Homogeneous Universe & Thermal History

R-W metric (flat): $ds^2 = -dt^2 + a^2(t)\,d\vec{x}^2$

Conformal time: $d\eta \equiv dt/a$

$$\Rightarrow ds^2 = a^2(\eta)\left(-d\eta^2 + d\vec{x}^2\right)$$

Perfect fluid: $P = w\rho$

$w = 0 \Leftrightarrow$ matter (dust)
$w = 1/3 \Leftrightarrow$ radiation
$w = -1 \Leftrightarrow$ Vacuum energy

Christoffel symbol for flat R-W metric: ($\dot{a} \equiv \frac{da}{dt}$ here)

$$\Gamma^{\alpha}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right)$$

$$\Rightarrow \Gamma^{0}_{00} = 0 \quad \text{and} \quad \Gamma^{0}_{0i} = 0 = \Gamma^{0}_{i0}$$

$$\Gamma^{0}_{ij} = \delta_{ij}\,\dot{a}a$$

$$\Gamma^{i}_{00} = 0$$

$$\Gamma^{i}_{0j} = \left(\frac{\dot{a}}{a}\right)\delta^{i}_{j}$$

$$\Gamma^{i}_{jk} = 0$$

Recall covariant stress-energy conservation: $\nabla_\mu T^{\mu\nu} = 0$

$$T^{\mu}{}_{\nu} = g_{\nu\alpha}T^{\mu\alpha} = (\rho + P)U^{\mu}U_{\nu} + P\delta^{\mu}{}_{\nu}$$

ρ = rest-frame (of fluid) energy density

P = " " pressure

Evaluate $\nu=0$ component of $\nabla_\mu T^\mu{}_\nu = 0$ for flat R-W:

$$\Rightarrow \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\alpha\mu} T^\alpha{}_0 - \Gamma^\alpha_{0\mu} T^\mu{}_\alpha = 0$$

$$\Rightarrow -\frac{d\rho}{dt} - 3\frac{\dot a}{a}\rho - \underset{\text{" }0}{\Gamma^0_{00}}\, T^0{}_0 - \underset{\text{" }\frac{\dot a}{a}\delta^j{}_i}{\Gamma^j_{0i}}\, \underset{\text{" }P\delta^i{}_j}{T^i{}_j} = 0$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} + 3\frac{\dot a}{a}(\rho + P) = 0}$$ Continuity Eq.

$$\Rightarrow \frac{d\rho}{dt} + 3H(\rho + P) = 0$$

$\Rightarrow$ Matter: $\rho_m \propto a^{-3}$

Rad.: $\rho_r \propto a^{-4}$

Vacuum: $\rho_\Lambda \propto a^0$

We also have Einstein's eq.: $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

Ricci tensor:

$$R_{00} = -3\left(\frac{\ddot a}{a}\right) \qquad R_{ij} = \delta_{ij}\left(2\dot a^2 + \ddot a a\right)$$

$$R_{0i} = 0$$

$\Rightarrow$ Ricci scalar: $R = R^{\mu}{}_{\mu} = 3\left(\frac{\ddot{a}}{a}\right) + \frac{1}{a^2}\delta^{ij}\left(2\dot{a}^2 + \ddot{a}a\right)\delta_{ij}$

$$\Rightarrow R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)$$

$\Rightarrow$ Einstein tensor: 00 component

$$G_{00} = R_{00} - \frac{R}{2}g_{00} = \frac{-3\ddot{a}}{a} + \frac{3\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 = 3\left(\frac{\dot{a}}{a}\right)^2$$

$\Rightarrow$ Einstein eq.: $\quad G_{00} = 8\pi G T_{00}$

$$\Rightarrow 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho$$

Define the <u>Hubble parameter</u>: $H(t) \equiv \frac{\dot{a}}{a}$

$$\Rightarrow \boxed{H^2(t) = \frac{8\pi G}{3}\rho(t)}$$

Friedmann Equation (1922)

Also have ij components of Einstein:

$$G_{ij} = 8\pi G T_{ij}$$

$$\Rightarrow -\delta_{ij}\left(2\ddot{a}a + \dot{a}^2\right) = 8\pi G T_{ij}$$

$$= 8\pi G\, g_{ik} T^{k}{}_{j}$$

$$= 8\pi G\, a^2\delta_{ik} P\, \delta^{k}{}_{j}$$

$$= 8\pi G a^2 P\, \delta_{ij}$$

$$\Rightarrow \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G P$$

Now use first Friedmann eq.: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$

$$\Rightarrow \boxed{\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3P)}$$

Second Friedmann Eq. (sometimes called Raychaudhuri Eq.)

Alt. derivation:

differentiate first Friedmann Eq. w.r.t. time and combine w/ continuity eq.

