Eigenvector non-orthogonality in non-Hermitian random matrices: theory and applications [1]

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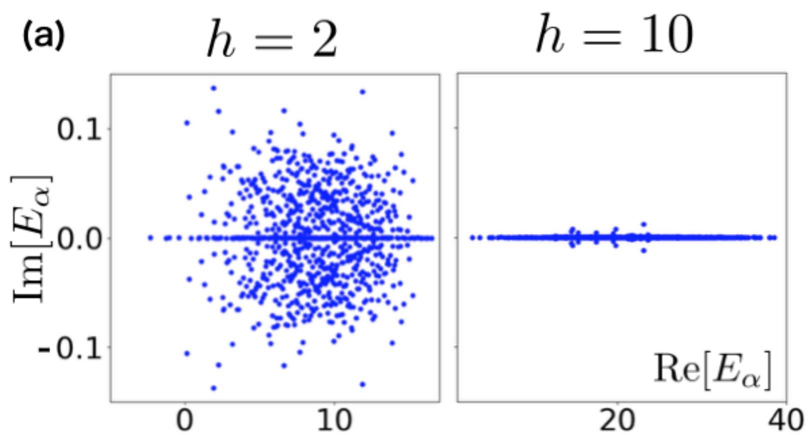
[1] **Works on the project:** **YF** *Commun. Math. Phys.* **363**, 579—603 (2018); **YF** & W. Tarnowski *Ann. Henri Poincare* **22** 309 – 330(2021); **YF** & M. Osman *J. Phys. A: Math. Theor.* **55** 224013 (2021); T. Würfel, M.J. Crumpton & **YF** *arXiv:2310.04307* & *arXiv:2402.09296*; **YF**, E Gudowska-Nowak, M.A. Nowak, W. Tarnowski *arXiv:2310.09018*

Effective **non-Hermitian** many-body Hamiltonians attracted recently a lot of attention, e.g. a time-reversal invariant many-body version of **Hatano-Nelson** model of hardcore bosons:

$$H = \sum_{k=1}^{L} \left[-t \left(e^{g} c_k^\dagger c_{k+1} + e^{-g} c_{k+1}^\dagger c_k \right) + h_k n_k + V n_k n_{k+1} \right]$$

with $n_k = c_k^\dagger c_k = (0,1)$ counting the number of particles on a lattice site k, while $h_k \in [-h, h]$ being random on-site potentials and V being non-random nearest-neighbour interaction. It has been shown to have a **complex-real** transition in eigenvalues as well as an MBL transition as the disorder strength increases:



Hamazaki, Kawabata, Ueda ’19.

Alternatively, non-time-reversal invariant manybody **gain-loss** model:

$$H = \sum_{k=1}^{L} \left[-t \left(c_k^\dagger c_{k+1} + c_{k+1}^\dagger c_k \right) + \left(h_k - i\gamma(-1)^k \right) n_k + V n_k n_{k+1} \right]$$

Natural reference point: **Ginibre** Ensembles, **real** for I and **complex** for II.

Left-right eigenvectors and eigenvalue condition numbers:

A (square) matrix X is **non-normal** if it does not commute with its Hermitian adjoint: $XX^* \neq X^*X$. Generically, **non-Hermitian random** matrices are non-normal. To each eigenvalue of a non-normal matrix λ_i, real or complex, correspond two eigenvectors: **left** $\mathbf{l}_i$ and **right** $\mathbf{r}_i$. The corresponding eigenproblems are

$$X\mathbf{r}_i = \lambda_i \mathbf{r}_i \text{ and } X^*\mathbf{l}_i = \overline{\lambda}_i \mathbf{l}_i.$$

The two sets can always be chosen **bi-**orthogonal: $(\mathbf{l}_i^* \mathbf{r}_j) = \delta_{ij}$.

