_\gamma \frac{dy \prod_{i=1}^{l} d\mu_i}{W} = \int_\Gamma \frac{\prod_{i=1}^{l} d\mu_i}{\partial_y W} = \int_\Gamma \frac{\prod_{i=1}^{\ell} d\mu_i}{P^{\frac{d-1}{d}}} = \int_\Gamma \prod_{ij} P_{ij}^{I} \prod_{ij} P_{ij}^{E} \prod_{i=1}^{l} d\mu_i \,, \tag{10}$$

are well defined. The significance for the application is that these period integrals over cycles $\Gamma \in H_l(M_l, \mathbb{Z})$ are building blocks for the amplitudes $I_G(\underline{a}) = Cf(\underline{a})\Phi_G(\underline{z})$

$$I_G(\underline{a}) = \int \sqrt{\left| \prod_{ij} P_{ij}^{I} \prod_{ij} P_{ij}^{E} \right|^2} \prod_{i=1}^{\ell} d\mu_i \wedge d\bar{\mu}_i = Cf(\underline{a}) \int_C \Omega(z) \wedge \bar{\Omega}(z) \,,$$

Claim 1: To each graph G we can associate a Calabi-Yau variety M_G whose periods determine I.

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Remark: Each $I_G(\underline{a})$ integral is an amplitude in the CFNT, i.e. $\Phi_G(\underline{z})$ has to be single valued.

Single valued Amplitudes and Volumes

The latter is a more general principle: Consider in $D = 4$ $N = 4$ SYM the four point one loop amplitude

$$\int \frac{\mathrm{d}^4\xi}{i\pi^2} \prod_{i=1}^{4} \frac{1}{(\xi - \alpha_i)^2} = -\frac{4}{\alpha_{13}^2\alpha_{24}^2} \frac{D(z)}{z - \bar{z}}$$

with z a cross ratio and $D(z) = \mathrm{Im}\big[\mathrm{Li}_2(z) + \log|z| \log(1 - z)\big]$ the single values Bloch Wigner Dilogarithm, which is the volume of a tetrahedron in hyperbolic 3 space.

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with z a cross ratio and $D(z)=\mathrm{Im}\big[\mathrm{Li}_2(z)+\log|z|\,\log(1-z)\big]$ the single values Bloch Wigner Dilogarithm, which is the volume of a tetrahedron in hyperbolic 3 space. One can consider a selfadjoint CY operator

$$\mathcal{L}=(\theta-1)^2(1-z)^2(\theta^2-z\theta^2)$$

so that in a integral monodromy basis of solutions $i\Pi^\dagger\Sigma\Pi=\pi^2 D(z)$.

The Yangian symmetry:

To each semi simple finite Lie Algebra g one can associate a Yangian extension $Y(g)$. E.g. for the conformal group in $D = 2$ is $S(3,1)$ and the Yangian algebra splits:

$$Y(SO(3,1)) = Y(Sl(2,\mathbb{R})) \oplus \overline{Y(Sl(2,\mathbb{R}))}.$$

The holomorphic Yangian is generated by the algebra

$$\begin{aligned} P_j^\mu &= -i\partial_{a_j}^\mu, & K_j^\mu &= -2ia_j^\mu(a_j^\nu \partial_{a_j,\nu} + \Delta_j) + ia_j^2 \partial_{a_j}^\mu \\ L_j^{\mu\nu} &= i(a_j^\mu \partial_{a_j}^\nu - a_j^\nu \partial_{a_j}^\mu), & D_j &= -i(a_j^\mu \partial_{a_j,\mu}), \end{aligned}$$

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in differentials w.r.t. to the external position, generates together with the permutation symmetries of the latter a differential ideal Y_G that annihilates the amplitude $I_G(\underline{a})$

Claim 3: Y_G is *equivalent* to the Picard-Fuchs differential ideal or the Gauss-Manin connection that describes the variation of the Hodge structure in the middle cohomology of M_G and annihilates the periods $\int_\Gamma \Omega(z)$.

Regular tilings and Calabi-Yau motives

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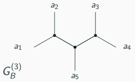
Conclusion: The symmetries determine the periods $\Pi(z)$ and if the monodromies are irreducible there is a unique single valued expression $e^{-K} = i^{\ell^2}\Pi(z)^\dagger\Sigma\Pi(z)$ that fixes up to a conformal factor, likewise determined by the symmetries, the amplitude.

