Stochastic Lattice Models for Clouds and Parameterization of Organized Convection

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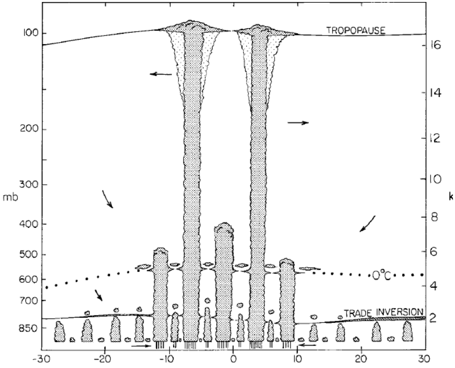
ICTP/PIMS summer school, July 1-19, 2024.

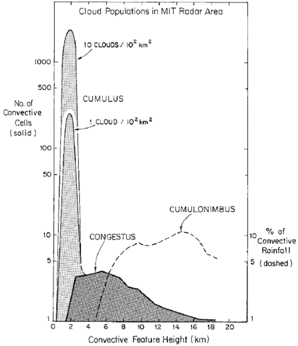
- Lecture 1: Stochastic modelling, Markov processes, Theory and Simulation (9 am - 10:30 am); Chapter 4 and Section 10.2.
- **Lecture 2: The stochastic multicloud model (11 am - 12:30 pm) ; Chapter 10: Sections 10.3 to 10.6**
- Hands-on activities (2:30 pm - 3:45 pm):
- Lecture 3: Waves and convective organization in the SMCM (4:00 pm - 5:15 pm) ; Chapter 11.

Introduction

- Organized convection in tropics involves three cloud types that interact with each other and with the environment
- Statistical-self similarity across-scales of tropical convective systems
- Deviation from quasi-equilibrium paradigm
- Impacts fidelity of GCMs with regards to tropical weather and climate
- Importance of missing subgrid variability due to organized convection as a multiscale process
- Design a stochastic models to mimic these subgrid scale interactions without substantial computational overhead in lieu of say super-parameterization

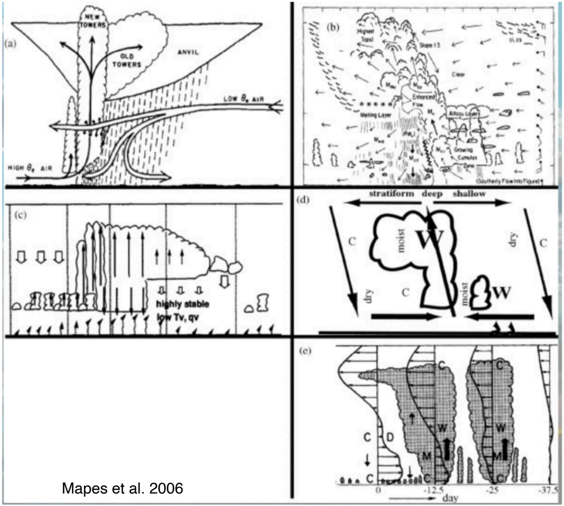
Tri-modal nature of tropical convection





Jonhson et al. 1999

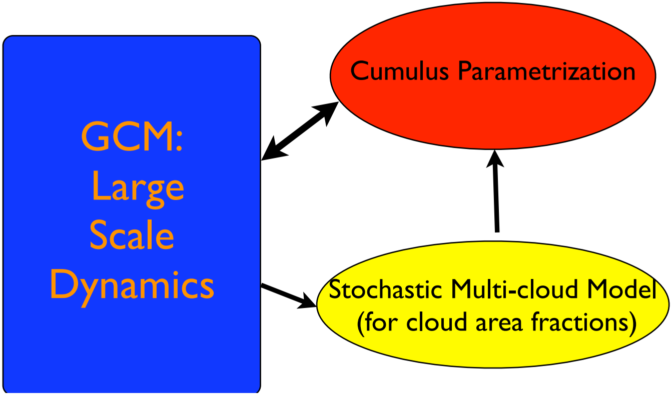
Self-similarity across scales

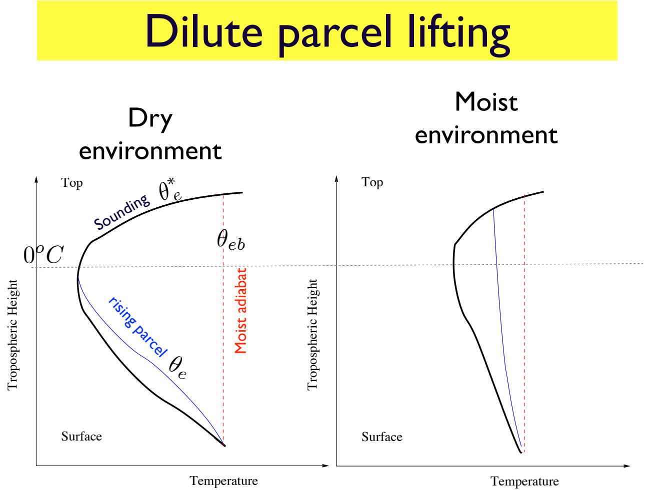


Mapes et al. 2006

The stochastic model to add on an existing cumulus scheme

Stochastic Multi-cloud Model to inform cumulus parametrization: represent the missing sub-grid scale variability

