Eigenstate correlations in spatially extended chaotic many-body quantum systems

John Chalker

Physics Department, Oxford University

Based on:

Dominik Hahn, David Luitz & JTC, arXiv:2309.12982

Takato Yoshimura, Sam Garratt & JTC, arXiv:2312.14234

Related work: Amos Chan, Andrea De Luca & JTC, PRL (2019)

Outline

Single particle systems

Eigenstate correlations and dynamics in chaotic/disordered systems

Many-body systems

Characterising quantum dynamics in spatially extended systems

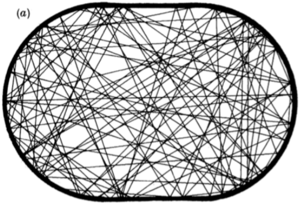
Implications for eigenstate correlations

Capturing correlations with Ansatz for joint eigenvector distribution

Quantum chaos in low-D systems

Semiclassical limit for chaotic systems

(a)



Berry's random wave conjecture:

wavefunctions
$\sim$ random superposition of plane waves

Model eigenfunctions as $\psi(\mathsf{r}) \propto \sum'_{\mathsf{k},|\mathsf{k}|=\mathsf{k}_\mathrm{F}} a_\mathsf{k} \cos(\mathsf{k}\cdot\mathsf{r}+\varphi_\mathsf{k})$

with random $a_\mathsf{k}, \varphi_\mathsf{k}$

Implies $[\psi(\mathsf{r}_1)\psi(\mathsf{r}_2)]_\mathrm{av} \propto J_0(k_\mathrm{F}|\mathsf{r}_1-\mathsf{r}_2|)$

Eigenstate correlations in spatially extended system

Disordered conductor

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathrm{r}) \quad \text{with} \quad H|n\rangle = E_n|n\rangle \quad \& \ V(\mathrm{r}) \text{ random}$$

Lowest order

$$[\langle \mathrm{r}_1|n\rangle\langle n|\mathrm{r}_2\rangle]_{\mathrm{av}} \sim J_0(|r_1 - r_2|)e^{-|\mathrm{r}_1-\mathrm{r}_2|/\ell} \qquad \text{short-range}$$

Lowest interesting order

$$[\langle \mathrm{r}_1|n\rangle\langle n|\mathrm{r}_2\rangle\langle \mathrm{r}_2|m\rangle\langle m|\mathrm{r}_1\rangle]_{\mathrm{av}} \quad \text{and} \quad [\langle \mathrm{r}_1|n\rangle\langle n|\mathrm{r}_1\rangle\langle \mathrm{r}_2|m\rangle\langle m|\mathrm{r}_2\rangle]_{\mathrm{av}}$$

long-range in metal (& gauge-invariant)

Relation to dynamics

Consider spreading wavepacket $\langle \mathrm{r}_2|e^{-iHt}|\mathrm{r}_1\rangle$

$$|\langle \mathrm{r}_2|e^{-iHt}|\mathrm{r}_1\rangle|^2 = \sum_{nm}\langle \mathrm{r}_2|n\rangle\langle n|\mathrm{r}_1\rangle\langle \mathrm{r}_2|m\rangle\langle m|\mathrm{r}_2\rangle e^{i(E_n-E_m)t}$$

$$\propto t^{-d/2}e^{-|\mathrm{r}_1-\mathrm{r}_2|^2/4Dt} \qquad \text{in diffusive conductor}$$

Many-body eigenstate correlations?

Analogue of Berry's random wave conjecture?

Semiclassical limit: dilute hard spheres, numerically inaccessible

E.g. for disordered spin chain in ergodic phase, couplings $\sim J$

$$H = J \sum_n \vec{\sigma}_n \cdot \vec{\sigma}_{n+1} + \sum_n \vec{h}_n \cdot \vec{\sigma}_n \qquad \vec{h}_n \text{ small \& random}$$

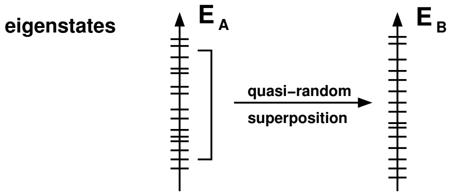
Eigenstate thermalisation hypothesis (ETH)

Deutsch (1991), Srednicki (1994), Rigol (2008) [also Peres *et al.* (1984)]

Picture:

Two similar systems A & B

Hamiltonians H_A & H_B



Eigenstate thermalisation hypothesis (ETH)