<u>Def.</u>: critical density $\rho_c = \frac{3H^2}{8\pi G}$ $\left(\rho_{c,0} \equiv \frac{3H_0^2}{8\pi G}\right)$

$\Rightarrow \rho = \rho_c \rightarrow$ flat universe $(k=0)$, as we will generally assume

Then $\Omega_m \equiv \frac{\rho_m}{\rho_c}$ (usually understood to be defined at $z=0$, today)

$$\left(\Leftrightarrow \Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{c,0}} = \Omega_m\right)$$

Similarly for Ω_r, Ω_Λ, etc.

The first Friedmann eq. can then be written

$$H^2(a) = H_0^2\left(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda \left(+\Omega_k a^{-2} + \Omega_v(a) + \dots\right)\right)$$

Obs. indicate: $\Omega_m \simeq 0.3$, $\Omega_\Lambda \simeq 0.7$, $\Omega_r \simeq 6\times 10^{-5}$

$\rightarrow \rho_m = \rho_r$ at $z_{eq} \simeq 3400$

and $\rho_\Lambda = \rho_m$ at $z_{\Lambda\text{-}m} = 0.3$

Evolution of Photon Bath

Recall geodesic eq. : $\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$

Consider massless particle (photon) in flat, expanding R-W metric.

How does this particle's energy change as the universe expands?

$$p^\alpha = (E, \vec{p}) \quad \text{4-momentum}$$

Use this to implicitly define parameter λ:

$$p^\alpha = \frac{dx^\alpha}{d\lambda}$$

Eliminate λ via noting: $\frac{d}{d\lambda} = \frac{dx^0}{d\lambda}\frac{d}{dx^0} = E\frac{d}{dt}$

Evaluate 0-component of geodesic eq.:

$$E\frac{dE}{dt} = -\Gamma^0_{ij}\, p^i p^j$$

(I used $\Gamma^0_{00} = 0$ and $\Gamma^0_{0i} = 0 = \Gamma^0_{i0}$)

Now use $\Gamma^0_{ij} = \delta_{ij}\dot{a}a$

$$\Rightarrow E\frac{dE}{dt} = -\delta_{ij}\dot{a}a\, p^i p^j$$

Massless particle: $g_{\mu\nu} p^\mu p^\nu = 0$

$(ds^2 = 0)$

$\Rightarrow -E^2 + \delta_{ij} a^2 p^i p^j = 0$

$\Rightarrow$ Plug in above:

$$\frac{dE}{dt} + \frac{\dot a}{a} E = 0$$

$\Rightarrow \frac{\dot E}{E} = -\frac{\dot a}{a} \Rightarrow \ln E = -\ln a + \text{const.}$

$$\Rightarrow \boxed{E(a) \propto \frac{1}{a}}$$

$\Rightarrow$ Massless particles lose energy as the universe expands

Why? Handwaving: $E \propto \lambda^{-1}$ ↪ wavelength

Physical wavelength: $\lambda \propto a$ $\Rightarrow$ light wave is "stretched" as the universe expands

$\Rightarrow E \propto a^{-1}$

Implication: consider photon emitted at frequency ν_{em} $(E = h\nu)$

$\Rightarrow$ observed at lower frequency ν_{obs}

$$\boxed{\frac{\nu_{obs}}{\nu_{em}} = \frac{a_{em}}{a_{obs}}}$$

Cosmological **Redshift** (z_{em})

$$z_{em} \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{\lambda_{obs}}{\lambda_{em}} - 1 = \frac{\nu_{em}}{\nu_{obs}} - 1$$

$$\Rightarrow 1 + z_{em} = \frac{\nu_{em}}{\nu_{obs}}$$

So if $a_{obs} = 1$ (photon observed today)

$$\Rightarrow \boxed{a_{em} = \frac{1}{1 + z_{em}}}$$

$\Rightarrow$ Link between redshift of photon obs. today and scale factor at time of its emission

Every time we measure a redshift, we measure the curvature of spacetime!

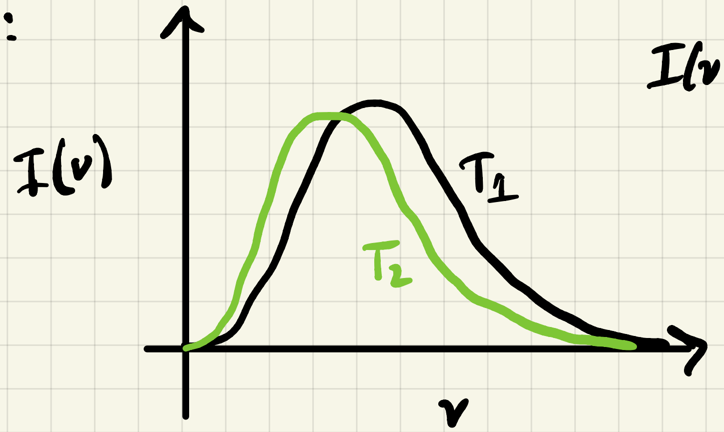
Implication for photon bath:

