

Introduction to thermodynamics

Dry thermodynamics

Equation of state for dry air

$$p = \rho R_d T$$

$$R_d = \frac{R^*}{M_a} = 287 \frac{\text{J}}{\text{K kg}}$$

$$M_a = 28.97 \text{ g/mole}$$

First law of thermodynamics for infinitesimal transformation

$$c_v dT + p dv = dq$$

$$v = \frac{1}{\rho}$$

For a cyclic transformation

$$\oint c_v dT + \oint p dv = \oint dq$$
$$W = Q$$

Dry thermodynamics

First law of thermodynamics using T and p as state variables

$$c_v dT + d(pv) - \frac{1}{\rho} dp = dq$$

$$\underbrace{(c_v + R_d)}_{c_p} dT - \frac{1}{\rho} dp = dq$$

$$c_p dT - \frac{1}{\rho} dp = dq$$

$$pv = R_d T$$

$$c_v = \frac{5}{2} R_d = 717 \frac{\text{J}}{\text{kg}}$$

$$c_p = \frac{7}{2} R_d = 1005 \frac{\text{J}}{\text{kg}}$$

Dry thermodynamics

Conserved quantities

1. **Enthalpy**, which is conserved under an adiabatic and isobaric process.

$$h = u + pV$$

$$dh - \frac{1}{\rho} dp = dq$$

$$dh = 0$$

$$u = c_v T$$

$$dq = 0$$

$$dp = 0$$

$$h = c_p T$$

Enthalpy is similar to internal energy but accounts for work that a TD system (air parcel) can do on its environment.

Dry thermodynamics

2. Static energy, which is conserved under an adiabatic process in which pressure changes hydrostatically.

$$d(h + gz) = dq$$

$$s = c_p T + gz$$

$$i) \quad dq = 0$$

$$ds = 0$$

$$-\frac{1}{\rho} dp = g dz$$



Dry thermodynamics

3. Entropy, which is conserved under an adiabatic process.

For a reversible process

$$d\eta = \frac{dq}{T}$$

$$c_p \frac{dT}{T} - R_d \frac{dp}{p} = \frac{dq}{T} = d\eta$$

$$c_p d \ln T - R_d d \ln p = 0$$

$$T p^{-\frac{R_d}{c_p}} = \text{constant}$$

For an isentropic process

$$d\eta = 0$$

$$d \left[\ln \left(T p^{-\frac{R_d}{c_p}} \right) \right] = 0$$

Dry thermodynamics

This is the basis to derive a temperature variable, the **potential temperature**, that is the temperature a parcel with T and p would have if brought adiabatically to a reference pressure p_0 .

$$T p^{-\frac{R+1}{c_p}} = \vartheta p_0^{-\frac{R+1}{c_p}} \quad \vartheta = T \left(\frac{p_0}{p} \right)^{\frac{R+1}{c_p}} \quad p_0 = 1000 \text{ hPa}$$

$$z = c_p \ln \vartheta + \text{const}$$

First law allows us to derive the dry adiabatic lapse rate.

$$ds = c_p dT + g dz = 0$$

$$\left(\frac{dT}{dz} \right)_{s, \vartheta} = - \frac{g}{c_p} \quad \Gamma_d = - \left(\frac{dT}{dz} \right)_{s, \vartheta} = \frac{g}{c_p} = 9.81 \frac{\text{K}}{\text{km}}$$