Star triangle relations and different realisations of CY motives

The star triangle is an exact relation that connects graphs with vertices of valency 3 to such with higher valency



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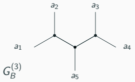

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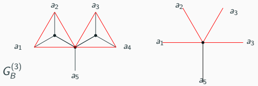

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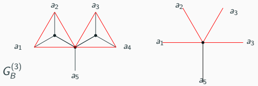

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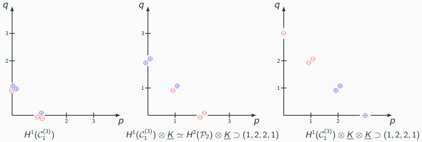
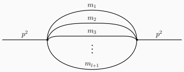


Figure 3: Relations of the Hodge structure $H^{q,p}$ of the genus three Picard curve $\mathcal{C}_1^{(3)}$, the Picard two-variety $\mathcal{P}_2$ and the Calabi-Yau three-variety over $K = \mathbb{Z}[e^{2\pi i/3}]$.

II. Applications to general QFT [1,2,3,4,5]

The maximal cut integrals of the ℓ-loop banana graph, a propagator correction, leads



directly to period integrals in a complete intersection Calabi-Yau $(\ell+1)$-fold geometry with $z_i = \frac{m_i^2}{p^2}$ complex parameters [1,2].

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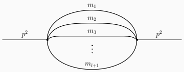

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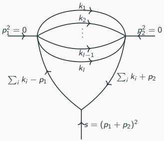
But the ice cone graph, a the vertex correction.



does not lead to period of a single CY $(\ell+1)$-fold [5].

Master Integrals and integration by parts relations

A systematic way to get the Gauss-Manin connection and the inhomogeneous term is to use the integration by by parts relations IBP relation between so called master integrals. Consider l-loop Feynman integrals in general dimensions $D \in \mathbb{R}_+$ of the form

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \tag{11}$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ are the propagators, q_j is the j^{th} momenta through D_j, $m_j^2 \in \mathbb{R}_+$ are masses, $i \cdot 0$ indicates the choice of contour/branchcut in $\mathbb{C}$. Subject to momentum conservation the q_j are linear in the external momenta $p_1, \ldots, p_E$, $\sum_{i=j}^{E} p_j = 0$ and the loop momenta k_r. We defined $\epsilon := \frac{D_{cr} - D}{2}$.

Master Integrals and integration by parts relations

The Feynman integral depends besides D (ϵ) on dot products of p_i and the masses m_j^2, written compactly in a vector $\underline{w} = (w_1, \ldots, w_N) = (p_{i_1} \cdot p_{i_2}, m_j^2)$ and dimensional analysis of $I_{\underline{\nu}}$ shows that it depends only on the ratios of two parameters x_i, we chose

$$z_k := \frac{w_k}{w_N} \qquad \text{for } 1 \leq k < N$$

and label now the parameters of the integrals $I_{\underline{\nu}}$ by the dimensionless parameters $\underline{z}$.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{x}, D)$ are called master integrals.

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The integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^\mu} \left(q_l^\mu \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \ .$$

relate the master integrals with different exponents $\underline{\nu}$.

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relate the master integrals with different exponents $\underline{\nu}$.

There is a finite region in the lattice that contains all non-vanishing master integrals. In a basis of master integrals one can express derivatives w.r.t. the z_k as a linear combination rational coefficients by the IBP relations.

Master Integrals and integration by parts relations

The basis of master integrals (graph cohomology) corresponds to the basis of the cohomology $H^{l-1}(M_l, \mathbb{Z})$.

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A complete set of IBP relations corresponds to the complete Picard Fuchs ideal of Gauss-Manin connection for the period integrals.

IBP relation summary:

The IBP relations characterise a suitable finite set of master integrals

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \ ,$$

with $D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ propagators and $(\underline{\nu}, D)$ in a finite region in $\mathbb{Z}^{p+1}$, by a first order Gauss Manin connection

$$\boxed{d\underline{I}(\underline{x}, \epsilon) = \mathbf{A}(\underline{x}, \epsilon)\underline{I}(\underline{x}, \epsilon)}$$

$\epsilon = (D_{cr} - D)/2$.