if we have a Planck (blackbody) photon distribution at some (early) time at temperature $T = T_1$, i.e.,:

$$f(p) = \frac{1}{e^{p/(kT_1)} - 1} = \frac{1}{e^{h\nu/(kT_1)} - 1} \qquad (\mu = 0 \text{ for photons})$$

then at a later time we will still have a blackbody distribution, but at $T_2 = \frac{T_1}{a}$, because the energy of each of the photons decreases by exactly the same factor:

$$I(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$



(Note: processes that inject energy or entropy into the bath can change the shape of the photon distribution – these are "CMB spectral distortions" – see final lecture.)

$\Rightarrow T \propto \frac{1}{a} \propto (1+z)$

Here T is the temperature of the photons, conventionally taken to define the "thermal bath" in the universe.

COBE-FIRAS (1990s): $T_{CMB}(z=0) = 2.726 \pm 0.001$ K

$\Rightarrow$ foundational measurement of physical cosmology!

$\Rightarrow$ tells us how much cosmic expansion has occurred since recombination

$\Rightarrow$ determines Ω_γ directly via $\rho_\gamma = \sigma T^4$

FIRAS data also show that $I(\nu)$ indeed is perfectly consistent with a blackbody (most perfect blackbody known in nature). Upper limits on spectral distortions

Recombination

$kT_{CMB}(z=0) \simeq 2.3 \times 10^{-4}$ eV and $T_{CMB} \propto (1+z)$

Recall ionization energy of hydrogen atom: $B_H \simeq 13.6$ eV

$\Rightarrow$ at high z, $kT_{CMB} > B_H$!

$\Rightarrow$ photons in the thermal bath were sufficiently energetic to keep H atoms ionized

At 10 keV $\gtrsim T \gtrsim 1$ eV: plasma consisting of γ, e^-, H nuclei, ^{4}He nuclei (and the decoupled ν and dark matter).

γ and e^- tightly coupled by Compton scattering.

e^- and H (i.e., p^+) tightly coupled by Coulomb scattering.

Very little neutral H around — plasma $T \gg 13.6$ eV.

As T decreased, eventually $e^- + p^+ \rightarrow H + \gamma$, i.e., e^- and p^+ combine to form neutral H ("recombination").

$\Rightarrow n_e$ decreased sharply

$\Rightarrow \gamma$ decoupled from the baryonic matter (photon decoupling) no longer in thermal equil

$\Rightarrow$ mean free path of photons became larger than the horizon

$\Rightarrow$ photons "free-stream": universe became transparent

$\Rightarrow$ these photons comprise the cosmic microwave background today.

Three-stage process:

1) Recombination (sharp decrease in n_e)
2) Photon-matter decoupling
3) Freeze-out of residual free electron fraction

(1) Recombination: $p^+ + e^- \leftrightarrow H + \gamma$

In chemical equil.: $\mu_H = \mu_p + \mu_e$ (recall $\mu_\gamma = 0$)
(at $T > 1$ eV)

We want to describe evolution of number densities of those species.

Key tool: Boltzmann Equation

We want to follow evolution of d.f. for species i $f_i(x^\mu, p^\mu)$ in presence of interactions:

$$\underbrace{\hat{L}}_{\text{Liouville operator}} f_i = \underbrace{\hat{C}_i}_{\text{collision operator}} [\{f_j\}]$$

→ can depend on d.f. of all other species j

↳ Non-rel. limit: total time derivative

$$\hat{L}_{NR} = \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\vec{x}}{dt}\frac{\partial}{\partial \vec{x}} + \frac{d\vec{p}}{dt}\frac{\partial}{\partial \vec{p}}$$

Particle species of mass m subject to force $\vec{F}$:

$$\hat{L}_{NR} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \frac{\vec{F}}{m}\cdot\frac{\partial}{\partial \vec{v}}$$

Relativistic generalization: total derivative w.r.t. affine parameter along worldline (recall similar derivatives appearing in geodesic eq.):

$$\hat{L}_{GR} = \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} + \underbrace{\frac{dp^\mu}{d\lambda}}_{\text{change in d.f. due to changes in mom. as particle traverses worldline}}\frac{\partial}{\partial p^\mu}$$

Normalize λ via $p^\mu = \frac{dx^\mu}{d\lambda}$ (effectively $\lambda \equiv$ proper time)

$\Rightarrow$ Geodesic eq.: $\frac{dp^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta$