Static stability

Starting from an atmosphere in hydrostatic equilibrium

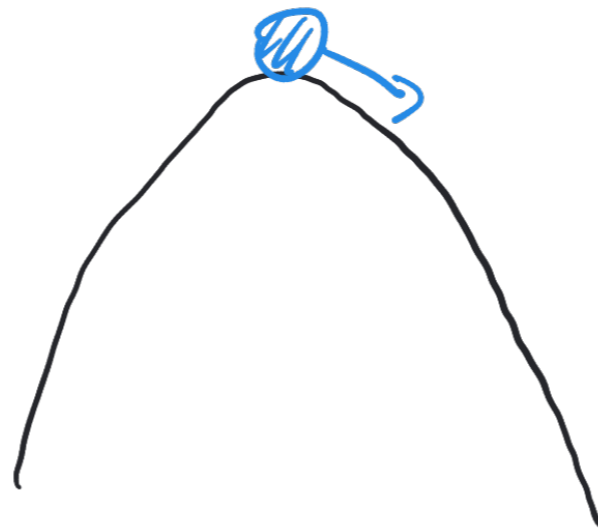
$$z = z_0$$



$$\uparrow \delta z$$



NEUTRAL



UNSTABLE



STABLE

Static stability

We are going to use the parcel method:

1. Parcel moves adiabatically (no mixing)
2. Pressure of parcel adjusts instantaneously to ambient pressure
3. Parcel does not disturb environment
4. Environment is in hydrostatic balance

T_E P_E S_E

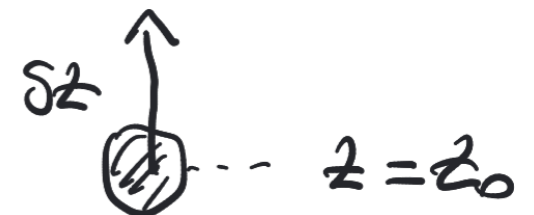
ENVIRONMENT

T_p P_p S_p

PARCEL

Condition 4 implies

$$-g - \frac{1}{S_E} \frac{dP_E}{dz} = 0$$



Static stability

$$\frac{dw}{dt} = -g - \frac{1}{\rho_p(z_0 + \delta z)} \frac{d\rho_p}{dz}$$

$$\rho_p(z_0 + \delta z) = \rho_e(z_0 + \delta z)$$

$$\frac{dw}{dt} = -g - \frac{1}{\rho_p} \frac{d\rho_e}{dz} = -g + g \frac{\rho_e}{\rho_p} = g \left(\frac{\rho_e - \rho_p}{\rho_p} \right)$$

Buoyancy force

B. F.	$\rho_p > \rho_e$	< 0	STABLE
	$\rho_p = \rho_e$	$= 0$	NEUTRAL
	$\rho_p < \rho_e$	> 0	UNSTABLE

Static stability

We want to derive conditions in terms of temperature and density profiles. We are going to do the following:

1. First order Taylor expansion of density around $z = 0$
2. Express density in terms of T and p (but pressure does not come into play, because p of the parcel adjusts to p of the environment)

Then, the vertical momentum equation becomes:

$$\frac{dw}{dt} = \frac{\rho' g}{\rho}$$

$$\left[\frac{d^2}{dt^2} + N^2 \right] \delta z = 0$$

N BRUNT - VAISALA
FREQUENCY

$$\delta z(t) = A \exp(iNt) + B \exp(-iNt)$$

if $N^2 > 0$

STABLE
OSCILLATORY SOLUTION

$$\gamma = \frac{2\pi}{N}$$

if $N^2 < 0$

UNSTABLE
EXPONENTIALLY GROWING SOLUTION

Static stability

$$N^2 = \frac{g}{T_0} \left[\frac{dT_e}{dz} - \frac{dT_p}{dz} \right] = \frac{g}{T_0} \left[\Gamma_e - \Gamma_p \right] =$$