Consider now a **perturbed** matrix $X' = X + \epsilon P$, with $\epsilon > 0$ controlling the magnitude of the perturbation P. To the leading order in ϵ the eigenvalues are shifted by

$$|\lambda_i(\epsilon) - \lambda_i(0)| = \epsilon |\mathbf{l}_i^* P \mathbf{r}_i| \leq \epsilon ||P||_2 \sqrt{(\mathbf{l}_i^* \mathbf{l}_i)(\mathbf{r}_i^* \mathbf{r}_i)} \quad ,$$

showing that the sensitivity of eigenvalues is mainly controlled by the **eigenvalue condition numbers**:

$$\kappa_i = \sqrt{(\mathbf{l}_i^* \mathbf{l}_i)(\mathbf{r}_i^* \mathbf{r}_i)} \geq 1,$$

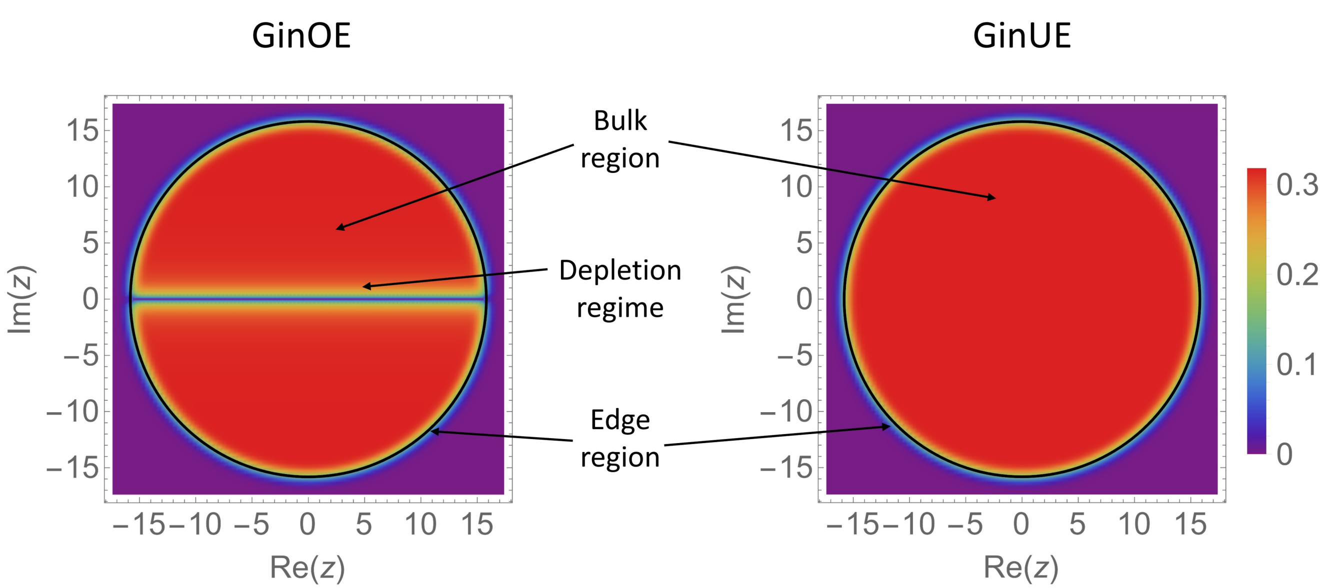
with $\kappa = 1$ only when X is normal. Thus, non-normal matrices are much more sensitive to perturbations of the matrix entries than their normal counterparts - “**ill-conditioned eigenvalues**".

Ginibre Gaussian Ensembles:

It is natural to ask how well-conditioned are eigenvalues of a '**typical**' $N \times N$ non-normal matrix randomly chosen according to a probability measure or "**ensemble**".

The simplest choice: all entries are i.i.d. normals $X_{j,k} \sim N^{-1/2}\mathcal{N}(0,1)$ for the **real Ginibre** or $\Re X_{j,k} \sim \Im X_{j,k} \sim N^{-1/2}\mathcal{N}(0,1/2)$ for the **complex Ginibre** ensembles.

