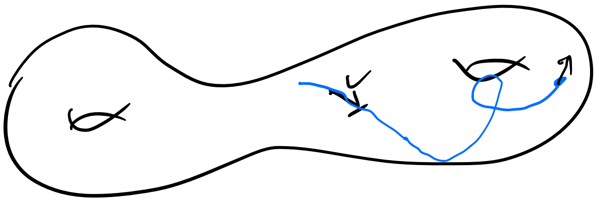
Geodesic flow

Given M manifold, $$F=(g_t): \overset{\substack{\text{unit tangent}\\ \text{bundle}\\ \downarrow}}{SM} \to SM$$
$$v \to f_t v$$
is given by moving at unit speed along geodesic determined by v.

A geodesic is a curve which (locally) minimizes distance



$v \in SM \leftrightarrow$ geodesic $c_v : \mathbb{R} \to X$ with

$\dot{c}_v(0) = v$

can thus identify SM with space of geodesics

$GM = \{ c : \mathbb{R} \to M : c \text{ is local isometry} \}$

Can also think of $(g_t) : GM \to GM$

$g_t c$ given by $(g_t c)(s) = c(s+t)$

d is Riem. distance on M

Metric on SM:

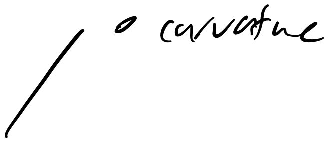
$d(v,w) = \max\{ d(g_s v, g_s w) : s \in [0,1] \}$

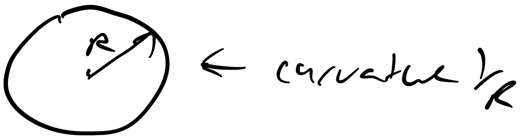
Curvature

Consider a surface

Gauss curvature defined using curvature of curves

↑ 2nd derivative

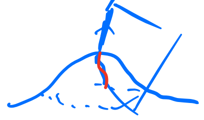




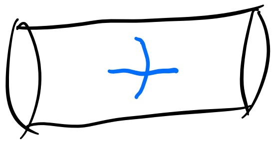
On surface at x: take smallest and largest signed curvatures through x, κ_1 and κ_2

Define $K(x) = \kappa_1 \kappa_2$

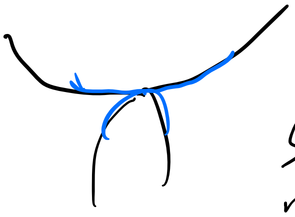
embedded in $\mathbb{R}^3$
intersect with plane
containing normal
direction. Get a curve.
compute curvature



Sphere $K(x) > 0$



cylinder
$K(x) = 0$



Saddle/
mountain pass $K(x) < 0$

Surfaces: completely classified by genus g
"number of holes"

Gauss-Bonnet: $\int K(x)\, d\mathrm{Vol}(x) = -4\pi(g-1)$

$\therefore$ Only genus $g \geq 2$ can have $K(x) < 0$
$\forall x$

Constant -ve curvature Model is $\mathbb{H}^2$

$\{x + iy,\ y > 0,\ x \in \mathbb{R}\}$

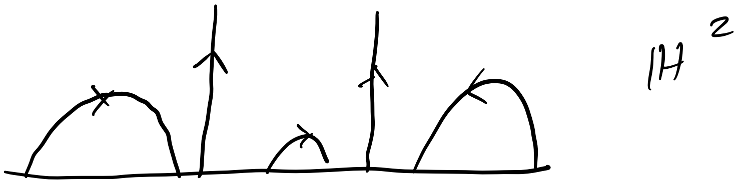
with metric $(ds)^2 = \frac{1}{y^2}(dx^2 + dy^2)$

Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, $z \to \dfrac{az+b}{cz+d}$ is an isometry

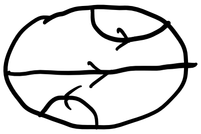
(This is all orientation-preserving isometries
Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ define same isometry
isometry group is $PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) / \pm Id$)

Knowing the isometries allows us to find the geodesics

- circles centred on real axis
- vertical lines



Projecting to disc model:



Boundary at ∞. $\partial^\infty \mathbb{H}^2$ is equivalence classes of geodesic rays which stay a bounded distance from each other. Topologically, the boundary is S^1.