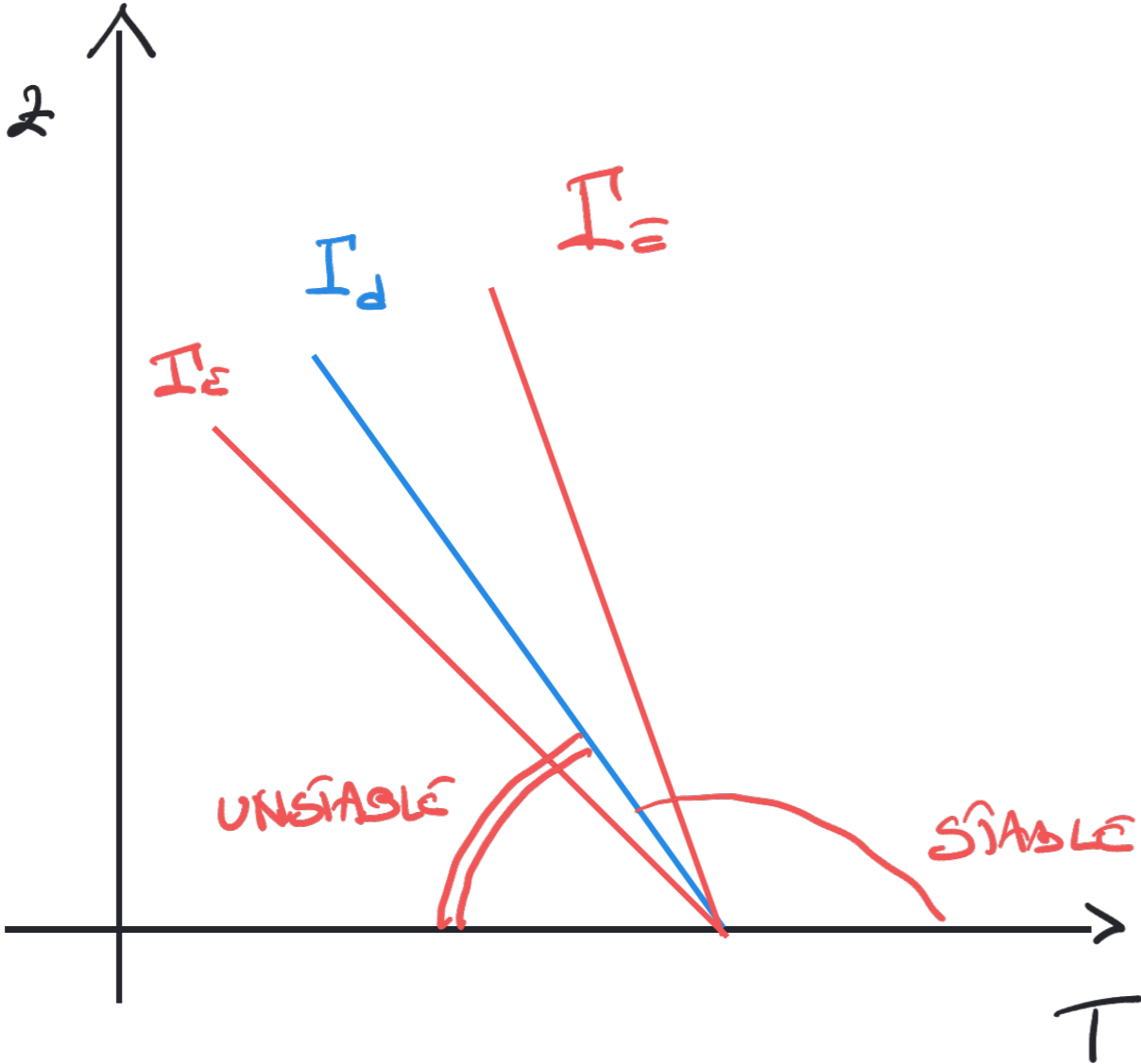
$$= \frac{g}{\vartheta_0} \frac{d\vartheta}{dz}$$

$$\Gamma_e = - \frac{dT_e}{dz}$$

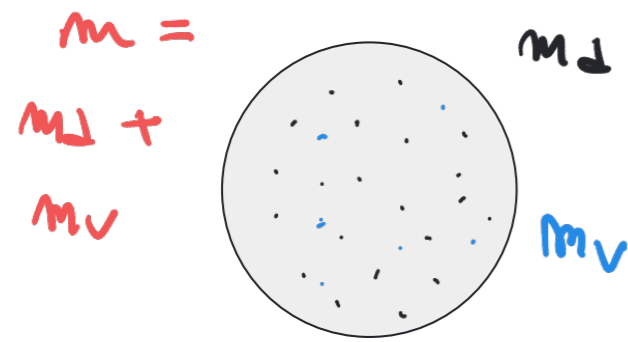
$$\Gamma_d = \Gamma_p = - \frac{dT_p}{dz}$$

$$N^2 \begin{cases} > 0 & \text{STABLE} \\ = 0 & \text{NEUTRAL} \\ < 0 & \text{UNSTABLE} \end{cases}$$

