Exercises

1. Let $f: M \to M$ and $g: N \to N$ be C^1 diffeomorphisms s.t.:

   (1) f is Anosov with decomposition $TM = E^s \oplus E^u$ and $\lambda \in (0,1)$ s.t.

   $$\begin{aligned} \|df^n v^s\| &\le \lambda^n \|v^s\|, \quad \forall v^s \in E^s \\ \|df^{-n} v^u\| &\le \lambda^n \|v^u\| \quad \forall v^u \in E^u \end{aligned} \quad , \forall n \geq 0$$

   (2) For all $v^{s/u} \in E^{s/u}$ unitary and $w \in TN$ unitary:

   $$\lambda^{-n} \|df^n v^s\| \le \|dg^n w\| \le \lambda^n \|df^n v^u\| , \quad \forall n \geq 0.$$

   Prove that $f \times g : M \times N \to M \times N$ is partially hyperbolic.

2. Let $f: M \to M$ be C^1 and $g_x : N \to N$, $x \in M$, a family of diffeos s.t.:

   (1) f is Anosov with decomposition $TM = E^s \oplus E^u$ and $\lambda \in (0,1)$ s.t.

   $$\begin{aligned} \|df^n v^s\| &\le \lambda^n \|v^s\|, \quad \forall v^s \in E^s \\ \|df^{-n} v^u\| &\le \lambda^n \|v^u\| \quad \forall v^u \in E^u \end{aligned} \quad , \forall n \geq 0$$

   (2) For all $v^{s/u} \in E_x^{s/u}$ unitary and $w \in T_y N$ unitary:

   $$\lambda^{-n} \|df_x^n v^s\| \le \|(dg_x^n)_y w\| \le \lambda^n \|df_x^n v^u\| , \quad \forall n \geq 0.$$

   Prove that $F : M \times N \to M \times N$ is partially hyperbolic.

   $$(x, y) \mapsto (f(x), g_x(y))$$

3. Let $g:\mathbb{T}^2\to\mathbb{T}^2$ be the CAT MAP and $f:\mathbb{T}^3\to\mathbb{T}^3$ be

$$f(x,t)=(g(x),t+\varphi(x))$$

where $\varphi: S^1\to\mathbb{R}$ is C^1. For $y\in W^s(g,x)$, let

$$\theta_x^s(y)=\sum_{n\geq 0}\varphi(f^n(x))-\varphi(f^n(y)).$$

Prove that

$$W^s(f,(x,t))=\{(y,t+\theta_x^s(y)): y\in W^s(g,x)\}.$$

Conclude an analogous result for the unstable direction.

4. Let $\mathcal{P}$ be a measurable partition and μ a probability measure. If $\{\mu_{\mathcal{P}(x)}\}_{x\in X}$ and $\{\nu_{\mathcal{P}(x)}\}_{x\in X}$ are both disintegrations of μ with respect to μ, then $\mu_{\mathcal{P}(x)}=\nu_{\mathcal{P}(x)}$ for μ-a.e. $x\in X$.

5. (Fubini for disintegration) Let $\mathcal{P}$ be a measurable partition, μ prob. measure and $\{\mu_{\mathcal{P}(x)}\}_{x\in X}$ the disintegration of μ with respect to $\mathcal{P}$. Prove that for every $\varphi: X\to\mathbb{R}$ measurable it holds:

$$\int\varphi\,d\mu=\int\left(\int\varphi\,d\mu_{\mathcal{P}(x)}\right)d\mu.$$

6. Let $v=(v_1,v_2)\in\mathbb{R}^2$ s.t. $v_2/v_1\notin\mathbb{Q}$, and let $\mathcal{P}$ be the partition of $\mathbb{T}^2$ s.t. $\mathcal{P}(x)=\{x+tv: v\in\mathbb{R}\}\subset\mathbb{T}^2$ is the line through x

in the direction of v.

(1) Prove that $\mathcal{P}$ is <u>NOT</u> measurable.

(2) Conclude that for the CAT MAP the partition into stable (or unstable) manifolds is NOT measurable.