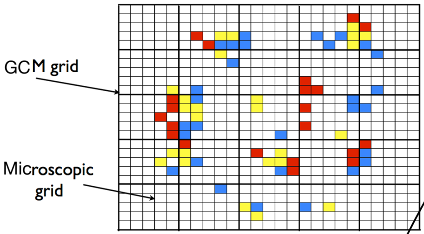


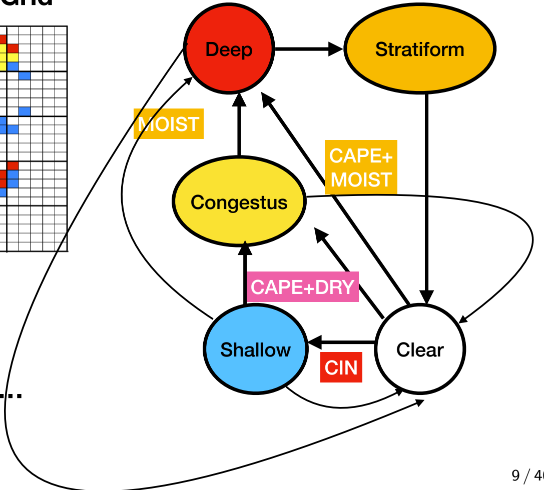

Dilute parcel lifting
Dry environment
Moist environment
Top
Sounding
θ_e^*
θ_{eb}
0^oC
rising parcel
θ_e
Moist adiabat
Tropospheric Height
Surface
Temperature
Top
Tropospheric Height
Surface
Temperature

Multi-type Particle Microscopic Lattice Model

Multidimensional Markov process on a Lattice with Conditional Probability Rules

Fine lattice overlaid on GCM Grid





Each lattice is either occupied by a cloud of certain type or is clear sky... time varying!

The devil is in the details:

Play simple: The three cloud type SCM.

$$X_t^i = \begin{cases} 0 & \text{if site } i \text{ is clear sky} \\ 1 & \text{if site } i \text{ is occupied by a congestus cloud} \\ 2 & \text{if site } i \text{ is occupied by a deep convective cloud} \\ 3 & \text{if site } i \text{ is occupied by a stratiform anvil.} \end{cases} \tag{1}$$

$$P_{lk}^i \equiv \text{Prob}\left\{X_{t+\Delta t}^i = k / X_t^i = l\right\} = R_{lk}^i \Delta t + o(\Delta t), \tag{2}$$

for $l, k = 0, 1, 2, 3,$ and $l \neq k$

$$P_{ll}^i \equiv \text{Prob}\left\{X_{t+\Delta t}^i = l / X_t^i = l\right\} = 1 - \sum_{k=0, k\neq l}^{3} P_{lk}^i, \tag{3}$$

$$= 1 - \Delta t \sum_{k=0, k\neq l}^{3} R_{lk}^i + O(\Delta t)$$

Intuitive Transition Rules

1. A clear site turns into a congestus site with high probability if CAPE is positive and the middle troposphere is dry.
2. A congestus or clear sky site turns into a deep convective site with high probability if CAPE is positive and the middle troposphere is moist.
3. A deep convective site turns into a stratiform site with high probability with a prescribed conversion rate, which may or may not depend on the state of the environment.
4. A cloudy site turns back to a clear sky with a certain probability according to a prescribed decay time scale for each cloud type.
5. It is very unlikely, during the short period of time Δt, for a clear sky or a congestus site to turn into a stratiform site, for a deep convective or stratiform site to turn into a congestus site, nor for a stratiform site to turn into a deep convective site.

The Matrix of transition rates

- Forbidden transitions

$$R_{03} = R_{13} = R_{21} = R_{31} = R_{32} = 0.$$

- Markov process at each site (independently on whether sites are connected or not)

$$R = \begin{bmatrix} -R_{01} - R_{02} & R_{01} & R_{02} & 0 \\ R_{10} & -R_{10} - R_{12} & R_{12} & 0 \\ R_{20} & 0 & R_{20} - R_{23} & R_{23} \\ R_{30} & 0 & 0 & -R_{30} \end{bmatrix}. \qquad (4)$$

Case when local interactions are ignored

- Based on intuitive rules, we set

$$
\begin{aligned}
R_{01} &= \frac{1}{\tau_{01}}\Gamma(C)\Gamma(D), \qquad & R_{02} &= \frac{1}{\tau_{02}}\Gamma(C)(1-\Gamma(D)), \\
R_{10} &= \frac{1}{\tau_{10}}\Gamma(D), \qquad & R_{12} &= \frac{1}{\tau_{12}}\Gamma(C)(1-\Gamma(D)), \\
R_{20} &= \frac{1}{\tau_{20}}(1-\Gamma(C)), \qquad & R_{23} &= 1/\tau_{23}, \qquad R_{30} = 1/\tau_{30}.
\end{aligned} \tag{5}
$$

where

$$
\Gamma(x) \equiv \begin{cases} 1 - e^{-x} & \text{if } x > 0 \\ 0 & \text{Otherwise.} \end{cases} \tag{6}
$$

- The transition time scales τ_{kl} are inferred from observations (radar data) using Bayesian machine learning: De La Chevrotiere et al. (2015,2016), Cardosso-Bihlo et al. (2019), ... **Carlos's talk next week!**

Academic example

Time	description	Case 1	Case 2
τ_{01}	formation of congestus	1 hour	3 hours
τ_{10}	decay of congestus	5 hours	2 hours
τ_{12}	conversion of congestus to deep	1 hour	2 hours
τ_{02}	formation of deep	2 hours	5 hours
τ_{23}	conversion of deep to stratiform	3 hours	0.5 hour
τ_{20}	decay of deep	5 hours	5 hours
τ_{30}	decay of stratiform	5 hours	24 hours