Static stability



Moist thermodynamics



$$p_d = \rho_d R_d T$$

$$e = \rho_v R_v T$$

$$R_v = 461.5 \frac{\text{J}}{\text{kg}}$$

Specific humidity

$$q_v = \frac{m_v}{m}$$

$$q_d = \frac{m_d}{m}$$

$$q_v + q_d = 1$$

Moist thermodynamics

Mixing ratio

$$r_v = \frac{m_v}{m_d}$$

$$q_v = \frac{r_v}{r_v + 1}$$

Moist thermodynamics

Equation of state for moist air

$$p = p_d + e = (s_d R_d + s_v R_v) T = \rho \left(\frac{s_d}{\rho} R_d + \frac{s_v}{\rho} R_v \right) T$$
$$= \rho (q_d R_d + q_v R_v) T$$

$$q_d = 1 - q_v$$

$$p = \rho \tilde{R} T$$

$$\tilde{R} = R_d \left[1 + \left(\frac{R_v}{R_d} - 1 \right) q_v \right]$$

$$\epsilon = \frac{R_d}{R_v} = 0.622$$

$$= R_d (1 + 0.61 q_v)$$

Moist thermodynamics

$$p = \rho R_d (1 + 0.61 q_v) T = \rho R_d T_v$$

$$T_v = T (1 + 0.61 q_v)$$

$$T_v > T$$

Virtual temperature is the temperature that dry air needs to be at to have same density at the same pressure as moist air.

Moist thermodynamics

Saturation

Pressure exerted by vapor at saturation with its liquid phase is just a function of temperature, through the Clausius-Clapeyron relation

