A Tale of Two Sigma Models

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based on arXiv:2408.12215

On the infrared limit of the O(3) nonlinear σ-model at $\theta = \pi$

2D nonlinear σ-models with Hermitian symmetric target admit a θ-term, which couples the field theory to the topological charge of its instanton gas. At the special coupling $\theta = \pi$, by what is nowadays attributed to a coupling-constant anomaly of Lieb-Schultz-Mattis type, such models have a degenerate ground state. Yet, the details of their non-trivial infrared limit have remained open in general. Here we suggest that non-perturbative renormalization group flow into the strong-coupling regime induces strong fluctuations of the θ-parameter, with the consequence that the instanton density is suppressed, the target-space topology effectively altered, and the target-space metric driven off reality and into geometrostasis. Assuming this heuristic scenario and combining it with a Cauchy process of target-space deformation, we present a detailed argument that the O(3) nonlinear σ-model at $\theta = \pi$, known to be the effective field theory for critical antiferromagnetic quantum spin chains with large half-integer spin, renormalizes to the conformal field theory of a U(1) boson with compactification radius $r = 1/\sqrt{2}$. A closely related scenario applies to Pruisken's nonlinear σ-model for the integer quantum Hall transition.

Motivation: Integer Quantum Hall Transition

Pruisken model: $\mathcal{L} = \frac{\sigma_{xx}}{8} \mathrm{Tr}\, \partial^\mu Q\, \partial_\mu Q + \frac{\sigma_{xy}}{8} \varepsilon^{\mu\nu} \mathrm{Tr}\, Q\, \partial_\mu Q\, \partial_\nu Q$?

RG beta function near criticality: $\beta = \frac{1}{\nu}(\sigma_{xy} - \sigma_{xy}^*) \frac{\partial}{\partial \sigma_{xy}} + y(\sigma_{xx} - \sigma_{xx}^*) \frac{\partial}{\partial \sigma_{xx}}$.
(Pruisken–Khmelnitskii)

Numerical Results from finite-size scaling assuming the PK conjecture:

Authors (year)		ν	y
Huckestein & Kramer	(1990)	2.34 ± 0.04	—
Huckestein	(1994)	—	-0.38 ± 0.04
Slevin & Ohtsuki	(2009)	2.59 ± 0.005	-0.17 ± 0.04
Amado, Malyshev et al.	(2011)	2.62 ± 0.01	$y = 0$ (log)
Obuse, Gruzberg & Evers	(2012)	2.62 ± 0.06	$-0.7 < y < 0$
Zhu, Wu, Bhatt, Wan	(2019)	2.48 ± 0.02	-4.3
Klümper et al.	(2019)	$2.37 < \nu < 3.28$	$-1.4 < y_p < -0.2$
Dresselhaus, Sbierski, Gruzberg	(2021)	$3.42 - 3.93$	0

Q: why are the numerical results so discordant ?

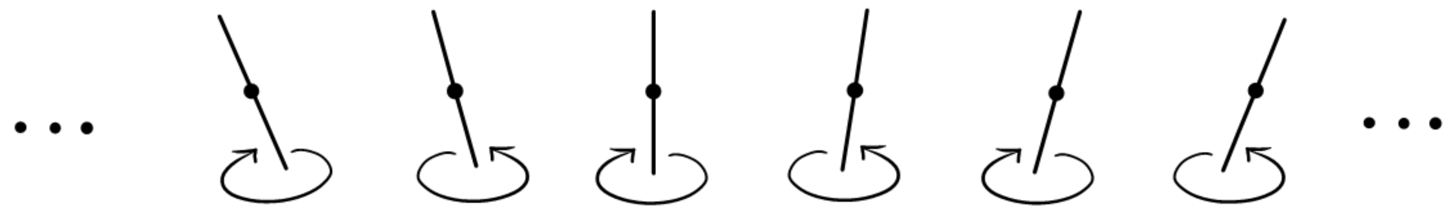
Outline

It was the best of times, it was the worst of times, ...

- O(3) nonlinear sigma model at $\theta = \pi$:
  - Anti-ferromagnetic quantum spin chains, Haldane conjecture
  - intuition from real-space renormalization (Kadanoff)
  - Wess-Zumino-Witten scenario, controversy
  - heuristic scenario for RG flow to U(1) boson model at $r = 1/\sqrt{2}$

- Pruisken nonlinear sigma model at $\theta = \pi$:
  - RG-fixed point with conformal symmetry now "derived"
  - How does the WZW anomalous term come about?