Space of geodesics on $\mathbb{H}^2$ can be identified with

$$\partial^\infty \mathbb{H}^2 \times \partial^\infty \mathbb{H}^2 \times \mathbb{R}$$

where you came from

where you're going

where $t=0$ is

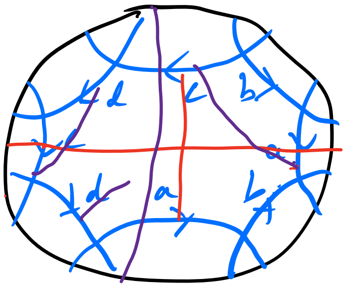
$c \in G\mathbb{H}^2$

Notation:

$c(x, \xi)$ is unique geodesic with $c(0)=x$, $c(\infty)=\xi$

$c(\eta, x)$ is unique geodesic with $c(-\infty)=\eta$, $c(0)=x$

To get compact constant -ve curvature manifold we quotient by discrete group of isometries



← gives genus 2

4g-gon gives genus g

Conversely every compact constant negative curvature surface M is $\mathbb{H}^2/\Gamma$

Γ Fuchsian group (cocompact)

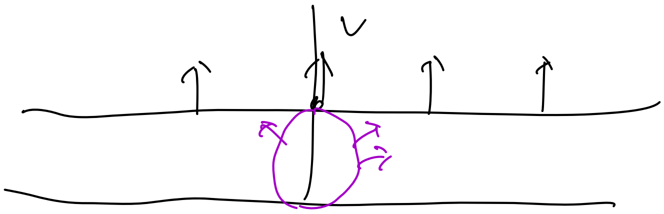
stable sets on $S\mathbb{H}^2$: For $v \in S\mathbb{H}^2$, consider

$$W^{ss}(v) = \{ w \in S\mathbb{H}^2 \text{ with } d(f_t v, f_t w) \to 0 \text{ as } t \to \infty \}$$

(i.e. the stable manifold)

For $c \in G\mathbb{H}^2$, consider $W^{ss}(c) = \{ c' \in G\mathbb{H}^2 \text{ with } d(c(t), c'(t)) \to 0 \}$
$W^{ss}(v)$ and $W^{ss}(c)$ are identified

Compute explicitly for vertical vector at $(0,1)$:



$W^{ss}(v)$ is normal vector field over horizontal line

$W^{uu}(v)$ " " over circle

Apply isometries to get all other

$W^s(v)$ and $W^{uu}(v)$

Get vector fields over horizontal lines and circles tangent to x-axis

In disc model, we get vector field over spheres tangent to $\partial \mathbb{H}^2$

(We can define $W^s(v)$ and $W^u(v)$ this way)

Define $E^s_v = T_v W^{ss}(v)$

$E^u_v = T_v W^{uu}(v)$

$E^0_v =$ flow direction

$E^s \oplus E^u \oplus E^0$ will be Anosov splitting

Hence, Anosov flow.

(Need to know exponential contraction / expansion ... soon...)

Haar measure on $PSL(2, \mathbb{R})$ descends to natural volume measure

- Liouville measure on SM

(f_t) is ergodic (Artin, Hopf)

Variable negative curvature

(M, g) M surface genus ≥ 2
is topologically $\mathbb{H}^2/\Gamma$
$\tilde{M}$ is universal cover: $\mathbb{H}^2$ equipped with metric $\tilde{g}$ which projects to g

$\partial\tilde{M}$ is still S^1

Hamiltonian flow, so natural volume on SM preserved by $F = \{f_t\}$

- called Liouville measure
- locally it is $\mathrm{Vol} \times \mathrm{Leb}|_{S^1}$

If (M, g) has -ve curvature, it is Anosov

$\Rightarrow$ 2 approaches

- geometric argument like before ↗ Need Busemann functions to describe W^s

OR

- Study Jacobi equation
  Get Anosov property by verifying cone condition

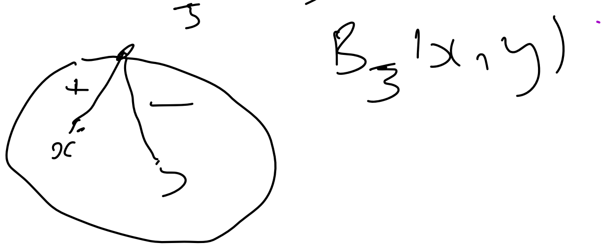
Anosov: ('60s) Volume-preserving Anosov diffeos/flows are ergodic (w.r.t. Liouville measure)

- Hopf argument
- Absolute continuity of foliations was the breakthrough

Busemann functions

- measures relative distance to a point at infinity

  i.e. how "out of phase" you are with a reference point traveling to infinity



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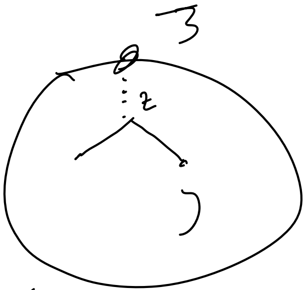
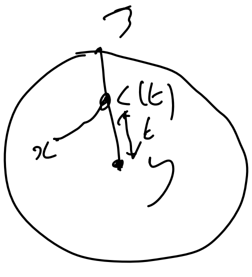
Define $$\boxed{B_\xi(x,y) = \lim_{z_t \to \xi} \left(d(x,z_t) - d(y,z_t) \right)}$$

