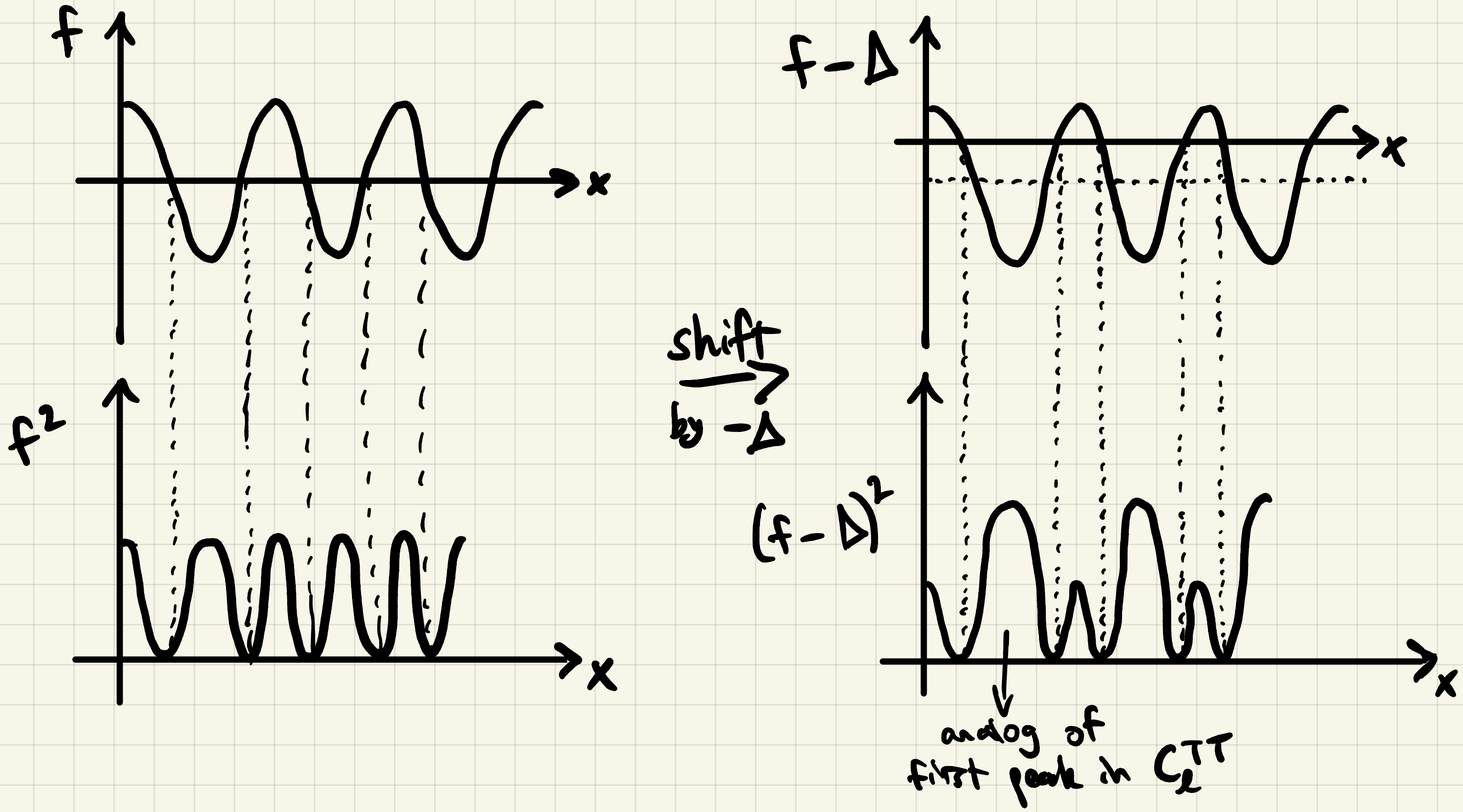
Schematic to understand effect on CMB temp. power spectrum: consider function $f(x) \propto \cos(x)$ and its square $f^2(x) \propto \cos^2(x)$, as well as shifted function $(f(x)-\Delta)$ and its square $(f(x)-\Delta)^2$




## Small scales (at decoupling): $k > k_{eq}$

First consider evolution of rad. perts. in the absence of baryons (in the rad.-dom. era).

