Anisotropy Basics: Random Fields on the Sphere

Basis for random fields on the sphere: spherical harmonics

$$\Rightarrow f(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}(\hat{n}) = \sum_{\ell m} f_{\ell m} Y_{\ell m}(\hat{n})$$

Recall $Y_{\ell m}$ from Q.M.: eigenstates (in position space) of $\hat{L}^2 = -\nabla^2$ and $\hat{L}_z = -i\partial_\phi$

$$\nabla^2 Y_{\ell m} = -\ell(\ell+1) Y_{\ell m} \qquad \ell \in \mathbb{Z} \geq 0$$

$$\partial_\phi Y_{\ell m} = im Y_{\ell m} \qquad |m| \leq \ell$$

Orthonormality: $\int d^2\hat{n}\, Y_{\ell m}(\hat{n}) Y^*_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\delta_{mm'}$

Phase convention: $Y^*_{\ell m} = (-1)^m Y_{\ell -m}$

$\Rightarrow$ If field is real $(f(\hat{n}) \in \mathbb{R}) \Rightarrow f^*_{\ell m} = (-1)^m f_{\ell -m}$

Statistical isotropy: implication for 2-pt. correlator of $f_{\ell m}$?

$\Rightarrow$ Turns out that $\langle f_{\ell m} f^*_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell$

$\Rightarrow C_\ell \equiv$ "angular power spectrum" of $f(\hat{n})$

Implication for 2-pt. correlator in position space?

$$\langle f(\hat{n}) f(\hat{n}') \rangle = \sum_{\ell m} \sum_{\ell' m'} \underbrace{\langle f_{\ell m} f^*_{\ell' m'} \rangle}_{= C_\ell \delta_{\ell\ell'}\delta_{mm'}} Y_{\ell m}(\hat{n}) Y^*_{\ell' m'}(\hat{n}')$$

$$= \sum_\ell C_\ell \underbrace{\sum_m Y_{\ell m}(\hat{n}) Y^*_{\ell m}(\hat{n}')}_{= \frac{2\ell+1}{4\pi} P_\ell(\underbrace{\hat{n}\cdot\hat{n}'}_{\equiv \cos\theta})}$$

$\hookrightarrow$ Legendre polynomial

$\Rightarrow$ addition theorem for spherical harmonics

$$C(\theta) \equiv \langle f(\hat{n}) f(\hat{n}') \rangle$$

$$= \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta)$$

$\Rightarrow$ 2-pt. correlator in real space only depends on angle between $\hat{n}$ and $\hat{n}'$, as required by statistical isotropy.

$\Rightarrow$ total variance in field $= C(\theta=0) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell$

Using orthogonality of $P_\ell(\cos\theta)$, can show that

$$C_\ell = 2\pi \int_{-1}^{1} d\cos\theta \, C(\theta) P_\ell(\cos\theta)$$

N.B. if we consider $\ell \gg 1 \Rightarrow \frac{2\ell+1}{4\pi} \approx \frac{\ell+1}{2\pi}$

and $\sum_\ell \rightarrow \int \frac{d\ell}{\ell} \cdot \ell = \int \ell \, d\ln\ell$

$$\Rightarrow \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \rightarrow \int d\ln\ell \, C_\ell \frac{\ell(\ell+1)}{2\pi}$$

$\Rightarrow$ Def. $\mathcal{D}_\ell \equiv \frac{\ell(\ell+1)}{2\pi} C_\ell$ $\Rightarrow$ contribution per decade in ℓ to total variance of the field.

Power spectrum intuition (slides)

Why the power spectrum? CMB is extremely well-approximated as a Gaussian random field (GRF)

$f(\vec{x}) \equiv$ random field with zero mean $\langle f(\vec{x}) \rangle = 0$

Probability of some field configuration is a functional of $f(\vec{x})$: $P[f(\vec{x})]$

GRF $\Rightarrow \Pr[f(\vec{x})]$ is a Gaussian functional of $f(\vec{x})$

Consider discretizing field $f(\vec{x})$ in N pixels (voxels)

$\Rightarrow$ represent as N-dim. vector $\vec{f} = [f(\vec{x}_1), f(\vec{x}_2), \ldots, f(\vec{x}_N)]^T$

$\Rightarrow$ PDF for $\vec{f}$ is a multi-variate Gaussian which is fully specified by the 2-pt. correlation function:

$$\underset{\substack{\uparrow \\ f_i \equiv f(\vec{x}_i)}}{\langle f_i f_j \rangle} = \xi(|\vec{x}_i - \vec{x}_j|) \equiv \xi_{ij}$$

$$\Rightarrow \Pr[\vec{f}] \propto \frac{e^{-\frac{1}{2} f_i \xi_{ij}^{-1} f_j}}{\sqrt{\det(\xi_{ij})}}$$

Since $f(\vec{k})$ is linear in $f(\vec{x})$, $\Pr[f(\vec{k})]$ is also a multi-variate Gaussian: $\Pr[f(\vec{k})] \propto e^{-f^2(k)/2P_f(k)} / \sqrt{\det(P_f(k))}$

Since different Fourier modes are uncorrelated, they are statistically independent for GRFs.