The stationary distribution, cloud area fractions, and the equilibrium statistics of the lattice model

- Equilibrium measure

$$\mathcal{P}_e R = 0 \qquad \text{(Steady state of forward equations)}$$

$$\mathcal{P}_e = \frac{1}{Z}\left(1, \frac{R_{01}}{R_{10}+R_{12}}, \frac{1}{R_{20}+R_{23}}\left(R_{02} + \frac{R_{12}R_{01}}{R_{10}+R_{12}}\right), \frac{R_{23}}{R_{30}}\frac{1}{R_{20}+R_{23}}\left(R_{02} + \frac{R_{12}R_{01}}{R_{10}+R_{12}}\right)\right) \tag{7}$$

- Cloud-type area fractions

$$\sigma_c = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{X_t^i=1\}}, \quad \sigma_d = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{X_t^i=2\}}, \quad \sigma_s = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{X_t^i=3\}} \tag{8}$$

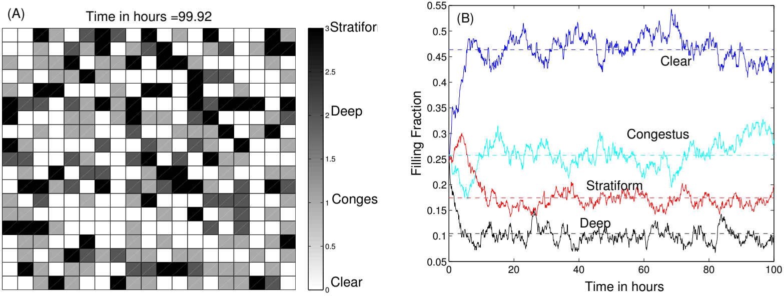
where

$$\mathbb{1}_{\{X_t^i=k\}} = \begin{cases} 1 & \text{if } X_t^i = k \\ 0 & \text{otherwise.} \end{cases}$$

The clear sky area fraction is given by

$$\sigma_{cs} \equiv 1 - \sigma_c - \sigma_d - \sigma_s.$$

Standalone example



Monte Carlo simulation SMCM with $n = 20$, $C = 0.25$, $D = 0.75$, Case 1. (A) A snapshot picture of one typical realization and (B) time series of the total coverages associated with each cloud type with the equilibrium values overlaid (dashed lines).

Coarse-grained process and mean-field equations

- Coarse grained process: $N = n \times n$ total number of lattice sites, N_c^t, N_d^t, and N_s^t are resp. the number of congestus, deep, and startiform sites at time t.
- Coverage fraction

$$\text{Prob}\{X_t^i = 1\} = \frac{N_c^t}{N} = \sigma_c^t, \quad \text{Prob}\{X_t^i = 2\} = \frac{N_d^t}{N} = \sigma_d^t, \tag{9}$$

$$\text{Prob}\{X_t^i = 3\} = \frac{N_s^t}{N} = \sigma_s^t, \quad \text{Prob}\{X_t^i = 0\} = \frac{N_{cs}^t}{N} = \sigma_{cs}^t, \quad \forall i = 1, 2, \cdots, N.$$

- Stochastic dynamics: Transition time is that of Min of i.i.d. Exp. R.V's:

$$\text{Prob}\{N_c^{t+\Delta t} = k + 1/N_c^t = k\} = N_{cs} R_{01} \Delta t + o(\Delta t). \tag{10}$$

$$\text{Prob}\{N_c^{t+\Delta t} = k - 1/N_c^t = k\} = N_c (R_{10} + R_{12}) \Delta t + o(\Delta t),$$

$$\text{Prob}\{N_d^{t+\Delta t} = k + 1/N_d^t = k\} = (N_{cs} R_{02} + N_c R_{12}) \Delta t + o(\Delta t),$$

$$\text{Prob}\{N_d^{t+\Delta t} = k - 1/N_d^t = k\} = N_d (R_{20} + R_{23}) \Delta t + o(\Delta t), \tag{11}$$

$$\text{Prob}\{N_s^{t+\Delta t} = k + 1/N_s^t = k\} = N_d R_{23} \Delta t + o(\Delta t), \tag{12}$$

$$\text{Prob}\{N_s^{t+\Delta t} = k - 1/N_s^t = k\} = N_s R_{30} \Delta t + o(\Delta t), \tag{13}$$

- Mixed population birth and death process with immigration.

Probability transition matrix, Kolmogorov Equations, and Gillespie's algorithm

- 3D vector $Z_t = (N_c^t, N_d^t, N_s^t)$,
  $\epsilon_1 = (1,0,0), \epsilon_2 = (0,1,0), \epsilon_3 = (0,0,1)$.
- Kolmogorov backward Equations,

$$\begin{aligned}\frac{d}{dt}\mathrm{P}_{\mathrm{i},\mathrm{j}} =& N_{cs}R_{01}\mathrm{P}_{\mathrm{i}+\epsilon_1,\mathrm{j}} + N_{cs}R_{02}\mathrm{P}_{\mathrm{i}+\epsilon_2,\mathrm{j}} + N_c R_{12}\mathrm{P}_{\mathrm{i}+\epsilon_2-\epsilon_1,\mathrm{j}} \\ &+ N_d R_{23}\mathrm{P}_{\mathrm{i}+\epsilon_3-\epsilon_2,\mathrm{j}} + N_c R_{10}\mathrm{P}_{\mathrm{i}-\epsilon_1,\mathrm{j}} + N_d R_{20}\mathrm{P}_{\mathrm{i}-\epsilon_2,\mathrm{j}} + N_s R_{30}\mathrm{P}_{\mathrm{i}-\epsilon_3,\mathrm{j}} \\ &- \left[N_{cs}(R_{01}+R_{02}) + N_c(R_{10}+R_{12}) + N_d(R_{23}+R_{20}) + N_s R_{30}\right]\mathrm{P}_{\mathrm{i},\mathrm{j}} \\ &\mathrm{i},\mathrm{j} \in \{0,1,\cdots N\}^3. \end{aligned} \tag{14}$$