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x},\epsilon) \to \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$

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$$\mathbf{A}(\underline{z};\epsilon)^{best} = [R_n(\underline{z};\epsilon)\mathbf{A}^{n-1} + dR_n(\underline{z};\epsilon)]R_n(\underline{z};\epsilon)^{-1} = \epsilon A(\underline{z})$$

$$\left[d_z - \epsilon \begin{pmatrix} & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & \\ * & \dots & * & A^1_{11} & \cdots & A^1_{1r_1} & & & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \mathbf{0} & \\ * & \dots & * & A^1_{r_1} & \cdots & A^1_{r_1 r_1} & & & \\ & \vdots & & & \ddots & & & & \\ * & \dots & * & & & & A^n_{11} & \cdots & A^n_{1r_1} \\ \vdots & \ddots & \vdots & & \mathbf{0} & & \vdots & \ddots & \vdots \\ * & \dots & * & & & & A^n_{r_2} & \cdots & A^n_{r_n r_n} \end{pmatrix} \right] \begin{pmatrix} I^{sub} \\ \Pi^1_1 \\ \vdots \\ \Pi^1_{r_1} \\ \vdots \\ \Pi_{n_1} \\ \vdots \\ \Pi_{n_r} \end{pmatrix}^{best} = 0$$

The blocks

Here $A^k_{ij}(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z and we typically have a situation, where the l-loop block in this improved IBP first order flat connection above is described by period integrals in the sense of Kontsevich and Zagier fullfilling the Gauss-Manin flat connection of a geometry X, which is typically a (non-smooth) Calabi-Yau manifold.

The blocks

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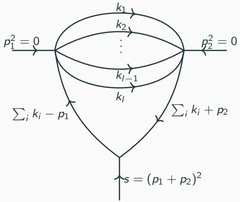
Example: From the (l+1)-loop ice-cone graph



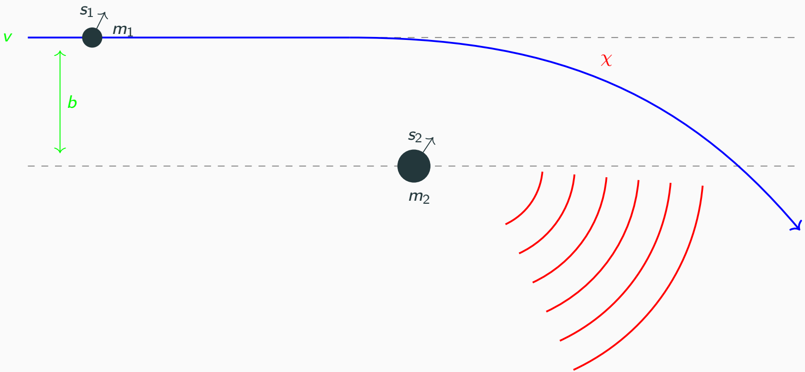
it is clear that it contains l-loop banana graph as block(s).

Dictionary for the blocks

	$l = (n+1)$**-loop in block integrals in** D_{cr} **dimensions**	**Calabi-Yau (CY) geometry**
1	Maximal cut integrals in D_{cr} dimensions	$(n, 0)$-form periods of CY manifolds or CY motives
2	Dimensionless ratios $z_i = {m_i}^2/p^2$	Unobstructed compl. moduli of M_n, or equi'ly Kähler moduli of the mirror W_n
3	Integration-by-parts (IBP) reduction	Griffiths reduction method
4	Integrand-basis for maximal cuts of of master integrals in D_{cr}	Middle (hyper) cohomology $H^n(M_n)$ M_n
5	Complete set of differential operators annihilating a given maximal cut in D_{cr} dimensions	Homogeneous Picard-Fuchs differential ideal (PFI) / Gauss-Manin (GM) connection

III. Worldline Quantum Field Theory approach to General Relativity [8]

Scattering of two black holes (BH) as starter to the description of BH mergers as the main sources for gravitational waves detected at LIGO, ...