Relevance to cosmology / CMB:

- Inflation predicts initial perturbations are very close to Gaussian (as required by actual data)
- Linear evolution preserves Gaussianity
- Non-linear evolution generates non-Gaussianity (NG)
- Searching for primordial NG is a very active research area

Goals for next ~2 lectures: develop understanding of the physics underlying the CMB angular power spectrum

The Inhomogeneous Universe

Conformal Newtonian gauge metric: $\Phi = \Phi(\eta, \vec{x})$ = Newt. potential

$$ds^2 = a^2(\eta)\left(-(1+2\Phi)\, d\eta^2 + (1-2\Phi)\, d\vec{x}^2\right)$$

Consider photon propagation along geodesic as above: define λ via $p^\mu = \frac{dx^\mu}{d\lambda}$

Important difference w.r.t. homog. case: the photon energy E measured by an observer in their local inertial frame now differs from p^0! In general, p^μ components are def. in coord. frame while physically we care about $p^{\mu'}$ measured in local inertial frame of observer (us!): $p^{\mu'} = (E, p^{i'})$ and $p^\mu = (p^0, p^i)$.

These are related via:

$$ds'^2 = ds^2 \quad \text{(invariant interval)}$$

$$\Rightarrow \eta_{\mu'\nu'} p^{\mu'} p^{\nu'} = g_{\mu\nu} p^\mu p^\nu$$

Minkowski

Take obs. to be at rest and orientation of coord. systems to align

$$\Rightarrow -E^2 + \delta_{ij} p^{i'} p^{j'} = g_{00} (p^0)^2 + g_{ij} p^i p^j$$

$$\Rightarrow E = \sqrt{-g_{00}}\, p^0 = \sqrt{a^2(1+2\Phi)}\, p^0 \simeq a(1+\Phi) p^0$$

$$\Rightarrow p^0 \simeq \frac{E}{a(1+\Phi)} \rightarrow p^0 \simeq \frac{E}{a}(1-\Phi)$$

and

$$p^i = \frac{E}{a}(1+\Phi)\, \underbrace{\hat{p}^i}_{\text{unit vector in propagation direction}}$$

Note: we could also intuitively "guess" the p^0 result by noting that Φ looks like a local perturbation of the scale factor: $\tilde{a}(\eta, \vec{x}) \simeq a(\eta)\left(1 + \Phi(\eta, \vec{x})\right)$

Now use these results in the geodesic eq. to determine evolution of E:

Note (like before) that $\frac{dp^0}{d\lambda} = \frac{dp^0}{d\eta}\frac{d\eta}{d\lambda} = \frac{dp^0}{d\eta}\frac{dx^0}{d\lambda} = \frac{dp^0}{d\eta} p^0$

Geodesic eq.: $\quad \frac{dp^0}{d\lambda} + \Gamma^0_{\mu\nu} p^\mu p^\nu = 0$

$$\Rightarrow \frac{dp^0}{d\eta} + \frac{1}{p^0}\Gamma^0_{\mu\nu} p^\mu p^\nu = 0$$

$$\Rightarrow \frac{dp^0}{d\eta} + \Gamma^0_{00} p^0 + 2\Gamma^0_{0i} p^i + \frac{1}{p^0}\Gamma^0_{ij} p^i p^j = 0$$

Exercise: compute $\Gamma^0_{\mu\nu}$ for this metric, to obtain the following: Here $\mathcal{H} \equiv \frac{da/d\eta}{a} = \frac{a'}{a}$

$$\Rightarrow \frac{dp^0}{d\eta} + (\mathcal{H} + \partial_\eta \Phi) p^0 + (2\partial_i \Phi) p^i + (-3\mathcal{H} - \partial_\eta \Phi) \delta_{ij} \frac{p^i p^j}{p^0}$$

