4.1 Entropy theory for geodesic flows: compact case

DRAFT 7/24/24

(M, g) compact Riemannian manifold

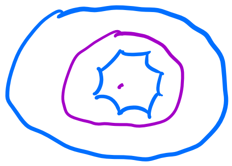
$h_{top}(g)$ is topological entropy of geodesic flow w.r.t. the metric g (i.e. of (f_t) on SM)
$\quad f_t = f_t^g$

Volume entropy for (M, g)

Define $\quad h_{vol}(g) = \lim_{R \to \infty} \frac{1}{R} \log \mathrm{Vol}(B(x, R))$

ball in universal cover $(\tilde{M}, \tilde{g})$

metric lifted from g



Thm (Manning) If curvature is negative (actually ≤ 0)

$$h_{top}(g) = h_{vol}(g)$$

Growth rate of closed geodesics

$$h^*(g) = \overline{\lim_{T \to \infty}} \frac{1}{T} \log \left(\begin{array}{c} \#\text{ closed geodesics} \\ \text{of length } \le T \end{array} \right)$$

<u>Thm</u> $\quad h^*(g) = h_{top}(g) \qquad$ [T, T+δ]

μ_L^g - Liouville measure for g (on S^gM)

unit tangent bundle w.r.t. g

V^g = total area of M (volume)

m^g = <u>normalized</u> Riemannian volume on M to be a prob. measure

i.e. $m^g = \frac{\mathrm{Vol}^g}{V^g}$ on M

Note for constant -ve curvature K Euler characteristic

$$h(\mu_L^g) = \sqrt{-K} = \left(\frac{-2\pi\,\chi(M)}{v^g}\right)^{1/2}$$

$$\overset{\shortparallel}{h_{top}(g)}$$

(Most natural as corollary that Anosov flows have unique MME: dynamical arguments)

Thm $\exists!$ MME μ_{BM}

Thm μ_L^g is ergodic

Proof Hopf argument ...

Also follows from more advanced result $\exists\, f : SM \to \mathbb{R}$ s.t. μ_L^g is the unique ES for f

(Katok)

Theorem (M, g) closed Riem. surface, Suppose $g_0 = \rho g$ where g_0 has constant -ve curvature with $v^{g_0} = v^g$

This happens for any g when genus ≥ 2 by uniformization)

Then $h_{top}(g) \geq h_{top}(g_0)$

$$\shortparallel \left(-\frac{2\pi\,\chi(M)}{v^g}\right)^{1/2}$$

with equality iff $\rho \equiv 1$

i.e. $g \to h_{top}(g)$ is minimized at constant neg. curvature

Comparing metrics

g_1, g_2 Riem. metrics

Definition

$$[g_1 ; g_2] = \int_{v \in S^{g_1}M} \|v\|_{g_2} \, d\mu_L^{g_1}(v)$$

Lemma 1 $[g_1 ; \rho g_1] = \int_M \rho^{1/2} \, dm^{g_1}$

[Recall: locally $\mu_L^g = m^g \times$ (Haar on S^1)]

Proof. The key ingredient in the computation is:

$$\|v\| = \sqrt{g(v,v)}$$

[Recall footpoint map $\pi : SM \to M$, $v = (x, \theta) \mapsto x$]

so $$\|v\|_{\rho g_1} = \sqrt{\rho(\pi v)\, g_1(v,v)} = \sqrt{\rho(\pi v)}\, \|v\|_{g_1} \quad \blacksquare$$

Details: Exercise

Lemma 2 If $g_2 = \rho g_1$ and g_1, g_2 have same volume

a) $[g_1 ; g_2] = [g_2 ; g_1]$

b) $\int_M \rho^{1/2} \, dm^{g_1} \le 1$ with equality if and only if $\rho \equiv 1$

$= [g_1 ; g_2]$ by Lemma 1

[If $\dim M = n$ get $\rho^{n/2}$]

Proof a) If $g_2 = \rho g_1$, then $\mathrm{Vol}_{g_2} = \rho \, \mathrm{Vol}_{g_1}$

so $$dm^{g_2} = \frac{V^{g_1}}{V^{g_2}} \rho \, dm^{g_1}$$

So if $V^{g_2} = V^{g_1}$, then
$$[g_1 ; g_2] = \int_M \rho^{1/2} \, dm^{g_1} = \int_M \rho^{-1/2} \, dm^{g_2} = [g_2 ; g_1]$$

b) $\int_M \rho \, dm^{g_1} = \int_M I \, dm^{g_2} = 1$

since m is prob. measure

So by Cauchy-Schwartz (or Jenssen) $\left(\int_M \rho^{1/2} dm^{g_1}\right)^2 \leq 1$

with equality iff $\rho = 1$ ∎

$\left[\int \rho^{1/2} \cdot 1 \, dm\right]^2$

$\leq \int \rho \, dm \int 1 \, dm$

only when one is scalar multiple of other

Combining Lemma 1 and 2

$$\boxed{[g_1 ; g_2] = [g_2 ; g_1] \leq 1 \text{ with equality iff } \rho = 1}$$

<u>Proof</u> It suffices to show $h_{top}(g) \geq h_{top}(g_0)$

where $g_0 = \rho g$, $V^{g_0} = V^{g}$, g_0 constant neg. curv.