Solution: $P(t) = \exp\{Rt\} \in \mathcal{S} \subset \mathbb{R}^{N^3 \times N^3}$ (and full matrix)

- Efficiently evolved with Gillespie's algorithm (next slide)

Multidimensional Gillespie's exact algorithm:

0) Let $\lambda = N_{cs}(R_{01} + R_{02}) + N_c(R_{10} + R_{12}) + N_d(R_{23} + R_{20}) + N_s R_{30}$.
1) Let $Z_t = (N_c, N_d, N_s)$ be the state of the system at time t, $0 \leq t < \Delta T$.
2) Generate a random number r_1 uniformly from $(0, 1)$. Set $s = -\frac{1}{\lambda}\ln(r_1)$.
3) If $t + s > \Delta T$, then no transition occurs. Set $t = \Delta T$. Stop.
4) If $t + s \leq \Delta T$. Divide the interval into seven subintervals $I_1, I_2, \cdots, I_7$ of sizes

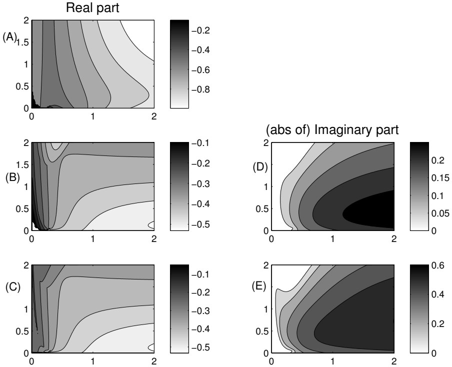
$$\frac{N_{cs}R_{01}}{\lambda}, \frac{N_{cs}R_{02}}{\lambda}, \frac{N_c R_{10}}{\lambda}, \frac{N_c R_{12}}{\lambda}, \frac{N_d R_{23}}{\lambda}, \frac{N_d R_{20}}{\lambda}, \frac{N_s R_{30}}{\lambda},$$

respectively.

5) Generate a random number r_2. Select the subinterval I_k such that $r_2 \in I_k$. Then make the corresponding transition as follows.
   - If $r_2 \in I_1$, then $N_c = N_c + 1$.
   - If $r_2 \in I_2$, then $N_d = N_d + 1$.
   - If $r_2 \in I_3$, then $N_c = N_c - 1$.
   - If $r_2 \in I_4$, then $N_c = N_c - 1$, $N_d = N_d + 1$.
   - If $r_2 \in I_5$, then $N_d = N_d - 1$, $N_s = N_s + 1$.
   - If $r_2 \in I_6$, then $N_d = N_d - 1$.
   - If $r_2 \in I_7$, then $N_s = N_s - 1$.
6) Set $t = t + s$. If $t < \Delta T$ got to 1.

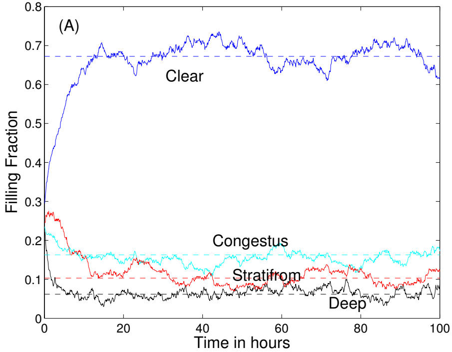
The mean field equations ... is damped

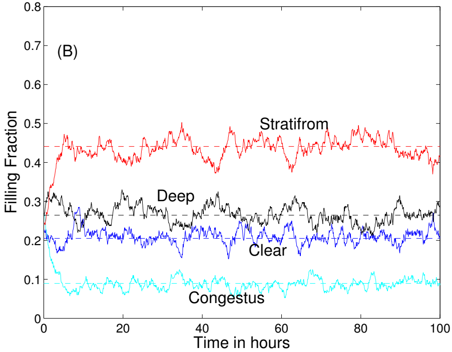
$$\begin{aligned}
\dot{\sigma}_c &= (1 - \sigma_c - \sigma_d - \sigma_s)R_{01} - \sigma_c(R_{10} + R_{12}) \\
\dot{\sigma}_d &= (1 - \sigma_c - \sigma_d - \sigma_s)R_{02} + \sigma_c R_{12} - \sigma_d(R_{20} + R_{23}) \\
\dot{\sigma}_s &= \sigma_d R_{23} - \sigma_s R_{30}.
\end{aligned} \tag{15}$$



Equilibrium eigenvalues of the mean field equations. (A),(B),(C): real part. (D): imaginary. (E): frequency/damping rate. CAPE C v.s dryness D on axises. Without external forcing the system relaxes quickly to its equilibrium measure.