Using $p^0 = \frac{E}{a}(1-\Phi)$ and $p^i = \frac{E}{a}(1+\Phi)\hat{p}^i$ here and keeping terms to first order (Ex.: do this), we obtain

$$\frac{dE/d\eta}{E} = \underbrace{-\mathcal{H} + \partial_\eta \Phi}_{\text{redshifting due to (perturbed) expansion}} \underbrace{- \hat{p}^i \partial_i \Phi}_{\text{gravitational redshifting}}$$

$$\Rightarrow \frac{d\ln(aE)}{d\eta} = \partial_\eta \Phi - \hat{p}^i \partial_i \Phi$$

(To obtain this, also used $\frac{d\Phi}{d\eta} = \partial_\eta \Phi + (\partial_i \Phi) \frac{dx^i}{d\eta}$
$= \partial_\eta \Phi + (\partial_i \Phi) \hat{p}^i$)

The grav. redshifting term can be rewritten as:

$$\hat{p}^i \partial_i \Phi = \frac{d\Phi}{d\eta} - \partial_\eta \Phi$$

$$\Rightarrow \frac{d\ln(aE)}{d\eta} = 2\partial_\eta \Phi - \frac{d\Phi}{d\eta}$$

Generalization of $E \propto \frac{1}{a}$ to perturbed universe

Integrate: (from η_e to η_0)

$$\Rightarrow \ln(a_0 E_0) - \ln(a_e E_e) = \Phi_e - \Phi_0 + 2\int_{\eta_e}^{\eta_0} d\eta (\partial_\eta \Phi)$$

Recall $a_0 \equiv 1$

Note that Φ_0 = local gravitational potential, which can only affect the monopole ($\ell=0$ mode) and is hence unobservable and can be set to zero WLOG.

Note: perturbation quantities here can be evaluated at the unperturbed last-scattering time (η_*), since corrections would be second order.

$$\Rightarrow \ln(a_0 E_0) = \ln(a_* E_*) + \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)$$

For photons, note that $E \propto T$ and that the distribution function (Bose-Einstein) is only a function of $\frac{E}{T}$, thus:

$$aE \propto aT \propto a(\bar{T} + \Delta T) \propto a\bar{T}\left(1 + \frac{\Delta T}{\bar{T}}\right)$$

$$\Rightarrow \ln(aE) = \ln\left(a\bar{T}\left(1 + \frac{\Delta T}{\bar{T}}\right)\right) + \text{const.}$$

$$= \ln(a\bar{T}) + \ln\left(1 + \frac{\Delta T}{\bar{T}}\right) + \text{const.}$$

$$\simeq \ln(a\bar{T}) + \frac{\Delta T}{\bar{T}} + \text{const.}$$ Taylor expand

Recall that $\bar{T} \propto 1/a$ so $\bar{T}_0 = a_* \bar{T}_*$ $\Rightarrow \ln(a_0 \bar{T}_0) = \ln(a_* \bar{T}_*)$

$$\Rightarrow \boxed{\frac{\Delta T_0}{\bar{T}_0} = \frac{\Delta T_*}{\bar{T}_*} + \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)}$$

Alternate derivation that clarifies connection to local perturbation in scale factor:

Start from $\frac{a_0 E_0}{a_e E_e} = e^{-\Phi_0 + \Phi_e + 2\int_{\eta_e}^{\eta_0} d\eta \, (\partial_\eta \Phi)}$

Taylor expand RHS (first order in perturbations):

$$\Rightarrow \frac{E_0}{a_e E_e} \simeq 1 - \Phi_0 + \Phi_e + 2\int_{\eta_e}^{\eta_0} d\eta \, (\partial_\eta \Phi)$$

Note that Φ_0 = local gravitational potential, which can only affect the monopole ($\ell=0$ mode) and is hence unobservable and can be set to zero WLOG.

$$\Rightarrow E_0 = a_e E_e \left(1 + \Phi_e + 2\int_{\eta_e}^{\eta_0} d\eta \, (\partial_\eta \Phi)\right)$$

Note: perturbation quantities on RHS can be evaluated at the unperturbed last-scattering time (η_*), since corrections would be second order. Not true for a_e!

$$\Rightarrow E_0 \simeq a_e E_* \left(1 + \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)\right)$$

For photons, note that $E \propto T$, so we have

$$\Rightarrow T_0 = a_e T_* \left(1 + \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)\right)$$

$$\Rightarrow \bar{T}_0 + \Delta T_0 \simeq (a_* + \Delta a_e) \bar{T}_* \left(1 + \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)\right)$$

$$\simeq a_* \bar{T}_* \left(1 + \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)\right) + \bar{T}_* \Delta a_e \quad \text{to first order}$$