Poisson: $\nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \Delta_r$ ($\Delta_r \equiv$ comoving density contrast)

$$\Rightarrow \Delta_r(\eta, \vec{k}) = -\frac{2}{3}(k\eta)^2 \, \Phi(\eta, \vec{k})$$ using $\mathcal{H} = \frac{1}{\eta}$ during RD

Use our sol. for $\Phi$ during rad. dom. to obtain:

$$\Delta_r(\eta, \vec{k}) = -4 R_i^k \, \frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})}$$ valid on all scales

Super-horizon limit: $k\eta \ll 1$
sub-horizon limit: $k\eta \gg 1$

$$\Rightarrow \Delta_r(\eta, \vec{k}) \simeq \begin{cases} -\frac{4}{9} R_i^k (k\eta)^2 & \text{super-horizon} \\ 4 R_i^k \cos\left(\frac{k\eta}{\sqrt{3}}\right) & \text{sub-horizon} \end{cases}$$

$\Rightarrow \Delta_r \propto \eta^2 \propto a^2$ on super-horizon scales

$\Delta_r$ oscillates with constant amplitude on sub-horizon scales

corresponds to $\delta_r = -\frac{4}{3} R_i$ on super-horizon scales
↳ Newtonian gauge

Now consider the matter-dom. era:

Perturbed cont. eq.: $\delta_r' = -\frac{4}{3} \vec{\nabla} \cdot \vec{v}_r + 4\Phi'$

Perturbed Euler eq.: $\vec{v}_r' = -\frac{1}{4} \vec{\nabla} \delta_r - \vec{\nabla} \Phi$

$$\Rightarrow \delta_r'' - \frac{1}{3} \nabla^2 \delta_r = \frac{4}{3} \nabla^2 \Phi + 4\Phi''$$

MD: growing mode sol. for $\Phi$ is $\Phi = \text{const.}$ on all scales

$$\Rightarrow \delta_r'' - \frac{1}{3} \nabla^2 \delta_r = \frac{4}{3} \nabla^2 \Phi = \text{const.} \Rightarrow \delta_r'' + \frac{1}{3} k^2 (\delta_r + 4\Phi) = 0$$

$\Rightarrow$ EOM of a S.H.O. with a constant driving force

Thus, sub-horizon perts. in rad. density oscillate with a constant amplitude around an offset equilibrium point, given by $-4\Phi_0(\vec{k})$, where $\Phi_0(\vec{k}) \equiv$ amplitude of potential in MD.

Note that $\Phi_0(\vec{k})$ is $\vec{k}$-dep. due to ICs and the non-trivial scale dependence of growth during RD era ("transfer function").

$$\Rightarrow \text{sol.}: \delta_r(\eta, \vec{k}) = C(\vec{k}) \cos\left(\frac{k\eta}{\sqrt{3}}\right) + D(\vec{k}) \sin\left(\frac{k\eta}{\sqrt{3}}\right) - 4\Phi_0(\vec{k})$$

On deep sub-horizon scales, $\Delta_r = \delta_r$

$\left(\Delta_r \equiv \frac{1}{\bar{\rho}_r}(\delta\rho_r + v_r \bar{\rho}_r')\right)$

and modes all entered during rad. dom., and by decoupling $\Phi$ had ~fully decayed away, so sol. is

$$\delta_r(\eta, \vec{k}) \simeq 4R_i(\vec{k}) \cos\left(\frac{k\eta}{\sqrt{3}}\right)$$

Sol. on intermediate scales is obtained by adding the const. potential offset above (as we also found earlier).

S-W: $S \equiv \frac{\delta_r}{4} + \Phi$

$\Phi \simeq 0$ on very small scales at decoupling

so $S \simeq \frac{\delta_r}{4}$ ("radiation driving" boosting amplitude — perts. don't have to fight against grav. potential as they bounce back in their oscillations)

$$\Rightarrow \delta_r(\eta, \vec{k}) \simeq R_i(\vec{k}) \cos(k r_s(\eta))$$

Synthesis:

$$S(\vec{k}, \eta^*) \simeq \begin{cases} \frac{1}{5} R_i(\vec{k}) \left((1+3R)\cos(k r_s^*) - 3R\right) & , \ k \ll k_{eq} \\ R_i(\vec{k}) \cos(k r_s^*) & , \ k \gg k_{eq} \end{cases}$$

$$\equiv T_s(k, \eta^*) R_i(\vec{k})$$

$$\Rightarrow \boxed{T_s(k, \eta^*) \simeq \begin{cases} \frac{1}{5}\left((1+3R)\cos(k r_s^*) - 3R\right) & , \ k \ll k_{eq} \\ \cos(k r_s^*) & , \ k \gg k_{eq} \end{cases}}$$