Local observable $O(x)$ e.g. spin operator at site x of spin chain

Matrix elements

$$\langle m|O(x)|n\rangle = \delta_{mn}\langle O(x)\rangle_E + \mathcal{N}^{-1/2} f(E,\omega) R_{mn}$$

$E = (E_m + E_n)/2$
$\omega = E_n - E_m$
$\mathcal{N} = e^S$ $S =$ entropy
$R_{mn} \sim$ random matrix

Generic state $|\psi\rangle = \sum_n c_n|n\rangle$

- Stat mech from diagonal elements

  $\langle\psi|O(x)|\psi\rangle$ fixed by $\langle\psi|H|\psi\rangle$

- Dynamics from off-diagonal elements

  $f(E,\omega)$ encodes $[O(x,t)O(x)]_{\mathrm{av}}$

Many-body correlations and dynamics in spatially extended systems?

Fock space spreading? fast process – limited information

Conserved densities

diffusively spreading or ballistically propagating modes

– reflected in $[O(x,t)O(x)]_{\mathrm{av}}$ & $f(E,\omega)$

Study Floquet systems

avoid all conserved densities (even energy)

retain fixed evolution operator

Dynamics at long times & distances

Operator spreading

$$O(x,t) = e^{iHt} O(x)\, e^{-iHt}$$

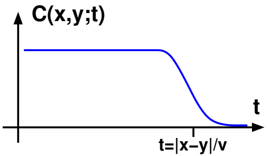
$O(x,t) =$ sum over strings of operators acting at many sites

How do lengths of operator strings grow with time?

probe via $|[O(y,t), O(x)]|^2$ and out-of-time order correlator (OTOC)

$$C(x,y;t) \equiv [O(y,t)O(x)O(y,t)O(x)]_{\mathrm{av}}$$

E.g. with $\mathrm{Tr}\, O(x) = 0$
and $O(x)^2 = \mathbb{1}$
& likewise for $O(y)$



butterfly velocity: v

OTOC implies correlations beyond ETH

Spectral decomposition

$$C(x,y,t) = \mathcal{N}^{-1} \sum_{abcd} \langle a|O(y)|b\rangle\langle b|O(x)|c\rangle\langle c|O(y)|d\rangle\langle d|O(x)|a\rangle e^{it(E_a - E_b + E_c - E_d)}$$

ETH (in its simplest form) suggests (wrongly!)

$C(x,y,t) \sim \mathcal{O}(\mathcal{N}^{-2})$ average $+\ \mathcal{O}(\mathcal{N}^{-1})$ fluctuations

Comparing single pcle and many-body cases

Equivalent roles played by

wavepacket spreading in single-particle systems

& operator spreading in many-body systems

Compare orders of correlators involved

using $e^{-iHt} \equiv \sum_n |n\rangle e^{-iE_n t} \langle n|$

Wavepacket $|\langle \mathrm{r}_2 | e^{-iHt} | \mathrm{r}_1 \rangle|^2$

4th order in $|n\rangle$ & $\langle n|$

OTOC $[O(y,t)O(x)O(y,t)O(x)]_{\mathrm{av}}$

8th order in $|n\rangle$ & $\langle n|$

Identifying minimal eigenfuction correlators – I

Simplify the OTOC

remove dependence on operator choice

– avge over complete sets of operators $\{O_j(x)\}$ & $\{O_k(y)\}$ acting at x & y

Outcome: scalars constructed from many-body eigenfunctions

Identifying minimal eigenfuction correlators – II

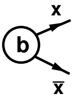
Direct approach to constructing scalars

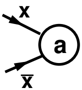
Schmidt decomposition of eigenfunction $|a\rangle$

subsystems x & $\overline{x}$ with $|a\rangle = \sum_{lm} a_{lm} |l\rangle_x \otimes |m\rangle_{\overline{x}}$

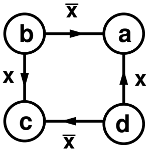
Central object: Schmidt matrix a_{lm} size $2^{\ell_x} \times 2^{\ell_{\overline{x}}}$

Pictorially

$\langle b| \equiv b^\dagger_{nk} =$ 

$|a\rangle \equiv a_{lm} =$ 

Combine e.g. $|a\rangle$, $|b\rangle$, $|c\rangle$ & $|d\rangle$ as $\mathrm{Tr}[ab^\dagger cd^\dagger] \equiv M_{abcd}(x)$



require invariance under $|a\rangle \to e^{i\alpha}|a\rangle$

must combine a & $a^\dagger$, b & $b^\dagger$ etc

hence (uniquely) $[M_{abcd}(x) M^*_{abcd}(y)]_{av}$

Relation to OTOC

Avge OTOC over complete sets of operators with fixed supports

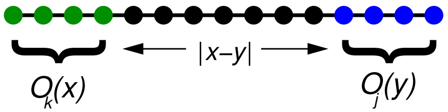
Notation:

$O_k(x)$ – kth operator from orthonormal set with support x.