Recall that $\bar{T} \propto 1/a$ so $\bar{T}_0 = a_* \bar{T}_*$

$$\Rightarrow \Delta T_0 \simeq \underbrace{a_* \bar{T}_*}_{=\bar{T}_0} \left(\Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi) \right) + \bar{T}_* \Delta a_e$$

$$\Rightarrow \frac{\Delta T_0}{\bar{T}_0} = \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)) + \frac{\Delta a_e}{a_*}$$

What is Δa_e? Must be determined by the local temp. being ~ 0.3 eV $(\sim \bar{T}_*)$.

$$\Rightarrow (\bar{\rho}_r + \delta\rho_r)(\eta_* + \Delta\eta) = \bar{\rho}_r(\eta_*) = \sigma T_*^4 \qquad ' \equiv \frac{d}{d\eta}$$

$$\Rightarrow \bar{\rho}_r(\eta_*) + \delta\rho_r(\eta_*) + \bar{\rho}_r'|_{\eta_*} \Delta\eta = \bar{\rho}_r(\eta_*)$$

$$\Rightarrow \delta\rho_r(\eta_*) = -\bar{\rho}_r'|_{\eta_*} \Delta\eta \quad \Rightarrow \quad \Delta\eta = -\frac{\delta\rho_r(\eta_*)}{\bar{\rho}_r'|_{\eta_*}}$$

$$\Rightarrow \Delta\eta = \frac{-\delta\rho_r}{\frac{d\bar{\rho}_r}{da} a'} = \frac{-\delta\rho_r}{\bar{\rho}_{r,0}(-4a^{-5})a'} = \frac{\delta\rho_r}{\bar{\rho}_r \cdot 4\left(\frac{a'}{a}\right)} = \frac{\delta_r}{4\mathcal{H}}$$

where $\delta_r \equiv \dfrac{\delta\rho_r}{\bar{\rho}_r}$ as usual.

Thus:

$$\Rightarrow \frac{\overset{= a_* + \Delta a_e = a_* + a_*' \Delta\eta}{a_e}}{a_*} \simeq \left(1 + \frac{a_*'}{a_*}\left(\frac{\delta_r^*}{4\mathcal{H}_*}\right)\right)$$

$$\Rightarrow \frac{a_e}{a_*} \simeq 1 + \frac{\delta_r^*}{4} \quad \Rightarrow \quad \frac{a_* + \Delta a_e}{a_*} = 1 + \frac{\delta_r^*}{4}$$

$$\Rightarrow \frac{\Delta a_e}{a_*} \simeq \frac{\delta_r^*}{4}$$

$$\Rightarrow \frac{\Delta T_0}{\bar{T}_0} = \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \,(\partial_\eta \Phi)) + \frac{\delta_r^*}{4} \quad \checkmark$$

$\Rightarrow$ shows immediately that $\frac{\Delta T_*}{\bar{T}_*} = \frac{\delta_r^*}{4}$ (by comparison to our result above)

In fact, in full generality there are two contributions to $\frac{\Delta T_*}{\bar{T}_*}$:

1) Radiation density perturbations: $\leftrightarrow$ local scale factor perturbations

$$\bar{\rho}_r \propto T^4 \Rightarrow \delta\rho_r \propto 4\bar{T}^3 \delta T \propto 4\bar{T}^4 \frac{\delta T}{\bar{T}} \propto 4\bar{\rho}_r \frac{\delta T}{\bar{T}}$$

$$\Rightarrow \frac{\delta T}{\bar{T}} \propto \frac{\delta\rho_r}{4\bar{\rho}_r} \Rightarrow \frac{\delta T}{\bar{T}} \propto \frac{\delta_r}{4}$$

But physically I think more useful to think in terms of local perturbation to scale factor, since the local photon temperature at decoupling is ~ 0.25 eV always: $\frac{\Delta a_*}{a_*} = \frac{\delta_r}{4} = \frac{\delta T}{\bar{T}}$

$\Rightarrow$ points w/ higher ρ_r (positive δ_r) reach 0.25 eV later, hence CMB last-scatters later (thus with larger a), thus redshifting less to today, and thus yielding positive observed temp. fluctuation (and vice versa for negative δ_r).