$\Rightarrow \hat{L}_{GR} = p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu}$ (assuming no external non-grav. forces)

Compute in FRW metric:

Homogeneity + isotropy $\Rightarrow f_i(\vec{x}, \vec{p}, t) \to f_i(E, t)$ (or $f_i(p,t)$)

Using $\Gamma^\mu_{\alpha\beta}$ for FRW, we find

$$\hat{L}f_i = E\frac{\partial f_i}{\partial t} - \frac{\dot{a}}{a}p^2 \frac{\partial f_i}{\partial E} = \hat{C}_i[\{f_j\}]$$

$E^2 = p^2 + m^2 \Rightarrow \frac{\partial E}{\partial p} = \frac{p}{\sqrt{p^2+m^2}} = \frac{p}{E}$

$$= E\frac{\partial f_i}{\partial t} - \frac{\dot{a}}{a}p^2 \frac{\partial f_i(p,t)}{\partial p}\frac{\partial p}{\partial E}$$

$\hat{\tilde{C}}_i = \frac{\hat{C}_i}{E}$

$$= E\left(\frac{\partial f_i}{\partial t} - \frac{\dot{a}}{a}p \frac{\partial f_i(p,t)}{\partial p}\right) \Leftrightarrow \frac{\partial f_i}{\partial t} - Hp\frac{\partial f_i}{\partial p} = \hat{\tilde{C}}_i[\{f_j\}]$$

(Dodelson 3.38)

If we drop assumption of homogeneity, this becomes $\frac{\partial f_i}{\partial t} - Hp\frac{\partial f_i}{\partial p} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} = \hat{\tilde{C}}_i[\{f_j\}]$

Using $\hat{L}f_i = E\frac{\partial f_i}{\partial t} - \frac{\dot{a}}{a}p^2\frac{\partial f_i}{\partial E}$, we can show that

$$\int \frac{d^3p_i}{(2\pi)^3}\frac{\hat{L}f_i}{E_i} = \frac{dn_i}{dt} + 3\left(\frac{\dot{a}}{a}\right)n_i = \frac{1}{a^3}\frac{d}{dt}\left(n_i a^3\right)$$

(integrate by parts and use $n_i = \frac{g}{(2\pi)^3}\int d^3p\, f(p)$)

Thus, the integrated (over mom.) Boltzmann eq. is

$$\frac{1}{a^3}\frac{d}{dt}\left(n_i a^3\right) = \int \frac{d^3p_i}{(2\pi)^3}\,\frac{\hat{C}_i[\{f_j\}]}{E_i} = \int \frac{d^3p_i}{(2\pi)^3}\,\hat{\tilde{C}}_i[\{f_j\}]$$

Define $\displaystyle\int_{p_i} \equiv \int \frac{d^3p_i}{(2\pi)^3}\,\frac{1}{2E_i} \Rightarrow \frac{1}{a^3}\frac{d}{dt}\left(n_j a^3\right) = 2\int_{p_i} \hat{C}_i[\{f_j\}]$

If no collisions $\Leftrightarrow \hat{C}_i = 0 \Rightarrow \frac{d}{dt}\left(n_i a^3\right) \Rightarrow n_i \propto a^{-3}$ ✓

$\Rightarrow$ Particle number conservation

<u>Collision Operator</u>: consider process $1+2 \longleftrightarrow 3+4$

$\frac{d}{dt}(n_1) \propto \Delta$(production − annihilation)

$$\frac{1}{a^3}\frac{d}{dt}\left(a^3 n_1\right) = \underbrace{\iiiint_{p_1 p_2 p_3 p_4}}_{\substack{\text{sum over all}\\ \text{momenta}}} (2\pi)^4 \underbrace{\delta^{(4)}\big(\overbrace{p^1+p^2-p^3-p^4}^{\text{4-momenta}}\big)}_{\substack{\text{energy and}\\ \text{mom. conservation}}} \underbrace{|\mathcal{M}|^2}_{\substack{\text{amplitude}\\ \text{(matrix element) for}\\ \text{process } 1+2\leftrightarrow 3+4\\ \text{(e.g., from QFT)}}} \underbrace{\left(f_3 f_4 - f_1 f_2\right)}_{\substack{\text{rate of production} \propto f_3 f_4\\ \text{rate of annihilation} \propto f_1 f_2}}$$

[neglecting Bose enhancement and Pauli blocking]

recall: one unit of phase space has volume $= (2\pi)^3 = (2\pi\hbar)^3$

Why the factor of $\frac{1}{2E}$? This arises because relativistically the phase-space integrals should be over 4-mom., subject to the mass-shell constraint: $E^2 = p^2 + m^2$

$$\int d^3p \int_0^\infty dE\, \delta\left(E^2 - p^2 - m^2\right) = \int d^3p \int_0^\infty dE\, \frac{\delta\left(E - \sqrt{p^2+m^2}\right)}{2E} = \int d^3p \left(\frac{1}{2E}\right)$$

(Recall that $\displaystyle\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$ where x_i are roots of g, i.e., $g(x_i) = 0$)

How to simplify further? In cosmology we generally have:

- System not in chemical equil., but still approx. in kinetic equil. (i.e., scattering is rapid enough that the d.f. of all species still take on B-E or F-D forms).