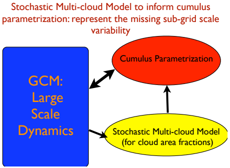
Stochastic oscillations depend on frequency-to-damping ratio

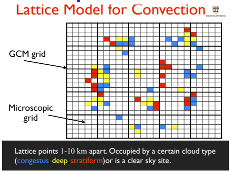




Stochastic oscillations: (a) small (< 0.1) and (b) large (≈ 0.6) frequency to damping ratio

Coupling the SMCM to a cumulus parameterization





- SMCM simulates evolution of convective area fraction of key cloud types
- Coupled to mass flux scheme in a straightforward manner, with the goal to break the quasi-equilibrium paradigm:

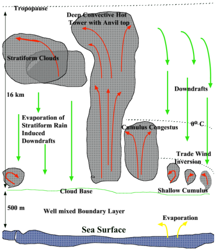
$$M_u = \sigma_u w_u$$

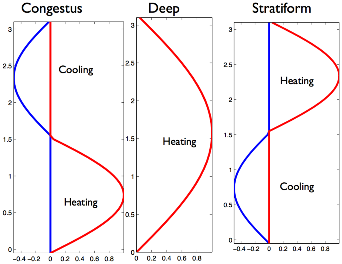
  (Peters et al. 2017; Dorrestijn et al. 2016)
- Here we will follow a simpler approach: different heating profiles associated with different cloud types, not necessarily well captured by existing mass-flux schemes.
- Success the Multi-cloud Model Paradigm in modelling Convectively Coupled Waves and MJO (both linear theory and non-linear simulations)

2 baroclinic mode Toy GCM

$$\mathrm{v} = \sqrt{2}\mathrm{v}_1 \cos(z) + \sqrt{2}\mathrm{v}_2 \cos(2z),$$
$$\theta = \sqrt{2}\theta_1 \sin(z) + 2\sqrt{2}\theta_2 \sin(2z).$$

$$\frac{\partial \mathrm{v}_j}{\partial t} + \beta y \mathrm{v}_j^{\perp} - \nabla \theta_j = -C_d(u_0)\mathrm{v}_j - \frac{1}{\tau_W}\mathrm{v}_j,\ j = 1, 2,$$
$$\frac{\partial \theta_1}{\partial t} - \operatorname{div} \mathrm{v}_1 = \bar{H}_d + \xi_s \bar{H}_c + \xi_s \bar{H}_s + S_1$$
$$\frac{\partial \theta_2}{\partial t} - \frac{1}{4}\operatorname{div} \mathrm{v}_2 = H_c - H_s + S_2,$$
$$\frac{\partial \theta_{eb}}{\partial t} = \frac{1}{h_b}(E - D)$$
$$\frac{\partial q}{\partial t} + \operatorname{div}\left((\mathrm{v}_1 + \tilde{\alpha}\mathrm{v}_2)q\right) + \tilde{Q}\ \operatorname{div}(\mathrm{v}_1 + \tilde{\lambda}\mathrm{v}_2) = -P + \frac{1}{H_T}D$$





Khouider and Majda, 2006; 2008: Simple models for convectively coupled waves

$$H_c = \sigma_c \frac{\bar{\alpha}\alpha_c}{H_m}\sqrt{CAPE_l^+}$$

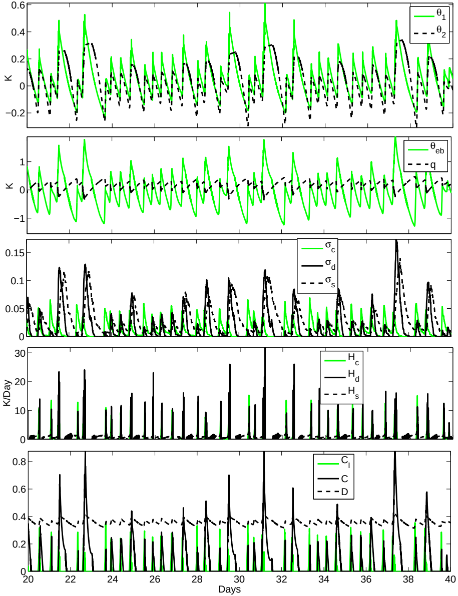
$$H_d = [\bar{\sigma}_d \bar{Q} + \frac{1}{\tau_c(\sigma_d)}(a_1\theta_{eb} + a_2 q - a_0(\theta_1 + \gamma_2\theta_2))]^+$$

$$\tau_c(\sigma_d) = \frac{\bar{\sigma}_d}{\sigma_d}\tau_c^0$$

$$H_s = \alpha_s[\bar{\sigma}_s \bar{Q} + \frac{1}{\tau_c(\sigma_s)}(a_1\theta_{eb} + a_2 q - a_0(\theta_1 + \gamma_2\theta_2))]^+$$