$$e_s(T)$$

$$\frac{1}{e_s} \frac{de_s}{dT} \approx \frac{L_v}{R_v T^2}$$

$$L_v = 2.5 \cdot 10^6 \frac{\text{J}}{\text{kg}}$$

Useful approximation for typical atmospheric temperatures is

$$e_s(T) = A \exp(\beta T)$$

$$A = 6.11 \text{ hPa}$$

$$\beta = 0.067 \text{ } ^\circ\text{C}^{-1}$$

Moist thermodynamics

Saturation specific humidity

$$q_s \sim r_s \sim \epsilon \frac{e_s(T)}{p}$$

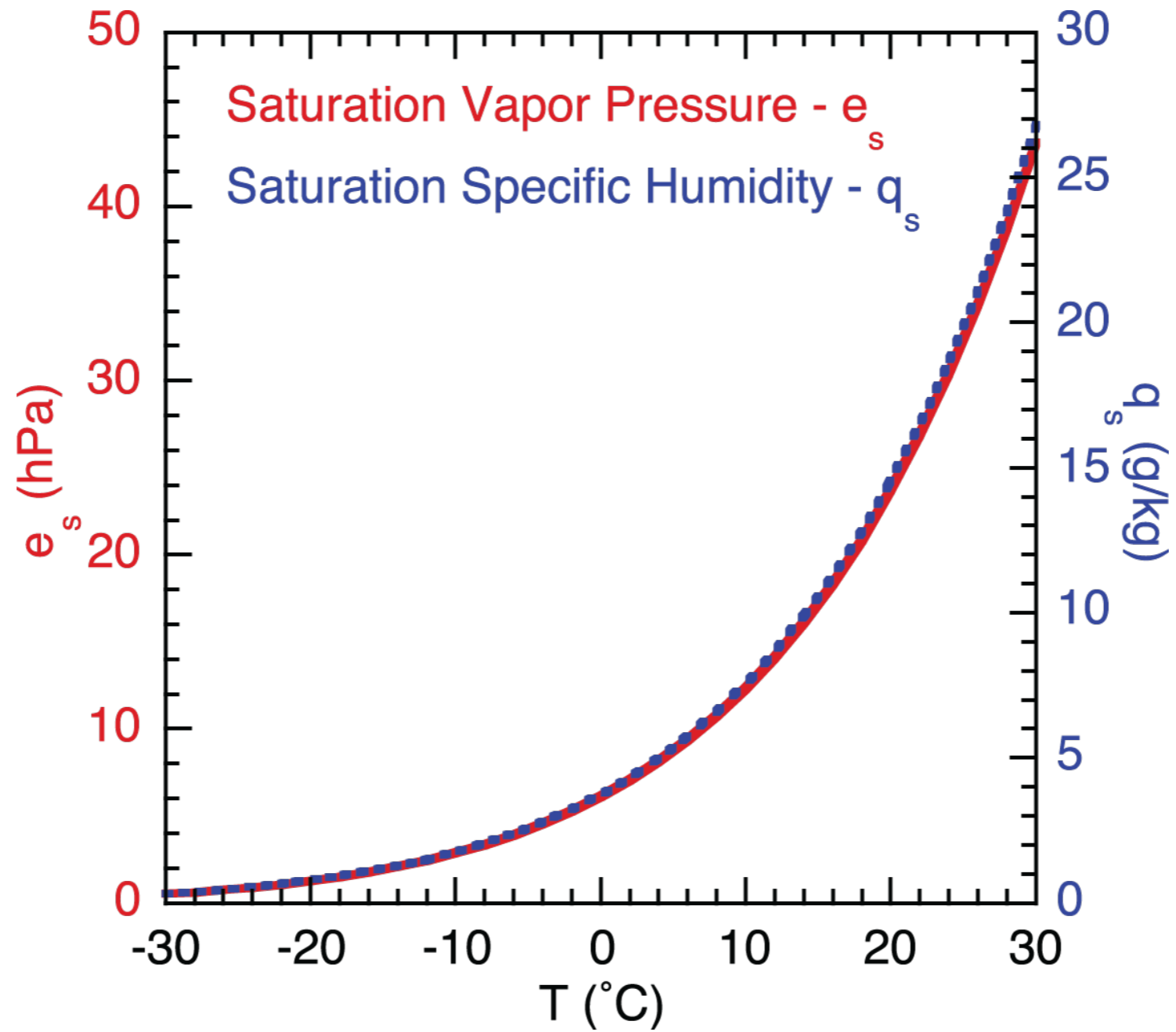
$$q_s(T, p)$$

And we can define relative humidity as:

$$RH = \frac{e}{e_s(T)} \%$$

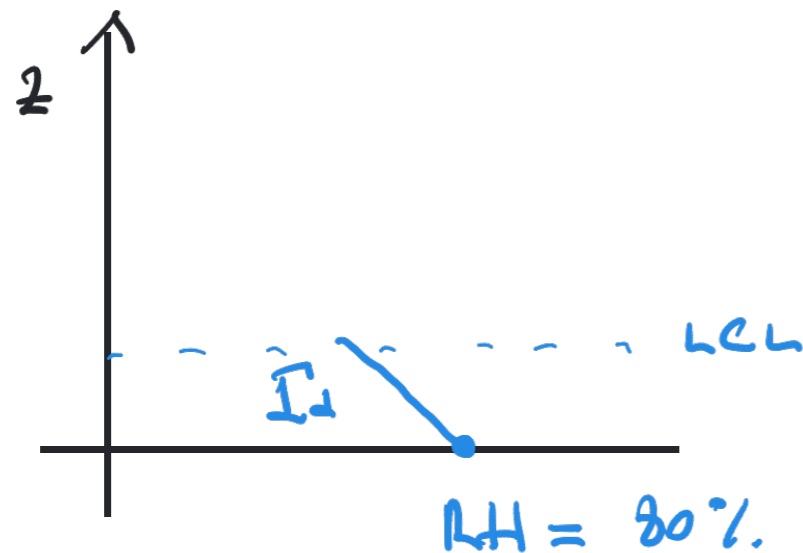
$$q_v \approx RH q_s(T, p)$$

Moist thermodynamics



Moist thermodynamics

What happens if we take an air parcel from the surface which is subsaturated (but not too far from saturation) and we lift it up?