$\Rightarrow$ surface of last scattering is "wrinkled" (in terms of a) [in Newtonian gauge! this is a gauge-dep. statement]

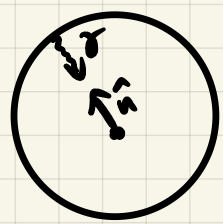
2) Doppler contribution: perturbations in T sourced by bulk velocities of e^- at last scattering

$\Rightarrow$ e^- moving toward us: positive $\frac{\delta T}{\bar{T}}$

e^- " away from us: negative $\frac{\delta T}{\bar{T}}$

Let $\hat{n} \equiv$ unit vector pointing from us to point on last-scattering surface

(note that γ is moving in $-\hat{n}$ direction)



$$\Rightarrow \frac{\delta T}{\bar{T}} = \vec{v}_e \cdot (-\hat{n})$$

Note that $\vec{v}_e \simeq \vec{v}_b$ since plasma is tightly coupled

$$\Rightarrow \frac{\delta T}{\bar{T}} = -\hat{n} \cdot \vec{v}_b$$

(This can also be derived from first principles via Boltzman approach.)

Put it all together:

$$\boxed{\frac{\Delta T_0}{\bar{T}_0} = \Phi_* + 2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)) + \frac{\delta_r^*}{4} - (\hat{n} \cdot \vec{v}_b)_*}$$

Interpretation:

1) Φ_*: gravitational redshift of CMB γ as they climb out of potentials at last scattering

($\Phi_* < 0 \Rightarrow$ matter overdensity $\Rightarrow -\delta T/\bar{T}$

$\Phi_* > 0 \Rightarrow$ " underdensity $\Rightarrow +\delta T/\bar{T}$)

2) $\frac{\delta_r^*}{4}$: intrinsic temp. perts. ($\leftrightarrow$ local scale factor perts.)

$\Rightarrow$ the combination $S_* \equiv \Phi_* + \frac{\delta_r^*}{4}$ is called the Sachs-Wolfe term [this combination is gauge-independent]

3) $2\int_{\eta_*}^{\eta_0} d\eta \, (\partial_\eta \Phi)$: "integrated Sachs-Wolfe" term

$\Rightarrow$ if $\partial_\eta \Phi \neq 0$, e.g., due to dark energy or radiation domination, then this term is non-zero

- rad. generates this at $z \gtrsim 1100 \Rightarrow$ early ISW
- DE generates this at $z \lesssim 1 \Rightarrow$ late ISW

Potentials decaying due to DE $\Rightarrow \Phi' > 0$ in collapsed regions $\Rightarrow + \delta T/\bar{T}$

($\Phi' < 0$ in voids $\Rightarrow - \delta T/\bar{T}$)

4) $-(\hat{n} \cdot \vec{v}_b)_*$: Doppler term

e^- velocity toward us $\Rightarrow + \delta T/\bar{T}$

" " away from us $\Rightarrow - \delta T/\bar{T}$

Notes:

- The S-W and Doppler terms are 3D fields evaluated at last scattering (at time η_* and position $(\eta_0 - \eta_*)\hat{n}$, where $\eta_0 - \eta_* = \chi_* \simeq 14$ Gpc is the distance to last scattering (in flat universe))
- Which contributions dominate?

Let's focus on large scales ($\ell \lesssim 100$):

- ISW generally small since universe has been matter-dominated for most of CMB-relevant history

· Doppler small since $\vec{v}_b$ vanishes on superhorizon scales

$\Rightarrow$ S-W will dominate

What term wins in S-W?

Superhorizon limit: $-2\Phi \simeq \delta_m = \frac{3}{4}\delta_r \Rightarrow \delta_r = \frac{-8}{3}\Phi$

$$\Rightarrow \left(\frac{\delta_r}{4} + \Phi\right)_* = \left(-\frac{2}{3}\Phi + \Phi\right)_* = \frac{\Phi_*}{3}$$

$$= \frac{1}{5}R \quad \left(\text{recall } \Phi = \frac{3}{5}R \text{ during matter dom.}\right)$$

$\Rightarrow$ grav. redshift term wins over $\frac{\delta_r}{4}$ term!

$\Rightarrow$ overdensity at last scattering ($\delta_r^* > 0$, $\delta_m^* > 0$, $\Phi_* < 0$) yields $-\frac{\delta T}{T}$ (cold spot) in CMB, and vice versa for underdensity

$\Rightarrow$ these are the hot and cold spots you see by eye in full-sky CMB maps (Planck, WMAP)

$\Rightarrow$ image of grav. field at last scattering!

Initial Conditions

Inflationary theories predict statistics of the comoving curvature perturbation R.

R is constant on superhorizon scales, thus allowing us to connect inflationary physics (when Fourier modes go outside the horizon) to late-time observables (after modes re-enter).