Thus, all we need to determine is $\mu(t)$. Since we are out of chemical equil., $\mu_1+\mu_2 \neq \mu_3+\mu_4$. Instead have to solve diff. eq. for each $\mu_i(t)$. But this will reduce to a single ODE — much easier than full Boltzmann eq.

- In all our applications, $T \ll E-\mu$. So the d.f. reduces to M-B form: $f(E) \simeq e^{-(E-\mu)/T} = e^{\mu/T} e^{-E/T}$

Plug in above:

$$f_3 f_4 - f_1 f_2 \simeq e^{-(E_1+E_2)/T}\left(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T}\right) \quad \circledast$$

using $E_1+E_2 = E_3+E_4$ (energy cons.).

Let's describe $\mu_i(t)$ using number density $n_i(t)$:

$$n_i = e^{\mu_i/T} n_i^{(0)} \quad \text{where } n_i^{(0)} \equiv \text{equilibrium number density}$$

$$n_i^{(0)} \equiv \frac{g_i}{(2\pi)^3}\int d^3p\, e^{-E_i(p)/T} \qquad \mu_i = T\ln\left(\frac{n_i}{n_i^{(0)}}\right) \longleftrightarrow \frac{n_i}{n_i^{(0)}} = e^{\mu_i/T}$$

Plug in to $\circledast$ above to obtain:

$$f_3 f_4 - f_1 f_2 \simeq e^{-(E_1+E_2)/T}\left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right)$$

To obtain final simplification of Boltzmann eq., define the thermally averaged cross-section:

$$\langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \iiiint_{p_1 p_2 p_3 p_4} (2\pi)^4 \delta^{(4)}(p^1+p^2-p^3-p^4)\, e^{-(E_1+E_2)/T} |M|^2$$

The integrated Boltzmann eq. then becomes:

$$\frac{1}{a^3}\frac{d}{dt}(a^3 n_1) = \langle\sigma v\rangle n_1^{(0)} n_2^{(0)} \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right)$$ ☆

$\Rightarrow$ simple ODE for number density!

Note: when reaction rates are large ($\Gamma \gg H$), we expect to be in chemical equil. $\Leftrightarrow n_i \propto a^{-3}$ (since $n_i \propto T^3$ in equil.) $\Leftrightarrow$ $\boxed{\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}}$ ($\Gamma \gg H$) Saha's Eq. ("nuclear statistical equil.")

(Concretely: LHS of Boltzmann is $\sim n_1 H$ since $dt^{-1} \sim H$, and RHS is $\sim n_1 n_2 \langle\sigma v\rangle \equiv n_1 \Gamma$. So if $n_2\langle\sigma v\rangle \equiv \Gamma \gg H$ then RHS $\gg$ LHS, and only way for eq. to hold is if the terms in parentheses on the RHS cancel.)

Now apply Saha to $e^- + p^+ \longleftrightarrow H + \gamma$:

In chemical equil.: $\mu_H = \mu_p + \mu_e$ (recall $\mu_\gamma = 0$)
(at $T > 1$ eV)

Apply Saha eq.: $\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$

For non-rel. particles, recall $n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-(m_i - \mu_i)/T}$
(note here that $T \ll m_e, m_p, m_H$)

Thus, $\frac{n_e n_p}{n_H} = \frac{g_e g_p}{g_H}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-(m_p + m_e - m_H)/T}$ (note $m_p \simeq m_H$)

Note that $B_H \equiv m_p + m_e - m_H$ = binding energy of atomic hydrogen
$= 13.6$ eV.

Recall $g_p = 2 = g_e$ (spin-1/2 fermions)

and $g_H = 4$ (e^- and p^+ spin anti-aligned: singlet state
e^- and p^+ spin aligned: triplet of states $\Big\} 1+3 = 4$)

No net electric charge in universe $\Rightarrow n_e = n_p$

Further simplification: ignore all nuclei other than p^+ (over 90% of nuclei, by number, are protons).