For the **real Ginibre** Ensemble of the order $\sqrt{N}$ eigenvalues are typically **purely real**, with the uniform density $\mathbb{E}\left[\rho(|z \in \mathbb{R}| < 1\right] = \frac{1}{\sqrt{2\pi}}$, the rest of eigenvalues coming in complex-conjugate pairs and forming the uniformly filled in unit circle. For the **complex Ginibre** ensemble all eigenvalues are with probability one complex.



'Diagonal' eigenvector overlaps for complex Ginibre matrices:

Characteristics of **non-orthogonality** in the set of left $\&$ right eigenvectors of complex Ginibre matrices have been originally addressed more than two decades ago by **J. Chalker** and **B. Mehlig** ('98, '00) who introduced the matrix of inner products $\mathcal{O}_{ij} = (\mathbf{l}_i^* \mathbf{l}_j)(\mathbf{r}_j^* \mathbf{r}_i)$, which they called "**eigenvector overlaps**". The diagonal 'overlaps' are simply the squared eigenvalue condition numbers.

They further associated with the diagonal elements of the overlap matrix the following single-point correlation function:

$$O_1(z) = \left\langle \frac{1}{N^2} \sum_{i=1}^{N} \mathcal{O}_{ii} \delta(z - \lambda_i) \right\rangle_{\mathbf{C.G.}}$$

where the angular brackets stand for the expectation with respect to the **complex Ginibre** ensemble, and then proceeded to computing the "**bulk**" value:

$$\lim_{N \to \infty} O_1(z) = \frac{1}{\pi}(1 - |z|^2) \text{ for } |z| < 1 \quad ,$$

implying that the typical eigenvalue condition number in the bulk has the order

$$\kappa_i^2 := \mathcal{O}_{ii} \sim O(N) \text{ as } N \gg 1$$

so is **parametrically larger** than for the normal matrices.

Distribution of ‘diagonal’ eigenvector overlaps for Ginibre ensembles:

Chalker and Mehlig also conjectured that the distribution of diagonal overlaps $\mathcal{O}_{ii}$ for **complex Ginibre** case is heavy-tailed: $\mathcal{P}\left(\mathcal{O}_{ii}\right) \sim \mathcal{O}_{ii}^{-3}$. This conjecture has been settled in 2018 by **Bourgade** $\&$ **Doubach** and **YF**, the latter paper further addressing overlaps for **real** eigenvalues of real Ginibre matrices.

Theorem:

Consider the (conditional) probability density function $\mathcal{P}_N(z,t)$ of the (scaled) ‘diagonal overlap’ factor $t = (\mathcal{O}_{ii} - 1)/N$ for eigenvectors corresponding to eigenvalues in the vicinity of a point $z = x + iy$ in the complex plane for $\beta = 2$ or on the real axis for $\beta = 1$. Then

$$\lim_{N\to\infty} \mathcal{P}_N(z,t) = \frac{\langle\rho(z)\rangle}{t} e^{-\frac{O_1^{(\beta)}(z)}{t\langle\rho(z)\rangle}} \left(\frac{O_1^{(\beta)}(z)}{t\langle\rho(z)\rangle}\right)^{\beta}, \quad |z| < 1.$$

where for $\beta = 1$ $\langle\rho(z)\rangle = \frac{1}{\sqrt{2\pi}}$ for the interval $|z| < 1$, whereas for $\beta = 2$ $\langle\rho(z)\rangle = \pi^{-1}$ inside the unit circle $|z| < 1$. Further $O_1^{(\beta=2)}(z) = \pi^{-1}(1 - |z|^2)$ and $O_1^{(\beta=1)}(z) = \frac{1}{2\sqrt{2\pi}}(1 - |z|^2)$ provide ’typical scale’ value for the diagonal overlap.