q_v

$$q_v < q_s(T, P)$$

As the parcel goes up, it gets closer and closer to saturation until it reaches its lifted condensation level (LCL) where

$$q_v = q_s(T_{LCL}, P_{LCL})$$

As the parcel is raised beyond its LCL, the excess water vapor will have to condense out and the associated latent heat release will partially offset the adiabatic cooling. Hence the rate of T decrease in a saturated ascent is smaller than the dry adiabatic one,

Moist conserved quantities

We now want to define moist quantities that are conserved in specific transformations and are the moist analogue of conserved quantities in dry thermodynamics. To do this we are going to modify the first law to account for latent heat released or absorbed as water changes phase.

$$c_p dT - \frac{1}{\rho} dp = dq$$

$$dq = -L_v dq_w$$

$$c_p dT - \frac{1}{\rho} dp + L_v dq_w = 0$$

Moist conserved quantities

1. **Moist enthalpy**, which is conserved under an adiabatic and isobaric process

$$h_m = c_p T + L_v q_v$$

2. **Moist static energy**, which is conserved under an adiabatic process in which pressure changes hydrostatically

$$S_m = h = c_p T + gz + L_v q_v$$

Notice how these variables are defined just as the dry counterparts, but now latent heat due to water phase changes is considered another energy form within the parcel that participates to energy transformation.

Moist conserved quantities

- 3. Moist entropy**, which is conserved under an adiabatic process

This leads to the definition of the equivalent potential temperature, which is the temperature an air parcel would have if it were brought adiabatically to a state where all the water is in the liquid phase, and one can show that

$$\theta_e \approx \theta \exp\left(\frac{L_v q_v}{c_p T}\right)$$

$$\theta_e > \theta$$

Saturated lapse rate

As we did for the dry adiabatic lapse rate, we can define the lapse rate following a saturated ascent. This is not just a constant and depends on the total amount of water. By definition, moist static energy is conserved in a moist adiabatic ascent:

$$d(c_p T + gz + h_v q_w) = 0$$

And for a saturated ascent $dq_w = dq_s(T, p)$

$$dq_s = \left(\frac{\partial q_s}{\partial p} \right)_T dp + \left(\frac{\partial q_s}{\partial T} \right)_p dT \quad dp \rightarrow dz$$

$$\Gamma_s = - \left. \frac{dT}{dz} \right|_{\text{saturation}} = \frac{g}{c_p} \left[\frac{1 + \frac{h_v q_s(T, p)}{R_v T}}{1 + \frac{h_v^2 q_s}{c_p R_v T^2}} \right]$$