Then applying conservation of baryon number yields $n_b \simeq \underset{\substack{\uparrow \\ p^+}}{n_p} + \underset{\substack{\uparrow \\ \text{neutral atomic H}}}{n_H} = n_e + n_H$.

$$\Rightarrow \frac{n_e n_p}{n_H} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B_H/T}$$

Def.: free electron fraction $X_e \equiv \dfrac{n_e}{n_b} = \dfrac{n_p}{n_b}$

Note that $n_b = n_p + n_n = \eta\, n_\gamma = \eta \cdot \dfrac{2\zeta(3)}{\pi^2} T^3$ where $\eta \equiv \dfrac{n_b}{n_\gamma} \approx 10^{-9}$

Exercise

$$\Rightarrow \boxed{\frac{1-X_e}{X_e^2} = \eta \cdot \frac{2\zeta(3)}{\pi^2}\left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T}}$$ "Saha's Eq."

Recombination is "complete" when 90% of e^- are in neutral H atoms, i.e., $X_e = 0.1$.

Can solve numerically to determine T corresponding to a given X_e.

Define "recombination time" as that when $X_e = 0.5$.

$\Rightarrow$ Numerical solution of Saha eq. $\Rightarrow T_{rec} \simeq 0.3$ eV

Recall $T(z) = T_0(1+z) = (2.726\text{ K})(1+z) \Rightarrow z_{rec} \simeq 1300$

Recall $1 + z_{eq} \simeq \Omega_r^{-1} = 3400 \Rightarrow$ Recombination occurs in matter-dominated era.

Important: since $\eta = 6\times 10^{-10}$ is extremely small, recombination does not start until $T \ll B_H = 13.6$ eV. Even when T is a bit below B_H, the Wien tail of the photon d.f. contains sufficiently many high-energy photons to keep H ionized. Same phenomenon as in delay in production of D (D "bottleneck") in Big Bang Nucleosynthesis.

Recombination era, cont.

(2) Photon-matter decoupling:

γ coupled tightly to plasma via Compton scattering:

$$e^- + \gamma \longleftrightarrow e^- + \gamma$$

The photon interaction rate for this process is:

$$\Gamma_\gamma \simeq n_e \sigma_T = n_b X_e \sigma_T \quad \text{where } \sigma_T = 2\times 10^{-3}\ \text{MeV}^{-2}$$

$$\Big(\sigma_T \equiv \text{Thomson cross-section} = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2}\Big)$$

$\alpha \to$ fine-structure constant $= 1/137$

$m_e \to e^-$ mass

Note that γ-nuclei interaction rate $\sim 1/m_{nuc}^2 \ll 1/m_e^2$, hence they are negligible.

$\Gamma_\gamma \propto n_e \Rightarrow \Gamma_\gamma$ drops as n_e decreases.

γ and e^- decouple when $\Gamma_\gamma \lesssim H$.

Recall that we are now in matter-dominated era, so $H(z) = H_0\sqrt{\underset{\Omega_{m,0}}{\Omega_m}(1+z)^3} \Rightarrow H(T) = H_0\sqrt{\Omega_m}\left(\frac{T}{T_0}\right)^{3/2}$

We have:

$$\Gamma_\gamma(T) = n_b \sigma_T X_e(T) = \frac{2}{\pi^2}\zeta(3)\,\eta\,\sigma_T X_e(T)\,T^3$$

Setting $\Gamma_\gamma = H \iff \frac{2}{\pi^2}\zeta(3)\,\eta\,\sigma_T X_e(T_D)\,T_D^{\,3} = H_0\sqrt{\Omega_m}\left(\frac{T_D}{T_0}\right)^{3/2}$

$$\Rightarrow X_e(T_D)\,T_D^{\,3/2} = \frac{\pi^2}{2\zeta(3)}\,\frac{H_0\sqrt{\Omega_m}}{\eta\,\sigma_T T_0^{\,3/2}}$$

Using Saha eq. to compute $X_e(T_D)$ and solving numerically, we obtain

$T_D \simeq 0.27$ eV $\simeq 3000$ K

More precise treatment yields $\underline{T_D \simeq 0.25 \text{ eV}} \simeq 2970$ K

$\Rightarrow \underline{z_D \simeq 1090}$

$\underline{t_D \simeq 370{,}000 \text{ years}}$

Note that $X_e(T_{rec}) \simeq 0.5 \rightarrow X_e(T_D) \simeq 10^{-3}$ even though $T_{rec} \simeq T_D$!

(3) Electron freeze-out: Apply integrated

Boltzmann eq. for reaction $e^- + p^+ \longleftrightarrow H + \gamma$:

$$\frac{1}{a^3}\frac{d}{dt}(a^3 n_1) = \langle \sigma v \rangle\, n_1^{(0)} n_2^{(0)} \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right)$$

$1 = e^-$
$2 = p^+$
$3 = H$
$4 = \gamma$

Recall $n_\gamma = n_\gamma^{(0)}$ since $\mu_\gamma = 0$.