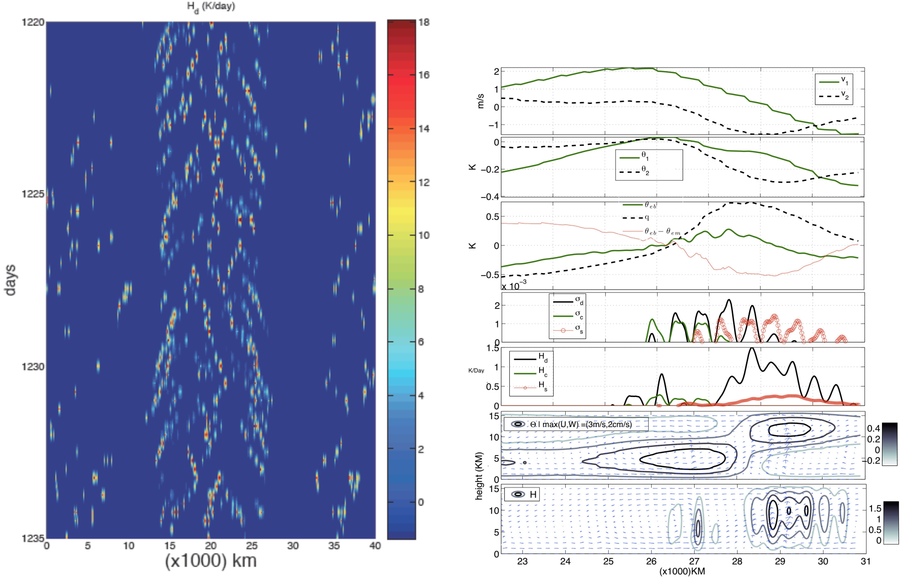
$$\tau_c(\sigma_s) = \frac{\bar{\sigma}_s}{\sigma_s}\tau_c^0$$

Single column run



Khouider et al. 2010.

Waves along the equator with warm pool forcing



Frenkel et al. 2013

SMCM with local interactions

Why local interactions?

- Emulate self-organization of cloud clusters at the meso-scale: MCS's Estern Pacific ITCZ; Coastal convection, etc.
- Represent unresolved effects of shear, sea-breeze, orography, etc. on organization of convection, turbulent mixing, radiation feedback, etc.
- Cloud-cloud interactions: can be both favourable and unfavourable to organization

The multi-type particle interacting system

- Multiple particle Hamiltonian

$$H(X,U) = \sum_{k=1}^{3}\sum_{l=k}^{3} E_{k,l}(X) + g_k(U)X$$

$$E_{k,l}(X) = -\sum_{i,j=1,i\neq j}^{N} J_{k,l}(|i-j|)\mathbb{1}_{\{X^i=k\}}\mathbb{1}_{\{X^j=l\}}, 1 \leq k \leq l \leq 3,$$

$$J_{k,l}(r) \neq 0 \iff r = 1 \text{ or } 0 < r \leq \sqrt{2},$$

- Choose Transition Rates to yield Gibbs equilibrium measure:

$$\begin{aligned} R_{02} + R_{01} &= \frac{\rho_1}{\rho_0} R_{10} e^{H_0 - H_1} + \frac{\rho_2}{\rho_0} R_{20} e^{H_0 - H_2} + \frac{\rho_3}{\rho_0} R_{30} e^{H_0 - H_3} \\ R_{01} &= \frac{\rho_1}{\rho_0}(R_{12} + R_{10}) e^{H_0 - H_1} \\ R_{02} &= \frac{\rho_2}{\rho_0}(R_{20} + R_{23}) e^{H_0 - H_2} - \frac{\rho_1}{\rho_0} R_{12} e^{H_0 - H_1} \\ R_{30} &= \frac{\rho_3}{\rho_2} R_{23} e^{H_2 - H_3}, \forall j = 1, \cdots, N, \forall X \in \Sigma. \end{aligned} \quad (16)$$

ρ is the "background" distribution counting for external potential, associated with background rates $\tilde{R}_{kl}(U)$. $H_k - H_j$ are local-interaction energy difference. Partial detailed balance.

- Equilibrium measure:

$$\mu(X_t) \propto e^{-H(X_t)}.$$

Coarse grained approximation

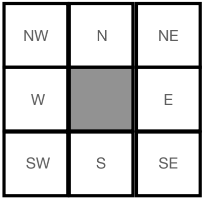
- Coarse-grained Hamiltonian as a conditional expectation:

$$\bar{H}(\bar{X}) = E[H(X)|\bar{X}] = \sum_{k=1}^{3}\sum_{l=1}^{3} E[E_{kl}(X)/\bar{X}] + \sum_{k=1}^{3} g_k(U) \sum_{i=1}^{M} N_k^i ,$$

$$E[E_{kl}(X)/\bar{X}] = -\sum_{k=1}^{3}\sum_{j=1}^{N}\sum_{l=1}^{3}\sum_{r=1,r\neq j}^{N} E\left[J_{kl}(|j-r|)\mathbb{1}_{\{X^j=k\}}\mathbb{1}_{\{X^j=l\}}/\bar{X}\right] .$$

- Uniform redistribution of particles, yields

$$E[E_{kl}(X)/\bar{X}] = -\frac{1}{Q^2}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{i=1}^{M} J_{kl}^0 \left[\left(n_b(q-2)^2 + 4(n_b - n_b')(q-2) + 4n_1 \right) N_l^i \right.$$
$$\left. + \left(n_b'(q-2) + n_2 \right)\left(N_l^{is} + N_l^{in} + N_l^{ie} + N_l^{iw} \right) + n_3 \left(N_l^{isw} + N_l^{ise} + N_l^{inw} + N_l^{ine} \right) \right] N_k^i ,$$



Retains interactions across coarse cells.