↑ Any $z \to \xi$



More often we think of y fixed and let $z \to \xi$ along $c(y,\xi)$



so

$$B_\xi(x,y) = \lim_{t\to\infty} \left(d(x,c(t)) - t \right)$$

where $c(t) = c(y,\xi)(t)$

(Let's think of y as a fixed origin "o")

We often write $B_{\underline{o},\xi}(x) = B_\xi(x,\underline{o})$.

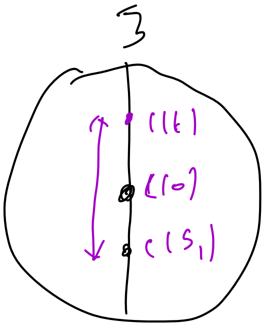
$$B_{\underline{o},\xi}(\cdot) : \tilde{X} \to \mathbb{R}$$

$$x \mapsto \lim_{t\to\infty} \left(d(x, c(t)) - t\right)$$

where $c = c(\underline{o},\xi)$

Properties:

1. For $c = c(\underline{o},\xi)$, $B_{\underline{o},\xi}(c(s)) = -s$



$s_1 < 0$: $d(c(s_1), c(t)) - t = t + |s_1| - t = -s_1$

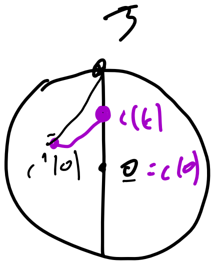
$s_2 \geq 0$, for t large

$$d(\underline{o}, c(t)) - t = (t - s_2) - t = -s_2$$

" $B_{\underline{o},\xi}(x) \begin{cases} > 0 & \text{when } x \text{ is "behind" } \underline{o} \\ < 0 & \text{"ahead" } \underline{o} \end{cases}$ "

2. let $c = c(\underline{0}, \xi)$ and let $c' \in Cm$ with $d(c(t), c(t)) \to 0$ as $t \to \infty$. Then $c'(\infty) = \xi$, by definition.

and $B_{\underline{0},\xi}(c'(0)) = 0$



Check: $B_{\underline{0},\xi}(c'(0)) = \lim \left(d(c'(0), c(t)) - t \right)$

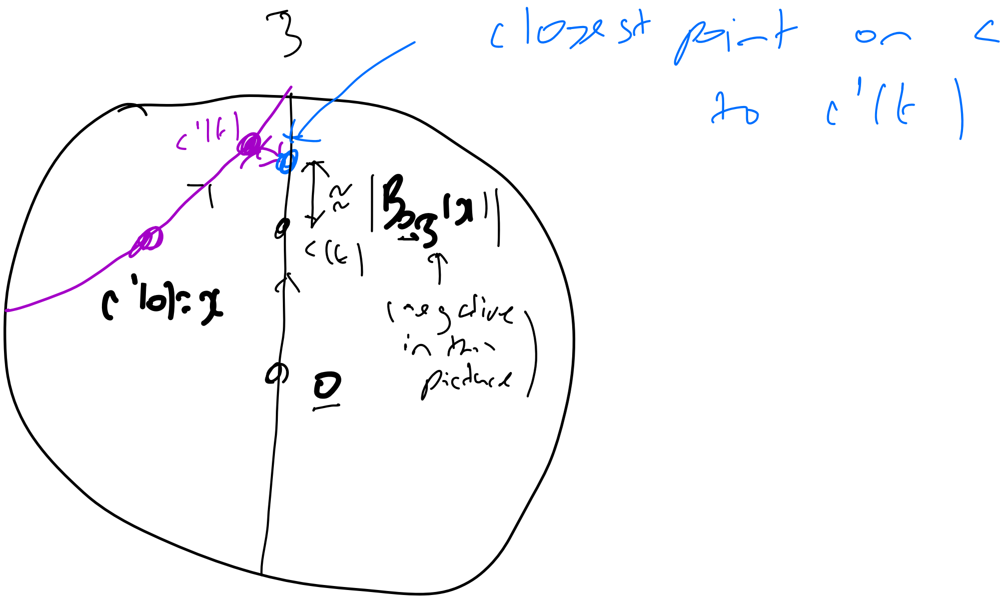
$$\leq \lim \; \underset{\overset{\shortparallel}{t}}{d(c'(0), c'(t))} + d(c'(t), c(t)) - t$$

$$= \lim \; d(c'(t), c(t)) = 0$$

Thus $B_{\underline{0},\xi}(c'(0)) = 0$

3. For $x \in X$, $B_{\underline{0},\xi}(x)$ can be thought of as taking the geodesic $c' = c'(x, \xi)$ and flowing towards ξ. Eventually the closest distance between c and c' becomes negligible. $B_{\underline{0},\xi}(x)$ is what is left over. It's like a signed distance in flow direction

i.e. $B_{\underline{0},\xi}(x)$ is (asymptotically) how much $\boxed{\text{time lag}}$ $c'(x, \xi)$ has compared with $c(0, \xi)$