Further simplification: $n_H \approx n_H^{(0)}$ during recombination (can improve using Saha approx. for n_H)

Recall $n_e = n_p$ (charge neutrality; neglect He here)

$$\Rightarrow \frac{1}{a^3}\frac{d}{dt}\left(a^3 n_e\right) \approx \langle \sigma v \rangle \left(\left(n_e^{(0)}\right)^2 - n_e^2 \right)$$

Calculating $\langle \sigma v \rangle$ is not simple; reasonable approx. is

$$\langle \sigma v \rangle \approx \sigma_T \sqrt{\frac{B_H}{T}}$$

As usual, def. new time variable: $x \equiv \dfrac{1\ \text{eV}}{T}$

Use usual steps to rewrite $\frac{d}{dt} \to \frac{d}{dx}$

Recall $n_e = n_b X_e$ and note that $a^3 n_b = \text{const.}$ (during this epoch)

Also use $H(T) \propto T^{3/2}$ (matter-dom.)

$$\Rightarrow \frac{dX_e}{dx} = \frac{\lambda}{x^2}\left(\left(X_e^{(0)}\right)^2 - X_e^2 \right) \quad \text{where } \lambda \equiv \left[\frac{n_b \langle \sigma v \rangle}{xH} \right] \Bigg|_{x=1}$$

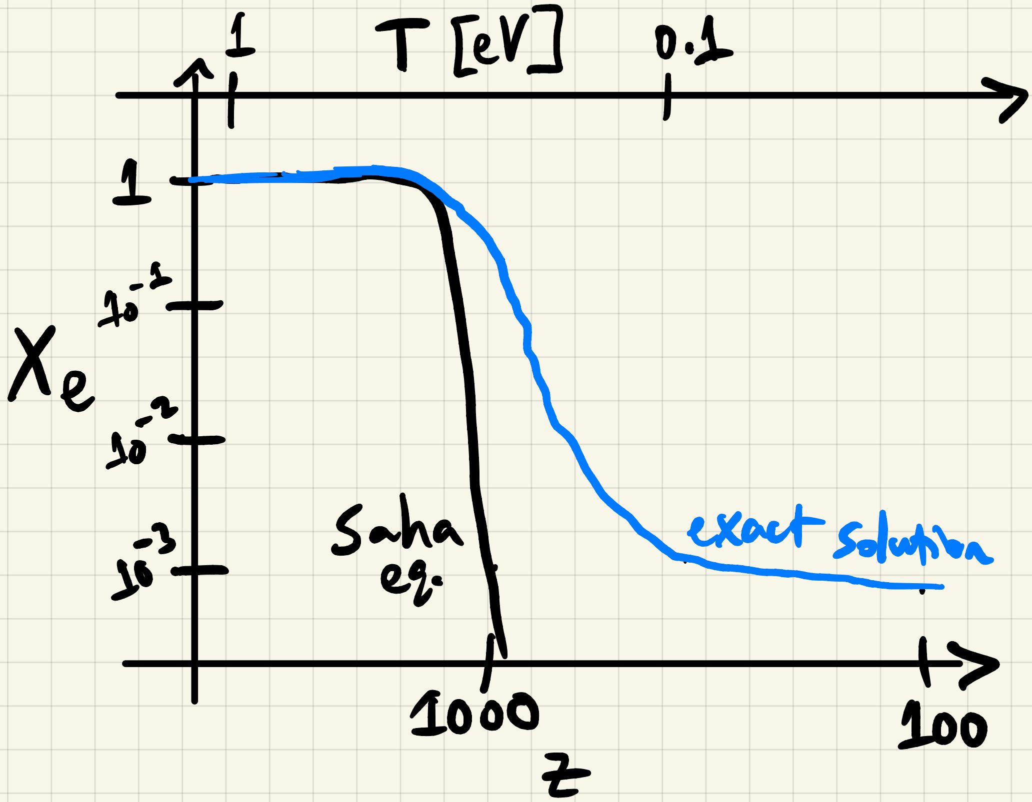
Putting in numbers, $\lambda \simeq 3900 \cdot \left(\dfrac{\Omega_b h}{0.03} \right)$

$\Rightarrow$ This is a "Riccati equation"

$\Rightarrow$ Result for electron freeze-out abundance:

$$X_e^{\infty} \simeq \frac{x_f}{\lambda} \simeq 0.9 \times 10^{-3} \left(\frac{x_f}{x_{rec}} \right) \left(\frac{0.03}{\Omega_b h} \right) \qquad \left(x_{rec} = \frac{1\ \text{eV}}{T_{rec}} \simeq 3.3 \right)$$

Numerical result: (next page)



1
T [eV]
0.1
1
10^{-1}
X_e
10^{-2}
Saha
eq.
exact solution
10^{-3}
1000
100
z

Timeline:

	Temp.	Redshift	Time
[M-R Equality]	0.75 eV	3400	60 kyr
Recombination	0.3 - 0.25 eV	1300 - 1090	260 - 380 kyr
γ-Matter Decoupling	0.25 eV	1090	380 kyr
Last scattering	0.25 eV	1090	380 kyr

Important point: recombination/decoupling are determined by the local plasma temperature T at each point in spacetime. The T at which these processes occur is set by the physics described above, which is universal. So why do we observe fluctuations in the CMB temperature across the sky?