Find

$$\sum_{jk}[O_j(y,t)O_k(x)O_j(y,t)O_k(x)]_{av}$$

$$\propto \sum_{abcd}[M_{abcd}(x)M^*_{abcd}(y)]_{av}e^{it(E_a-E_b+E_c-E_d)}$$



ETH & eigenstate correlations

ETH Ansatz

$$\langle a|O_k(x)|b\rangle = \delta_{ab}\langle O_k(x)\rangle_E + \mathcal{N}^{-1/2} f(E,\omega) R_{ab}$$

$f(E,\omega)$ encodes autocorrelation function

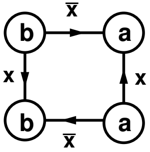
$$C(x,t) = \left[O_k(x,t)O_k(x)\right]_{\rm av} = \mathcal{N}^{-1}\sum_{ab}|\langle a|O(x)|b\rangle|^2 e^{i(E_a-E_b)t}$$

Take avge of autocorrelation function over operators $O_k(x)$ with fixed support x

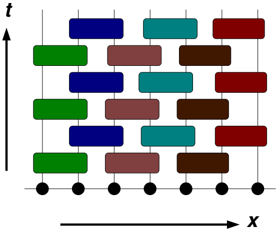
$$\sum_k \left[O_k(x,t)O_k(x)\right]_{\rm av} \propto \sum_{ab} M_{abba}(x) e^{i(E_a-E_b)t}$$

Pictorially

$M_{abba}(x) \equiv$



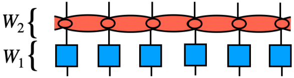
Calculations: Floquet quantum circuits



- $H \Rightarrow H(t)$ periodic in t
- No conserved densities
- Random gates $\Rightarrow$ ensemble of systems
- Choices: local Hilbert space dimension q
  + circuit architecture

Numerical results: $q = 2$ brickwork circuit with Haar gates

Analytical results: $q \to \infty$ for:



W_2 – nearest-neighbour coupling, strength ε

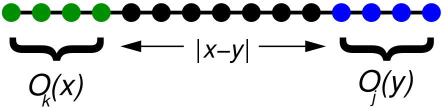
W_1 – single-site Haar scramblers

Expected behaviour

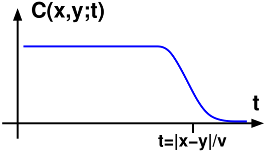
Operator-averaged OTOC

$$\sum_{jk} [O_j(y,t) O_k(x) O_j(y,t) O_k(x)]_{av}$$

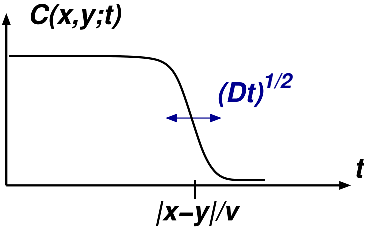
with



Recall form of OTOC



Results for solvable Floquet quantum circuit circuit



Curve: error function

Parameters:

velocity $v = 1 - \rho$

diffusion constant $D = \frac{1}{2}\rho(1-\rho)$

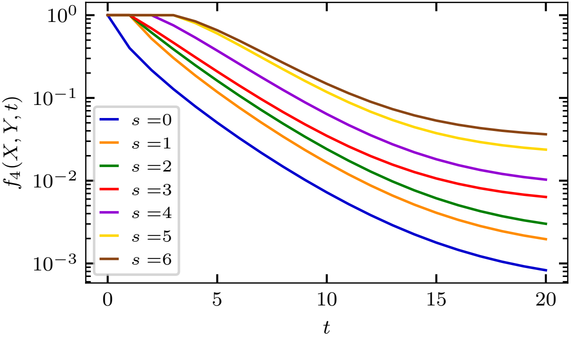
with $\rho = e^{-2\varepsilon}$ ε: nn coupling

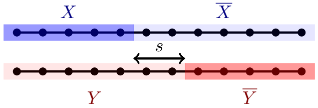
Same behaviour in RUCs (circuits random in time, not Floquet):