Mean field equations

In the limit of a large micro and macro lattice sizes we get

$$\begin{aligned}\frac{\partial \sigma_1}{\partial t} &= \alpha_{01}\sigma_0 - (\alpha_{10} + \alpha_{12})\sigma_1 \\ \frac{\partial \sigma_2}{\partial t} &= \alpha_{02}\sigma_0 + \alpha_{12}\sigma_1 - (\alpha_{20} + \alpha_{23})\sigma_2 \\ \frac{\partial \sigma_3}{\partial t} &= \alpha_{23}\sigma_2 - \alpha_{30}\sigma_3,\end{aligned} \tag{17}$$

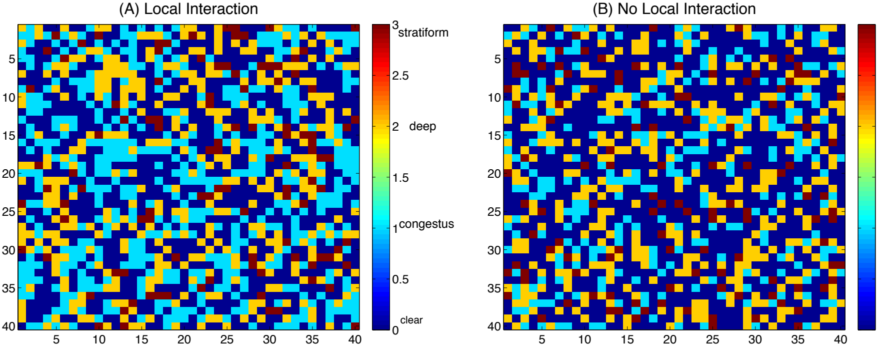
with

$$\begin{aligned}&\alpha_{k0} = \tilde{R}_{k0}, k = 1, 2, 3,\ \alpha_{01} = \tilde{R}_{01}e^{\Gamma_{10}},\ \alpha_{12} = \tilde{R}_{12}e^{\Gamma_{12}},\ \ \alpha_{23} = \tilde{R}_{23}e^{\Gamma_{23}}, \\ &\alpha_{02} = \frac{1}{\rho_0}(\rho_2\tilde{R}_{20} - \rho_1\tilde{R}_{12})e^{\Gamma_{20}} + \frac{\rho_3}{\rho_0}\tilde{R}_{30}e^{\Gamma_{30}}\end{aligned} \tag{18}$$

$$\Gamma_{0k}(x) = \sum_{l=1}^{3} J_{kl} * \sigma_l(x) \text{ and } \Gamma_{kl}(x) = \Gamma_{0l}(x) - \Gamma_{0k}(x), \tag{19}$$

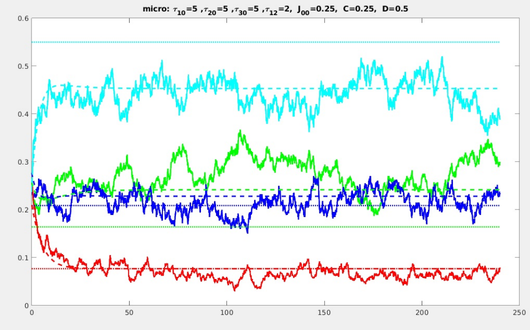
with $f * g(x) = \int f(x - y)g(y)\, dy$ is the convolution operation.

Numerical tests

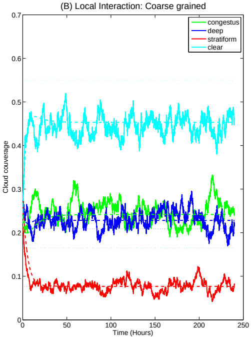


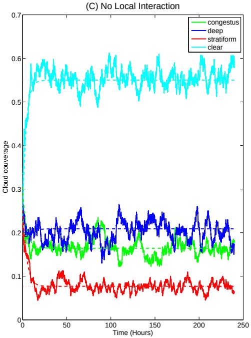
Snapshot of the multicloud microscopic lattice model. (a) With local interactions and (b) without local interactions. $n = 40$.

$$\text{Interaction matrix } J_0 = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.125 & 0.05 \\ 0 & 0.05 & 0.125 \end{bmatrix}$$



dash: Mean field
dots: background

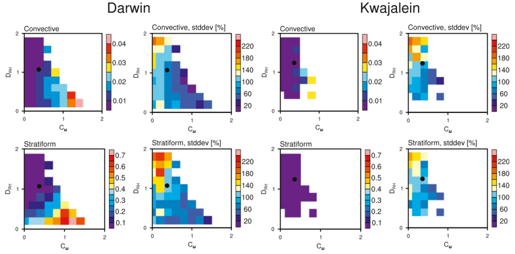




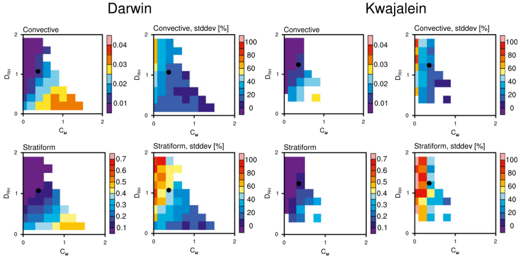
Parameter calibration using obs data

- Transition probabilities of cloud types are uncertain
- Use brute-force computations using LES data. (Dorrestijn et al. 2016)
- The formalism allows to reduce the task to learning key parameters: transition times and CAPE0, CIN0, reference values, etc.
- What is dequate cost function:
  - Cheap and dirty: Constraint the equilibrium distribution (Peters et al. 2017)
  - Precise but highly expensive
  - Midway pathways exists: Carlos Sevilla's thesis

Data from Darwin and Kwajalein



Radar data



SMCM

Regimes of forced v.s random convection: regions of dry and/low instability tend to display high CAF variance and low mean CAF

The Bayesian Inference Procedure

De La Chevrotiere et al. (2016; 2017)