$\Rightarrow$ different points reach this T at (slightly) different times

$\Rightarrow$ hence different amounts of expansion between us and each point on last-scattering surface

Precise Treatment: Beyond Equilibrium

The Saha eq. assumes thermal equilibrium holds throughout recombination/decoupling, but this is not actually true. When the $e^- + p^+ \leftrightarrow H + \gamma$ interaction rate drops below Hubble, we must use the full Boltzmann eq. to describe the evolution.

In addition, recombination dynamics are subtle: direct recombination to H ground state (1s) is very inefficient because γ is emitted with $E \gtrsim 13.6$ eV, which then ionizes nearby H atom, hence no net recombination.

Process instead proceeds mainly via 2 channels:

- Recombination from continuum $\rightarrow 2s \rightarrow 1s$
  $\downarrow$
  "two-photon decay" — 2γ emission, thus each γ not energetic enough to ionize nearby atom
- " " continuum $\rightarrow 2p \rightarrow 1s$
  $\downarrow$
  "resonance escape" — Lyα γ emitted, but if it is not absorbed too quickly by another H, it can redshift out of Lyα line entirely

$\Rightarrow$ Overall impact: recombination is delayed relative to Saha expectation ($z_{rec} \approx 1270$ rather than 1380)

Visibility function:

Recall optical depth

$$\tau(z) = \int_{t(z)}^{t_0} dt\, n_e(t)\, \sigma_T$$

$$= \int_0^z dz\, \sigma_T \frac{n_e(z)}{H(z)\cdot(1+z)}$$

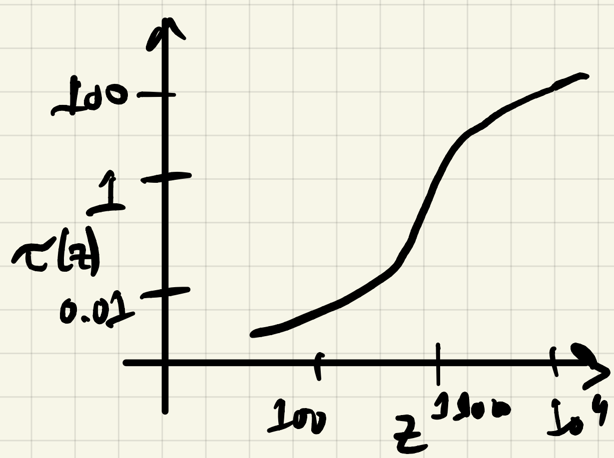
Prob. that a photon did not scatter off an e^- between today and redshift z is

$$P(z) = e^{-\tau(z)}$$

Prob. that photon last scattered in redshift interval $[z, z+dz]$ is

$$g(z)dz \equiv P(z) - P(z+dz)$$

↑
"visibility function"



$$\Rightarrow g(z) = -\frac{d}{dz}\left(e^{-\tau(z)}\right)$$

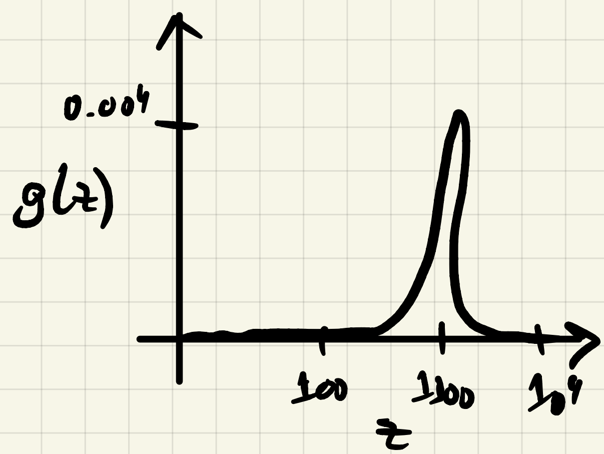
$$= \frac{d\tau}{dz} e^{-\tau(z)}$$

$\Rightarrow$ sharp peak at

$z \simeq 1080$

(in conformal time, peak is at $z \simeq 1090$)

(width $\Delta z \simeq 80$)



$\Rightarrow$ we see the intersection of this snapshot in time with our past light cone, thus defining a 2D "surface of last scattering" (nearly spherical, up to the anisotropies to be discussed below).