- $B_{\underline{0},\xi}(\cdot)$ is Lipschitz and convex (see wiki - Buseman functions)

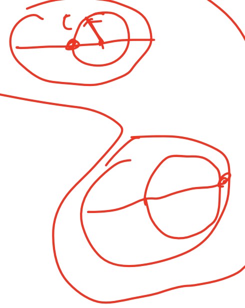
Buseman functions let us *define* $W^{ss}(c)$

For a point p and $\xi \in \partial X$

define $\boxed{H_{p,\xi} = \{ x : B_{p,\xi}(x) = 0 \}}$.

$H_{p,\xi}$ is the *horosphere* for ξ at p.

Alternate construct. take ball radius t around $c(t)$. Take limit



"horosphere" means "sphere touching infinity". This construction shows this

The (strong) stable set of geodesics for $c = c(p, \xi)$ is

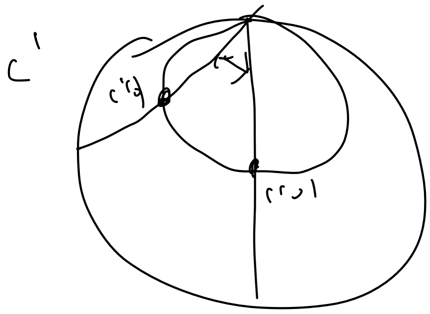
$$W^{ss}(c) := \{ c' : c'(\infty) = \xi ,\ c'(0) \in H_{p,\xi} \}$$

In neg. curv., this agrees with dynamical defn. $\{ c' : d(c(t), c'(t)) \to 0 \}$

This is because:

["Really requires neg. curv., not true in non-pos. curv. Can have flat strip with $B_\xi(\cdot) = 0$"]

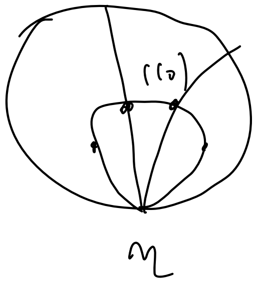
Geometry fact: In neg. curv., two asymptotic rays converge exponentially in nearest distance



Thus for $c' \in W^{ss}(c)$, dist. between geodesics $\to 0$ and "time lag" is 0

The unstable set of geodesics for $c(\eta, 0)$ is

$$W^{uu}(c) = \{ c' : c'(-\infty) = \eta ,\ c(0) \in H_{p,\eta} \}$$



For $v \in SH^2$, we thus constructed $W^{uu}(v)$ and $W^{ss}(v)$ on which distance is expanded resp. contracted | exponentially

This construction upstairs projects down to SM

Gives $T^1 M = E^s \oplus E^u \oplus E^0$

$$\uparrow \boxed{\text{Anosov flow}}$$

Qn: What did we gain by Buseman function definition of $W^{ss}(v)$?

Ans: regularity

and explicit characterization we can understand by further analysis of Buseman functions

— The definition

$$W^{ss}(x) = \{y : d(f_t x, f_t y) \to 0\}$$

can always be made in any setting!! But it may be useless. Why is $W^{ss}(x) \neq \emptyset$? Why does it have any structure?

- The Busemann functions have nice regularity and hence so does $H_{p,\xi}$ and hence so does $W^{ss}(v)$ defined using $H_{p,\xi}$

- Lipschitz regularity of $B_{\xi,0}(x)$ is easy in x

- Hölder regularity as ξ varies is also elementary (if a little harder)

  [Corresponds to Hölder regularity of $x \to E^s_x$ in hyperbolic dynamics]

- For a non-positive curvature manifold, the Busemann functions and hence $H_{p,\xi}$ are regular (geometry gives that each $W^{ss}(v)$

is at least a C^2 manifold)

↑ smoothness was needed to define

$$E^s_v = T W^{ss}(v) \text{, etc.}$$

- **Note** Another advantage of the geometry approach is W^{ss}, W^{uu} are defined for some interesting settings <u>beyond</u> the manifold case

e.g. CAT(-1) spaces

↑ flow is no longer Anosov in the conventional smooth sense but there is enough

Hölder regularity to study it dynamically from a metric point of view.