Nahum, Vijay & Haah (2018)

von Keyserlingk, Rakovszky, Pollmann & Sondhi (2018)

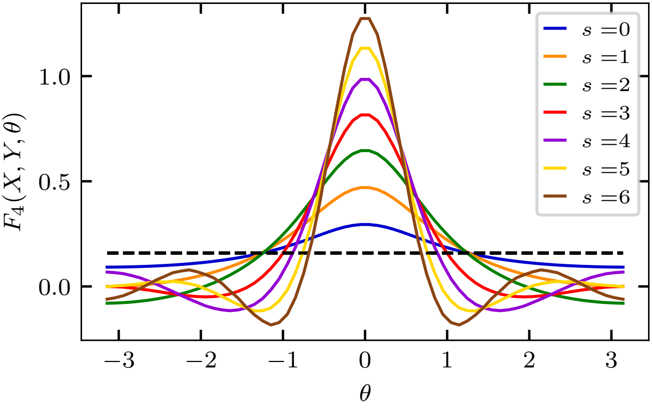
Exact diagonalisation: time domain





Operator-averaged OTOC $[O_j(y,t)O_k(x)O_j(y,t)O_k(x)]_{av}$ vs time t
for $s = 0, 1, \ldots 6$

Exact diagonalisation: energy domain



Normalised $[M_{abcd}(x)M^*_{abcd}(y)]_{av}$ vs $E = (E_a - E_b + E_c - E_d)$
for $s = 0, 1, \ldots 6$

Sharp structure in E for large s

Alternative to ETH

complementary to Free Probability theory, Pappalardi *et al.* (2022 & 2023)

Recall formulation of ETH

Statistical Ansatz for matrix elements

$$\langle a|O(x)|b\rangle = \delta_{ab}\langle O(x)\rangle_E + \mathcal{N}^{-1/2} f(E,\omega) R_{ab}$$

Replace with statistical Ansatz for joint eigenvector distribution

Individual many-body eigenvectors $\vec{a}$

Floquet $\Rightarrow \langle O(x)\rangle_E$ indept of $E \Rightarrow$ isotropic distribution for $\vec{a}$

Pairs of eigenvectors $\vec{a}$ and $\vec{b}$

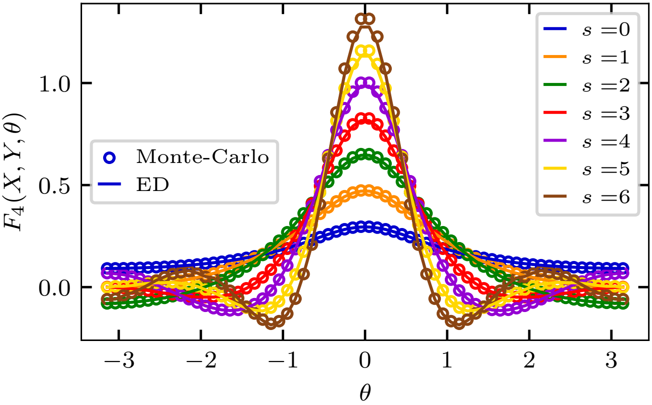
Floquet circuit models $f(E,\omega) \sim \omega$-indept $\Rightarrow \vec{a},\ \vec{b} \sim$ uncorrelated

Maximum entropy form for joint distribution of four vectors

$$P(\vec{a},\vec{b},\vec{c},\vec{d}) \propto e^{-\sum_{xy} M^*_{abcd}(x) G(x,y,E) M_{abcd}(y)} \times P_{\text{Haar}}(\vec{a},\vec{b},\vec{c},\vec{d})$$

– correlations parameterised via $G(x,y,E)$ vs $E \equiv E_a - E_b + E_c - E_d$

Testing the Ansatz



Normalised $[M_{abcd}(x)M^*_{abcd}(y)]_{av}$ vs $E = (E_a - E_b + E_c - E_d)$
for separation $s = 0, 1, \ldots 6$ between subsystems x & y

Monte Carlo sampling of Ansatz (points)
vs exact diagonalisation (lines)

Summary

Many-body quantum dynamics in chaotic systems

- Characterised by operator growth (and entanglement spreading)
- Encoded in correlator between sets of 4 eigenfunctions
- Unique correlator at this order
  with structure at long-distance/small energy

$$[M_{abcd}(x)M^*_{abcd}(y)]_{av}$$

- Joint distribution of eigenvectors builds in these correlations
  and is replacement for ETH