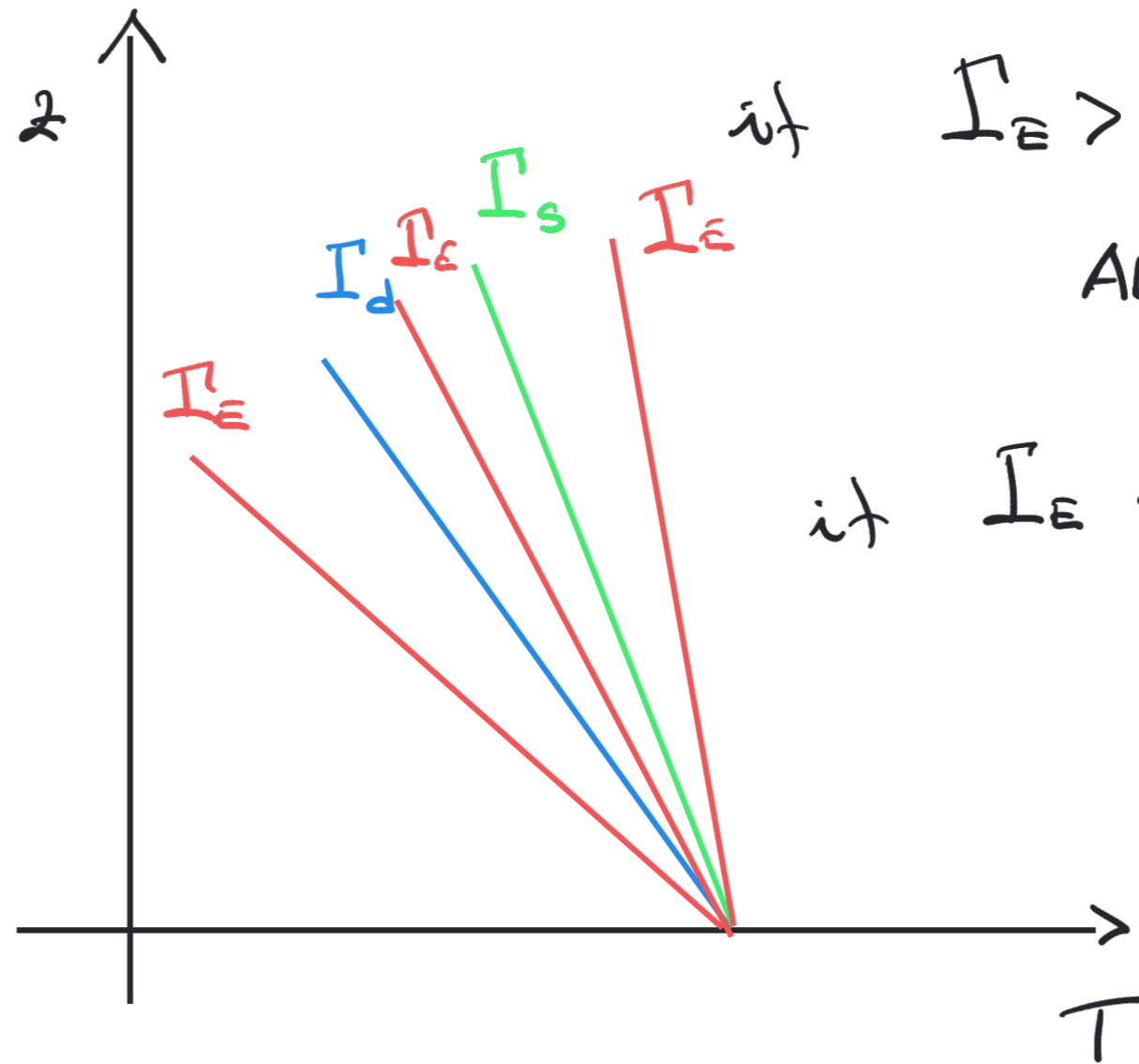
Saturated lapse rate

Two important points about the saturated lapse rate:

1. It is not a constant but changes with height
2. It is smaller than the dry adiabatic lapse rate

$$\Gamma_s \sim 6.5 - 7 \text{ K / km}$$

Conditional instability

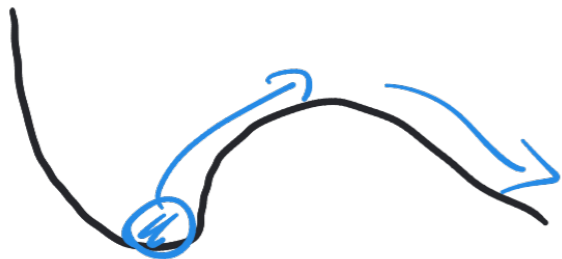
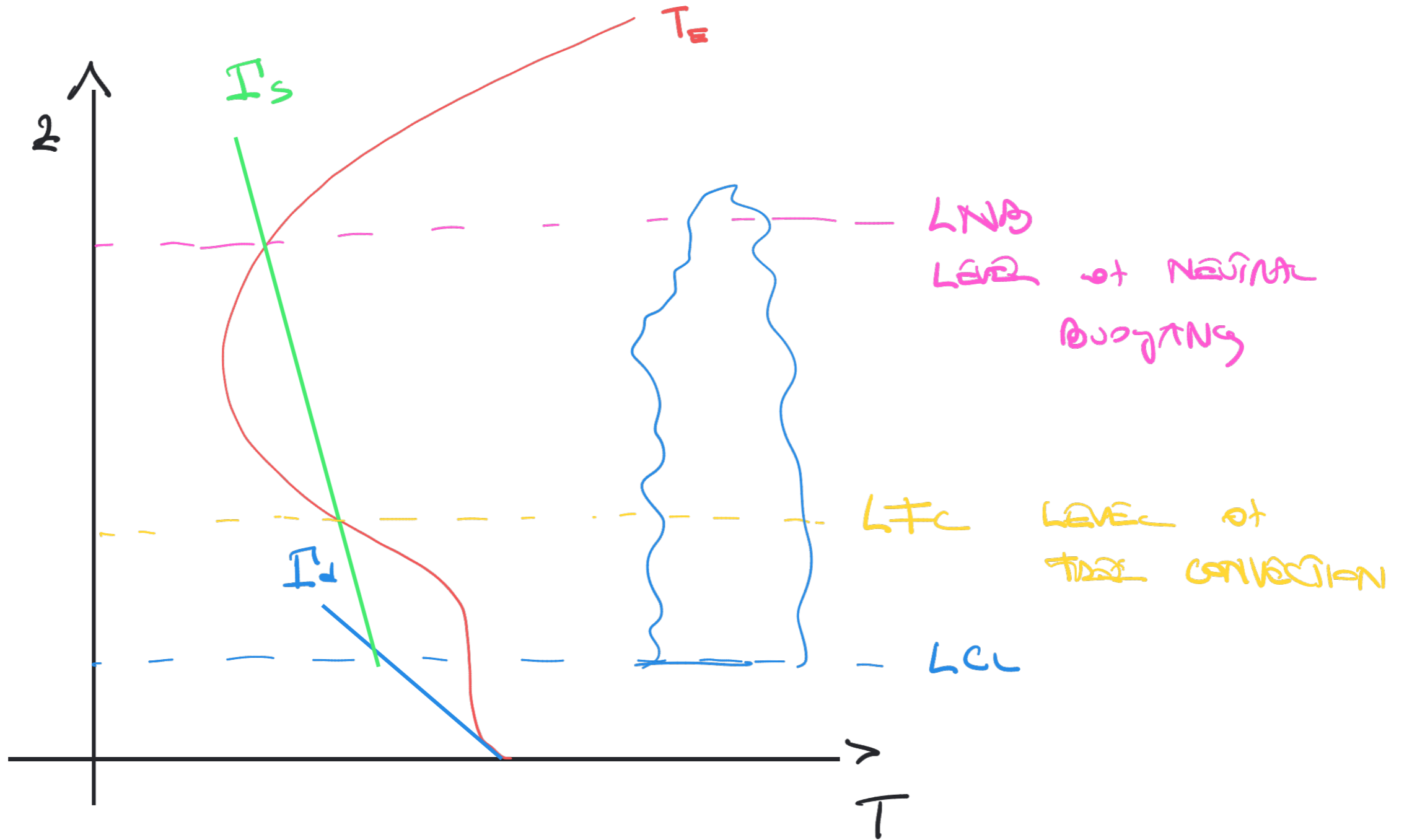


if $\Gamma_E > \Gamma_d > \Gamma_s$
 ABSOLUTE INSTABILITY

if $\Gamma_E < \Gamma_s - \Gamma_d$
 ABSOLUTE STABILITY

if $\Gamma_s < \Gamma_E < \Gamma_d$ CONDITIONAL INSTABILITY

Finite-amplitude instability



C-C scaling