Note: For $\beta = 2$ ‘**typical scale=mean**’, whereas for $\beta = 1$ the mean **does not exist!**

Eigenvector overlaps in Ginibre: further works and applications:

Studies of Chalker-Mehlig overlaps in all scaling regimes of the complex plane:
(**i**) at the edge, where non-orthogonality is parametrically weaker: $\mathcal{O}_{ii} \sim \sqrt{N}$
(**ii**) in the depletion regime of GinOE close to the real axis, and in the **weak non-Hermiticity** regimes. **YF, Tarnowski**'21, **Würfel- Crumpton-YF**'23

In the bulk more general off-diagonal correlation function has been evaluated in various regimes (**Chalker** & **Mehlig**'99, **Walters** & **Simm**'15 **Bourgade** & **Doubach**'18)

$$O_2(z_1, z_2) = \left\langle \frac{1}{N} \sum_{k \neq l}^{N} \mathcal{O}_{kl} \delta(z_1 - \lambda_k) \delta(z_2 - \lambda_l) \right\rangle_{GC} .$$

Numerics shows those eigenvector correlations are markedly different between the delocalized and localized phases: **Ghosh, Kulkarni, Roy**'23

It was argued eigenvector non-orthogonality may lead to a **violation** of **Eigenstates Thermalization** : **Cipolloni** & **Kudler-Flam**'23.

Enhancement of entropy production for driven multivariate linear systems:
$dx_i = \sum_{j=1}^{N} A_{ij} x_j \, dt + dW_j(t)$ **YF**,**Gudowska-Nowak**,**Nowak**,**Tarnowski**'23

Applications to quantum chaotic scattering, both theoretically (**Schomerus et al.** '00, **YF, Savin** '12; **YF, Osman** '22) and experimentally **Gros** et al.'14, **Davy, Genack** '19.

Non-orthogonality factors for rank-one non-Hermitian deformations:

Theorem **YVF, M. Osman** , '21:

Consider $\mathcal{H} = H - i\gamma \mathbf{e} \otimes \mathbf{e}^T$, *with* $H \in GUE$ *or* $H \in GOE$ *and define the (conditional) probability density of the non-orthogonality factor* $t = O_{nn} - 1$ *corresponding to eigenvalues in the vicinity of a point* $z = X - iY,\ Y > 0$ *in the complex plane*

$$\mathcal{P}(t; z) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(O_{nn} - 1 - t)\delta(z - z_n) \right\rangle$$

Then for $H \in GUE$ as $N \to \infty$ the limiting density $\mathcal{P}_y^{(2)}(t) := \lim_{N\to\infty} \frac{1}{\pi\rho N} \mathcal{P}(t; z = X - i\frac{y}{\pi\rho N})$ takes the following form

$$\mathcal{P}_y^{(2)}(t) = \frac{16}{t^3}\, e^{-2gy}\, \mathbb{L}_2\, e^{-2gy\left(1+\frac{2}{t}\right)} I_0\left(\frac{4y}{t}\sqrt{(g^2-1)(1+t)}\right)$$

where we defined $g = \frac{1}{2\pi\rho_{sc}(x)}\left(\gamma + \frac{1}{\gamma}\right)$, $I_\nu(x)$ stands for the modified Bessel function and $\mathbb{L}_2$ is a differential operator acting on functions $f(y)$ as

$$\mathbb{L}_2\, f(y) = \left\{1 + \left(\frac{\sinh 2y}{2y}\right)^2 + \frac{1}{2y}\left(1 - \frac{\sinh 4y}{4y}\right)\frac{d}{dy} + \frac{1}{4}\left(\left(\frac{\sinh 2y}{2y}\right)^2 - 1\right)\frac{d^2}{dy^2}\right\} y^2 f(y).$$

Similar explicit result is also available for $H \in GOE$.

The **heavy-tail** asymptotics $\mathcal{P}_y^{(2)}(t) \sim t^{-3}$ seems the most **universal** feature of statistics of diagonal overlaps in the complex plane.