Exercises

1. Let $\alpha \in \mathbb{R}$, and let $f: S^1 \to S^1$, $f(x) = x + \alpha$. Observe that $\mu = \text{Leb}$ is invariant. Prove that μ is ergodic iff $\alpha \notin \mathbb{Q}$.

2. Let $f = $ CAT MAP, and note that $\mu = \text{Leb}$ is invariant. Prove that μ is ergodic.

Hint for 1 and 2: use Fourier series

(★) 3. Let $f: \mathbb{T}^3 \to \mathbb{T}^3$, $f(x,t) = (g(x), t+\alpha)$, where $g = $ CAT MAP and $\alpha \notin \mathbb{Q}$. Prove that $\mu = \text{Leb}$ is ergodic.

4. Let $f: \mathbb{T}^2 \to \mathbb{T}^2$, $f(x,y) = (x+\alpha, x+y)$. Calculate the Lyapunov exponents of f.

5. Let $\varphi_t : M \to M$ be a flow generated by a vector field X. Let $f = \varphi_1$ (time-one map). Prove that $\lambda(f, X(x)) = 0, \forall x \in M$.

(★) 6. Find an example of an Anosov diffeo f for which there exits $v \in TM$ s.t.

$$\varliminf_{n\to\infty} \frac{1}{n} \log \|df^n v\| < \varlimsup_{n\to\infty} \frac{1}{n} \log \|df^n v\|.$$

7. Let $f = $ CAT MAP $\times$ Id on $\mathbb{T}^3$. Prove that f is PH.

8. Let $E^s \oplus E^c \oplus E^u$ be df-invariant decomposition. Show that (3) in the definition of PH is equivalent to

(3') $\exists N > 0$ s.t. $\forall v^{s/c/u} \in E^{s/c/u}$ unitary:

$$2\|df^N v^s\| \leq \|df^N v^c\| \leq \frac{1}{2}\|df^N v^u\|.$$