For large T, consider $Per^{g_0}(T)$

We know $\frac{1}{T} \log \# Per^{g_0}(T) \to h_{top}(g_0)$

We also know that $\mu_L^{g_0}$ is the unique MME for $(f_t^{g_0})$. For $\varphi \in C(SM)$ let $\varepsilon > 0$ and define

$$= \int_0^{L(\gamma)} \varphi(f_s^{g_0} \dot\gamma(0))\,ds$$

$$Per^{g_0}(t, \varepsilon, \varphi) = \left\{ \gamma \in Per^{g_0}(t) : \left| \frac{\int \varphi \, d\mu_\gamma}{L^{g_0}(\gamma)} - \int \varphi \, d\mu_L^{g_0} \right| < \varepsilon \right\}$$

<u>Claim</u>: $\dfrac{\# Per^{g_0}(t, \varepsilon, \varphi)}{\# Per(t)} \to 1$ as $t \to \infty$

<u>Sketch proof</u>. If not, there are "enough" (i.e. $\geq \delta \# Per(t)$) γ's <u>not</u> in $Per^{g_0}(t, \varepsilon, \varphi)$ to construct an MME m. By construction $\int \varphi \, dm \neq \int \varphi \, d\mu_L^{g_0}$. By uniqueness of the MME, this is a contradiction ∎

Now assume T is large enough so that

$$\# Per(T, \varepsilon, \rho^{-1/2}) > (1-\varepsilon) \# Per(T)$$

(where $g_0 = \rho g$)

Note that $\int_{S^{g_0}M} \rho^{-1/2}\, d\mu_L^{g_0} = \int_M \rho^{-1/2}\, dm^{g_0}$

$$= \int_M \rho^{1/2}\, dm^{g}$$

$$\overset{\text{Lecture 28 Lemma 1}}{=} [g, \rho g] = [g, g_0]$$

Note that $L^g(\gamma) = \int_0^{L^{g_0}(\gamma)} \| f_s^{g_0} \dot\gamma(0) \|_g \, ds$

$$= \int \sqrt{\rho^{-1} g_0(\cdot,\cdot)}\, ds$$

$$= \int_0^{L^{g_0}(\gamma)} \rho^{-1/2}\, ds$$

We thus have $\boxed{L^g(\gamma) = L^{g_0}(\gamma) \cdot \left(\text{average of } \rho^{-1/2} \text{ along } \gamma\right)}$

For $\gamma \in Per^{g_0}(T, \varepsilon, \rho^{-1/2})$

then $L^g(\gamma) \le T([g, g_0] + \varepsilon)$

Replace each such γ by g-shortest curve in its homotopy class. This gives an injective map from $Per^{g_0}(T,\varepsilon,\rho^{-1/2})$ to $Per^g(T([g,g_0]+\varepsilon))$

Thus

$$\# Per^g(T([g,g_0]+\varepsilon)) \ge \# Per^{g_0}(T,\varepsilon,\rho^{-1/2})$$

$$\ge (1-\varepsilon) \# Per^{g_0}(t)$$

It follows that

$$h^*(g) \ge [g, g_0]^{-1} h^*(g_0)$$

Recall $[g, g_0] \le 1$ with equality iff $g = g_0$

Thus $h^*(g) \ge h^*(g_0)$ with equality iff $\rho = 1$ ■

Digression

Key result for $h(\mu_L^{g}) \le h(\mu_L^{g_0})$

If $g_2 = \rho g_1$ both with neg. curv. Then

$$\boxed{h^*(g_2) \geq \frac{h(\mu_L^{g_1})}{[g_1; g_2]}}$$

[Equality is free when $\rho = 1$ (constant neg. curvature]

Proof of Katok's thm given Key result:

Let $g_0 = \rho g_1$ with same volume V and g_0 has constant neg. curv.