- Find a distribution of the parameters

$$\boldsymbol{\theta} = (\tau_{01}, \tau_{10}, \tau_{12}, \tau_{02}, \tau_{23}, \tau_{20}, \tau_{30}).$$

given convective area fraction data and large scale predictors — time series:

$$\boldsymbol{x}_t, \boldsymbol{u}_t,\ 1 \leq t \leq T$$

Prior distribution: $\pi(\boldsymbol{\theta}) \longrightarrow$ posterior distribution:$\pi(\boldsymbol{\theta}|\boldsymbol{x}_t)$

Data information encoded model *likelihood:* $f(\boldsymbol{x}_t|\boldsymbol{\theta}; \boldsymbol{u}_t)$;

- Bayes's theorem:

$$\pi(\boldsymbol{\theta}|\boldsymbol{x}_t) = \frac{f(\boldsymbol{x}_t|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int f(\boldsymbol{x}_t|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}}. \tag{20}$$

Computing the likelihood

- Markov process

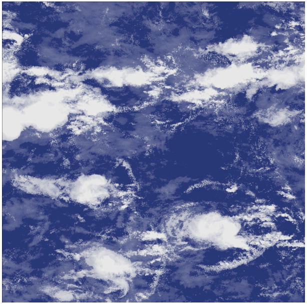
$$f(\boldsymbol{x}_{1:T}|\boldsymbol{u}_{1:T},\boldsymbol{\theta}) = \prod_{t=1}^{T} f_{t-1}(\boldsymbol{x}_t|\boldsymbol{x}_{t-1},\boldsymbol{u}_{t-1},\boldsymbol{\theta})$$
$$= \prod_{t=1}^{T} \mathbb{1}^{*}_{\phi(\boldsymbol{N}_c^{t-1},\boldsymbol{N}_d^{t-1},\boldsymbol{N}_s^{t-1})} \exp[R(\boldsymbol{u}_{t-1},\boldsymbol{\theta})\Delta t]\mathbb{1}_{\phi(\boldsymbol{N}_c^{t},\boldsymbol{N}_d^{t},\boldsymbol{N}_s^{t})} \quad (21)$$

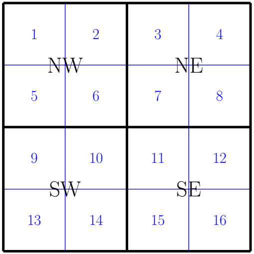
  ϕ is the injection of the state space $\mathcal{S} \subset \mathbb{R}^3$ into the integers.
- MCMC allows sampling of posterior without knowing it explicitly.
- Computational burden is in exponential matrix: $exp[R(\boldsymbol{u}_{t-1},\boldsymbol{\theta})\Delta t]$
- Uniformization technique, stable and accurate but still inefficient!
- Convergence of MCMC can also be slow
- Alternatives: Variational inference in lieu of MCMC;
- There are also way to approximate f ... sample statistics

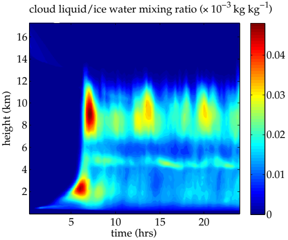
Giga LES data

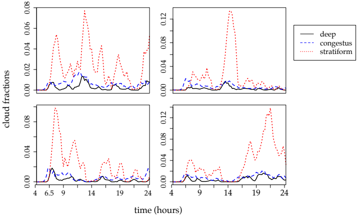
Khairoutdinov et al. (2001).

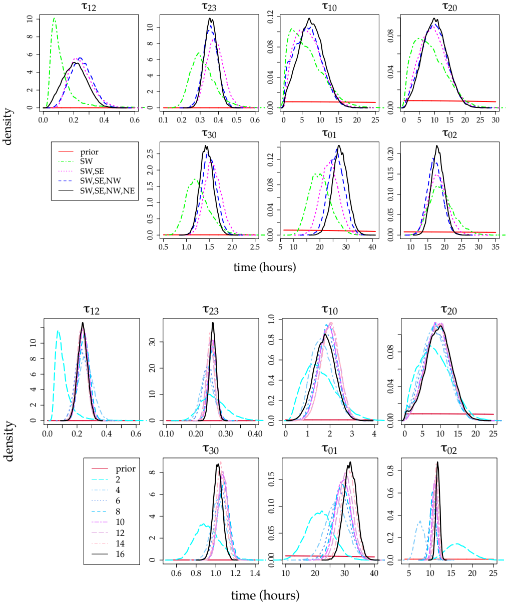
24 hr simulation 2048 × 2048 × 256 grid; Domain size: 204.8 × 204.8 km^2 × 27 km; data saved every 15 minutes.











Sequence of posterior distributions based on sequential Bayesian inference using a 2 $\times$ 2 (top set of panels) and 4 $\times$ 4 domain partitions (bottom panels)

Comparison of inferred time scales

Parameter	Mean (SD) [hours]		
	2×2 Partition	4×4 Partition	Peters et al.
τ_{01}	27.686 (8.233)	31.789 (4.795)	1
τ_{10}	7.426 (11.155)	1.761 (0.224)	1
τ_{12}	0.208 (0.006)	0.238 (0.001)	3
τ_{02}	17.950 (3.507)	11.821 (0.211)	4
τ_{23}	0.359 (0.001)	0.2570 (0.0001)	0.13
τ_{20}	10.126 (15.674)	9.551 (13.146)	5
τ_{30}	1.444 (0.021)	1.021 (0.002)	5