# Damping : Diffusion

Important feature not captured in our simple analysis: "diffusion damping" ("Silk damping")

$\Rightarrow$ arises because $\gamma$ diffuse out of high-density regions into low-density regions, which reduces $\delta_\gamma$ (density contrast)

$\Rightarrow$ suppresses $\delta_\gamma$ on very small scales (high $k$)

$\Rightarrow$ treatment requires going beyond tight-coupling approx.

Approx.: $\gamma$ mean free path $\ell_p = \frac{1}{a n_e \sigma_T} = |\tau'|^{-1}$

$\Rightarrow$ number of scatterings $\simeq d\eta/\ell_p \equiv N$

$\Rightarrow$ random-walk dist. $= \sqrt{N}\,\ell_p = \sqrt{\frac{d\eta}{\ell_p}}\,\ell_p \simeq \sqrt{d\eta\,|\tau'|^{-1}}$
$\equiv L_D$

$\Rightarrow$ damping scale : $k_D^{*\,-2} \simeq L_D^2 \simeq \int_0^{\eta_*} d\eta\,|\tau'|^{-1}$

Result: $k_D^{*\,-1} \simeq 7$ Mpc $\Rightarrow \ell_D \simeq \frac{k_D^{*\,-1}\chi_*}{\sqrt{2}} \simeq 1400$

$\Rightarrow$ really a combination of thermal conduction and photon viscosity

# Damping: Landau

Decoupling is not actually instantaneous

For high-$k$ modes, the duration of recombination is comparable to the fluctuation wavelength

Gaussian approx. to width of visibility function $g(\eta)$:

$$\sigma_g \simeq 16 \text{ Mpc}$$

Have to average S-W (and other) source terms over this finite width $\Rightarrow$ leads to damping of high-$k$ modes

$$\Rightarrow k_L^{*-1} \simeq \frac{c_s^* \sigma_g}{\sqrt{2}} = \frac{\sigma_g}{\sqrt{6(1+R_*)}} \simeq 5 \ \text{Mpc}$$

Combine Silk + Landau:

$$\Rightarrow k_{\text{damp}}^{*\,-1} = \sqrt{k_D^{*\,-2} + k_L^{*\,-2}} \simeq 9 \ \text{Mpc}$$

$$\Rightarrow \ell_{\text{damp}} \simeq \frac{k_{\text{damp}}^{*-1} \chi_*}{\sqrt{2}} \simeq 1100$$

Include in transfer function:

$$\boxed{T_s(k,\eta^*) \simeq e^{-\frac{k^2}{k_{\text{damp}}^{*\,2}}} \times \begin{cases} \frac{1}{5}\left((1+3R)\cos(kr_s^*) - 3R\right) & , \ k \ll k_{eq} \\ \cos(kr_s^*) & , \ k \gg k_{eq} \end{cases}}$$

$\Rightarrow$ (approximate) S-W transfer function

## Reionization:

Consider $\gamma$ coming from some direction with temperature $T = \bar{T}(1 + \frac{\Delta T}{\bar{T}})$.

They scatter off free $e^-$ during reionization: only $e^{-\tau}$ get through to us.

A fraction $(1-e^{-\tau})$ get scattered into this line of sight from all other directions (photon number is conserved by Thomson scattering). These $\gamma$ will have temperature $\bar{T}$ on average.

$\Rightarrow$ resulting $T$ we see $= \bar{T}(1+\frac{\Delta T}{\bar{T}})e^{-\tau} + \bar{T}(1-e^{-\tau})$

$$= \bar{T}\left(1 + \frac{\Delta T}{\bar{T}} e^{-\tau}\right) \equiv T_{obs}$$

$\Rightarrow$ We observe anisotropy $\frac{\Delta T}{\bar{T}} e^{-\tau}$

$\Rightarrow$ $C_\ell^{TT}$ is suppressed by $e^{-2\tau}$ for all modes within the horizon during reionization ($\ell \gtrsim 50$)