Worldline Quantum Field Theory approach to General Relativity

The action for the scattering process

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^{\mu} \dot{x}_i^{\nu}\right] + S_{\mathrm{EH}}$$

is expanded in Post Minkowskian (PM) approximation in the Worldline Quantum Field Theory (WQFT) approach around the non-interacting background configurations

$$x_i^{\mu} = b_i^{\mu} + v_i^{\mu} \tau + z_i^{\mu}(\tau) \,, \quad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}(x) \;.$$

Worldline Quantum Field Theory approach to General Relativity

The goal is to calculate from the initial data: the impact parameter $b^\mu = b_1^\mu - b_2^\mu$ and the incoming velocities v_1, v_2 the physical quantity of interest, which is the radiation induces change in the momentum say $\Delta p_1^\mu = m_1 \int \mathrm{d}\tau \ddot{x}(\tau)$ of the first particle.

In the PM approximation the latter can be expanded in the gravitational coupling G

$$\Delta p_1^\mu = \sum_{n=1}^{\infty} G^n \Delta p^{(n)\,\mu}(x) \ .$$

At each order the contributions $\Delta p^{(n)\,\mu}(x)$ are calculated in the WQFT approach in the Swinger-Keldysh in-in formalism in terms of a Feynman graph expansion with retarded propagators. Here $x = \gamma - \sqrt{\gamma^2 - 1}$ with γ the Lorentz factor of the relative velocities is the only parameter.

Worldline Quantum Field Theory approach to General Relativity

In the 4PM approximation the Feynman integral in the 1SF sector

(K3.0)

involve bilinear of elliptic function which are periods of the $K3$

$$Y^2 = X(X-1)(X-x)Z(Z-1)(Z-1/x)\,.$$

In the 5PM approximation we find in [8] that in the 5PM approximation the following graphs in the 1SF sector

(CY3)

Worldline Quantum Field Theory approach to General Relativity

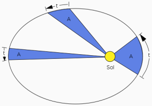
The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

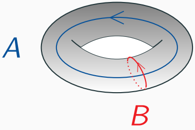
$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0,\ j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$. The periods of the above $K3$ and CY threefold determine all special functions that are necessary to solve for $\Delta p^{(5)\,\mu}(x)$ in the 1SF sector.

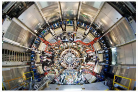
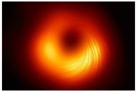
In the 5PM 2SF further different CY and K3 appear.

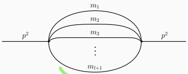
Conclusion and Outlook

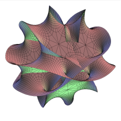




Conclusion and Outlook

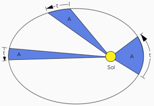



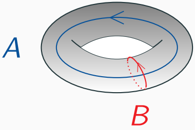




Physics

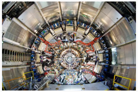
Mathematics

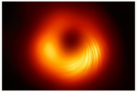


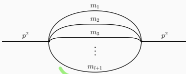


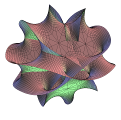
Conclusion and Outlook

String Theory



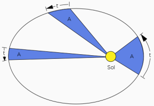


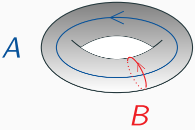




Physics

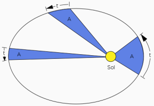
Mathematics

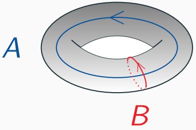




Enumerative Geometry

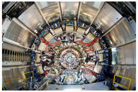
Conclusion and Outlook

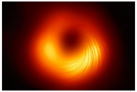


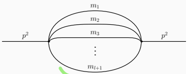


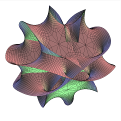
String Theory

Black Hole Physics









Physics

Mathematics

Enumerative Geometry

Arithmetic Geometry

Sublementary Material desingularization of Fishnet CY

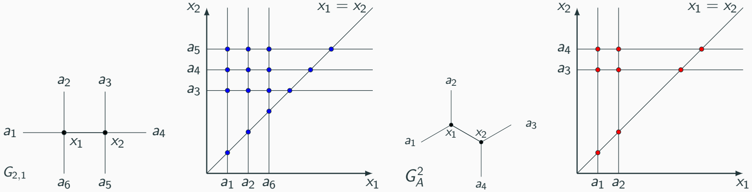


Figure 4: Singularities of the $K3$ denoted for the valence 4 graph $M_{G_{1,2}}$ and the valence 3 graph $M_{G_A^2}$. Note that 3 of the a_i can be set to $0, 1, \infty$ by a diagonal $PSL(2, \mathbb{C})$ acting on the projective plane in which the a_i lie