Then $\left(\frac{-2\pi\chi(M)}{V}\right)^{1/2} = h_{top}(g_0) = h^*(g_0)$

$$\overset{\text{Key result}}{\geq} \frac{h(\mu_L^{g_1})}{[g_1; g_0]}$$

By lemma 2, $[g_0; g_1] = [g_1; g_2] \le 1$

Thus $h(\mu_L^{g_1}) \le \left(\frac{-2\pi\chi(M)}{V}\right)^{1/2} [g_1; g_0]$

$$\le \left(\frac{-2\pi\chi(M)}{V}\right)^{1/2}$$

Equality holds only if $[g_1; g_0] = 1 \iff \rho = 1$ ■

Time permitting

4.2 The non-compact world

Setting: upstairs

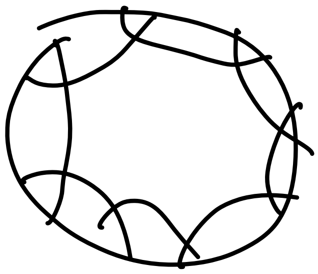
Let X be a simply connected complete manifold with dimension ≥ 2 and pinched negative curvature.

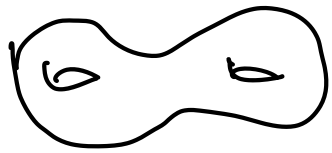
Defn Pinched curvature if $\exists\, a, b > 0$ so that
$$-b^2 \leq K \leq -a^2 < 0.$$
where K is any sectional curvature

Let Γ be a non-elementary Kleinian group (i.e. discrete group of isometries)

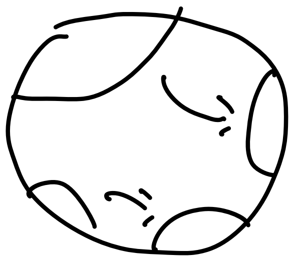
Let $X_0 = X/\Gamma$

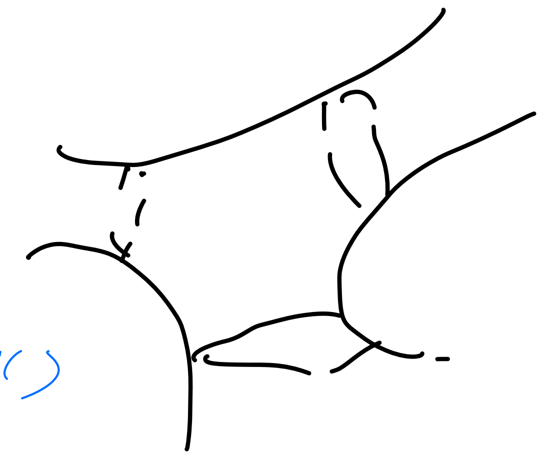
Eg. 1: Hyperbolic octagon





Eg 2. Schottky surface

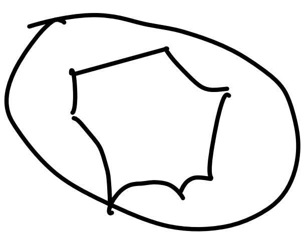


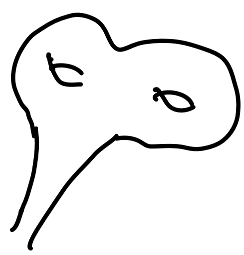


Consider only geodesics which don't escape up a funnel. This is a Cantor set

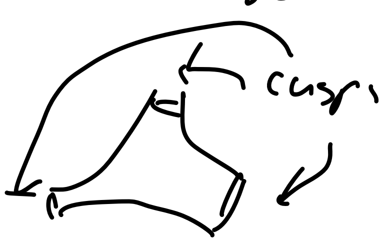
funnels

e.g. 3 A surface with cusps





Can also get



or

e.g. 4 Cusps & funnel's in same system

e.g. 5. Higher dims, cusps can have topology

e.g. 6. Allow the curvature to vary in all above examples

Define $h^{VP}(F) = \sup\{h_\mu : \mu \in M_F(T^1 X_0)\}$ and MME as μ with $h_\mu = h^{VP}(F)$

<u>Theorem</u> (Otal-Peigné's Principe Variationnel)

(X_0, F) has a unique MME or there is no MME

Trichotomy $X_0 = \tilde{X}/\Gamma$ is $\left\{\begin{array}{l} \text{positive recurrent} \nearrow \exists \text{ finite Gibbs measure it's unique MME} \\ \text{null recurrent} \Rightarrow \exists \text{ infinite Gibbs measure} \\ \text{transient} \searrow \text{best you can do is a totally dissipative infinite measure} \end{array}\right.$

Defns. PR : Schapira-Pit (Defn '19)

Also exists <u>Strong</u> PR (Defn '22)

The rest:.. Dilsavor-T.. are coining this terminology in this setting! (TBA)

Defn: Gibbs measure

$$\mu(B_n(x,\varepsilon)) \in [C^{-1}, C]\, e^{-nh} \quad \forall x$$

for all

This trichotomy mirrors a better known trichotomy in the theory of countable state SFT's (Sarig, etc.)