# 1. Introduction and Results

**1.1 Context** These lectures live in the world of hyperbolic or "beyond hyperbolic" dynamics.

Dynamics "happens" in
- celestial mechanics
- number theory
- $\boxed{\text{geometry}}$

Dynamics as its own field is relatively recent ... 60's/70's saw explosive growth!

Examples and connections with other fields are central.

("compact for lectures 1, 2, 3")

**Setup** $X$ metric space (often a manifold)

Continuous (often smooth) flow $\mathcal{F} = (f_t) : X \to X$

(or discrete time: $f : X \to X$ and we consider $(f^n) : X \to X$)

Orbit $\quad \mathcal{O}_{\mathcal{F}}(x) = \{ f_t x : t \in \mathbb{R} \}$

Central question in dynamics: What do orbits do?

"With knowledge of present (i.e. with an initial condition) what can we say about distant future (i.e. asymptotic behavior)"

Answer: Usually probabilistic
↓ in terms of dynamically-invariant prob. measures

$\mathcal{M}_f(X) = \{ \text{prob. measures on } X \text{ s.t. } \mu(f_t A) = \mu(A) \ \forall t \in \mathbb{R}, \ \forall \text{ sets } A \}$

This perspective is: Ergodic Theory

My interest: Ergodic Theory in Geometry → Ergodic Geometry

("high complexity" / "chaotic" / "positive entropy" dynamics: system is deterministic but after a long time you "appear" random)

Defn $\mu \in M_F(X)$ is <u>ergodic</u> if every $F$-inv. set (i.e. $f_t A = A$ $\forall t$) satisfies $\mu(A) = 0$ or $\mu(A) = 1$.

<u>Ergodic</u> <u>Theorem</u>

For $\mu$ ergodic,

$$\boxed{\text{time average} = \text{space average}}$$

i.e. for $\varphi : X \to \mathbb{R}$ continuous

$$\frac{1}{t} \int_0^t \varphi(f_s x)\, ds \to \int \varphi \, d\mu$$

for $\mu$-a.e. $x \in X$.

# 1.2 Examples

(1) Symbolic spaces $X \subseteq \{0, \dots, k-1\}^{\mathbb{Z}}$ closed $\sigma$-invariant

or $X \subseteq \{0, \dots, k-1\}_{i=0}^{\infty}$

$\sigma : X \to X$

(2) Expanding maps

e.g. $x \to dx \pmod 1$ $d \geq 2$, $d \in \mathbb{N}$

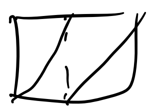
on interval or circle

(3) Hyperbolic maps $f$

e.g. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ acting on $\mathbb{T}^2$

uniformly expanding / contracting directions

(4) Non-uniform expansion or hyperbolicity $x \to x + x^{1+\alpha}$, $\alpha > 1$



$f(0) = 0$ $f'(0) = 1$

(5) Partial hyperbolicity:

Admits expanding, contracting and intermediate directions : ref Lima / Poletti

(6) Complex dynamics – rational maps, etc

ref: Lomonaco

7 Geodesic flow on negative curvature surfaces

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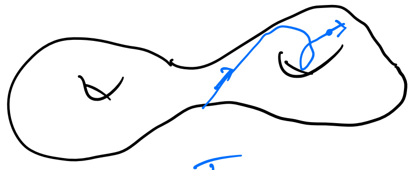


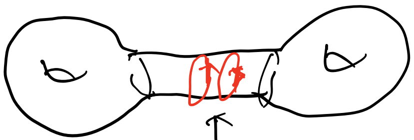
Turns out to be uniformly hyperbolic flow

surfaces with 2 or more holes have negative curvature

- CHAOTIC

Different journeys through holes

8 Geodesic flow on non-positive curvature manifolds



flat cylinder

9 Geodesic flows with weak curvature / structure assumptions

One of the top experts is Khadim War (IMPA)

PhD: ICTP

# 1.3 Entropy: The master invariant

A. Katok: "Entropy: the most glorious number in dynamics"

M. Pollicott: "Entropy - it's just a number"

Entropy comes in two flavors:

topological

$h_{top}(X, F)$, write $h$

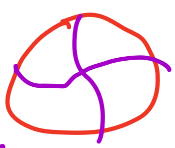
measure-theoretic

for each $\mu \in M_F(X)$

$h(\mu, X, F)$

write $h(\mu)$

e.g. chop space into 4 parts we can observe



$f_1 : X \to X$

At most $4$ length 1 orbits observed

$\leq 4^2$ length 2 orbits observed

measures exponential complexity of the orbit structure

It turns out that

$$h = \sup\{ h(\mu) : \mu \in M_F(X) \}$$

A measure achieving this sup is called measure of maximal entropy (MME)

For $f: X \to \mathbb{R}$ "weight" or "potential" function

can consider

$$P(f) := \sup\left( h(\mu) + \int f \, d\mu \right)$$

A measure achieving the sup is called equilibrium state (ES)

<u>Big</u> <u>questions</u>

Identify the "important" invariant measures

An MME is a "special" measure

$\boxed{\text{Does MME exist? Is it unique?}}$

— BREAK

- If $Vol \in M_F(X)$ "volume preserving"

When does $Vol = MME$?

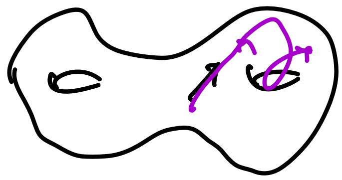
Characterize Vol dynamically when $Vol \neq MME$? Is Vol an ES?

- Get detailed statistical info on the system from MME/ES theory

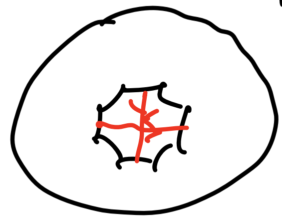
- $\boxed{\text{Can entropy theory "see" the geometry?}}$

## 1.4 Ergodic theory in negative curvature

$X$ compact negative curvature surface



$\mathbb{H}^2$



- Prototype of uniformly hyperbolic flow
  "Anosov flow"
- Theorem $\exists!$ MME (70's Bowen)
  (Dynamical techniques: "symbolic or specification approach")
- Patterson-Sullivan theory:
  The MME can be constructed "geometrically" (via the boundary at infinity)

- Volume entropy $h_{vol}$: natural / simple quantity

<u>Theorem</u> $h_{vol} = h$

- the <u>Liouville</u> <u>measure</u> $\mu_L$ is the natural "volume" measure for the geodesic flow

consider $\{ g : (M, g)$ is a neg. curv. surface, $\mathrm{vol}^g(M) = 1 \}$

write $\begin{cases} h^{top}(g) = h & \text{for geodesic flow for } (M, g) \\ h^L(g) = h(\mu_L^g) \end{cases}$

<u>Theorem</u> (Katok, entropy rigidity)

$\mu_L^g = \mu_{MME}^g$ iff the surface $(M, g)$ has <u>constant</u> negative curvature

(this is major open problem in higher dimensions)

Follows from Katok's theorems:

1) $h^{top}(g)$ is minimized at constant curvature

2) $h^{L}(g)$ is maximized at constant curvature "$\mu_L$ is dumb" - Katok

- Non-compact $M$ ← Need geometric approach

<u>The</u>orem (Otal-Peigné)

For a manifold $M$ (non-compact)

with $\underbrace{-b^2 \leq \text{curv} \leq -a^2 < 0}_{\text{"Pinched negative"}}$

the geodesic flow has either 0 or 1 MME