Anti-ferromagnetic quantum spin chains



Heisenberg model (translation-invariant, isotropic):

$$H = J \sum_n S_n \cdot S_{n+1} \quad (J > 0).$$

Haldane Conjecture

F.D.M. Haldane

Nobel Prize Physics 2016

- $|S| \in \mathbb{N} + 1/2$: degenerate ground state (presumably with massless excitations)

  [Lieb – Schultz – Mattis 1961 ; Affleck – Lieb 1986]

- $|S| \in \mathbb{N}$: gapped excitation spectrum

  [Haldane 1983 ; Affleck – Kennedy – Lieb – Tasaki 1987]

Heuristic motivation for Haldane Conjecture:

for $|S|$ large, one has controlled derivation of O(3) nonlinear σ-model as an effective field theory:

$$\mathcal{L}^{O(3)} = \frac{\beta}{8} \mathrm{Tr}\, \partial^\mu Q\, \partial_\mu Q + i\,\Theta\, \mathcal{L}_{top},$$

imaginary-valued two-form term

$$\mathcal{L}_{top} = \frac{\varepsilon^{\mu\nu}}{16\pi i} \mathrm{Tr}\, Q\, \partial_\mu Q\, \partial_\nu Q .$$

Matrix field: $Q = u\, \sigma_3\, u^{-1} = \sum m^a \sigma_a$, $\quad u \in \mathcal{U}(2)$.

Target space: $\mathcal{U}(2) / \mathcal{U}(1) \times \mathcal{U}(1) \cong S^2$.

Couplings (dimensionless):

$\beta = |S|$ (spin stiffness), $\quad \Theta = 2\pi\, |S|$ (topological angle).

Some $O(3)_{\theta=\pi}$ model history

- Polyakov & Wiegmann (PL-B 1983) used the correspondence with a Yang-Baxter integrable interacting fermion model with N flavors ($N \to \infty$) to show that $O(3)_{\theta=\pi}$ possesses scale and conformal invariance.

- Zamolodchikov & Zamolodchikov (NPB 1992) constructed a factorized S-matrix (Yang-Baxter) with $SU_L(2) \times SU_R(2)$ symmetry, which they conjecturally attributed to $O(3)_{\theta=\pi}$.

- Quote from Cordova-Freed-Lam-Seiberg (SciPost Phys 2020): "$O(3)_{\theta=\pi}$ is believed to be a gapless WZW model."

- ArXiv:2405.14627 — $O(3)_{\theta=\pi}$ treated by functional RG á la Wetterich.

Fixed point of renormalization-group flow = ?

O(3) nonlinear σ – model
$\theta = \pi$

RG

?

SU(2) Wess–Zumino–Witten model, level $k=1$

$$\frac{d}{d\ln a}\left(\frac{1}{T} g_{ij}\right) = \frac{D-2}{T} g_{ij} - Ric_{ij} - \frac{1}{2} T R_{ipqr} R_j{}^{pqr} + O(T^2)$$
$$= 0 \overset{D=2}{\Longrightarrow} \mathbb{R},\ S^1,\ G.$$

$\cong$

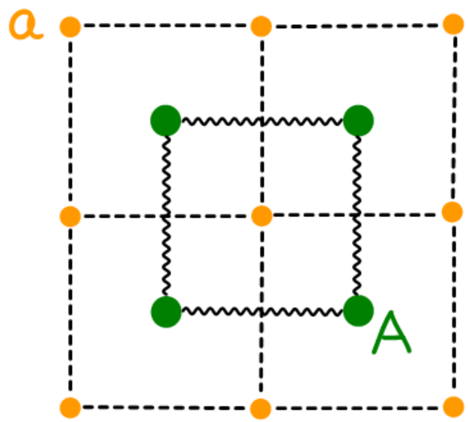
U(1) boson (xy–model) radius $r = 1/\sqrt{2}$

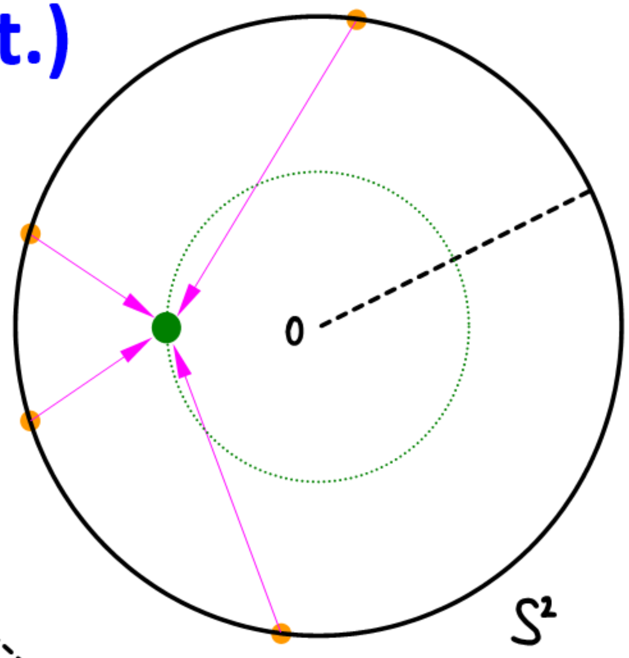
P. Ginsparg, NPB 1988
"Curiosities at $c=1$"

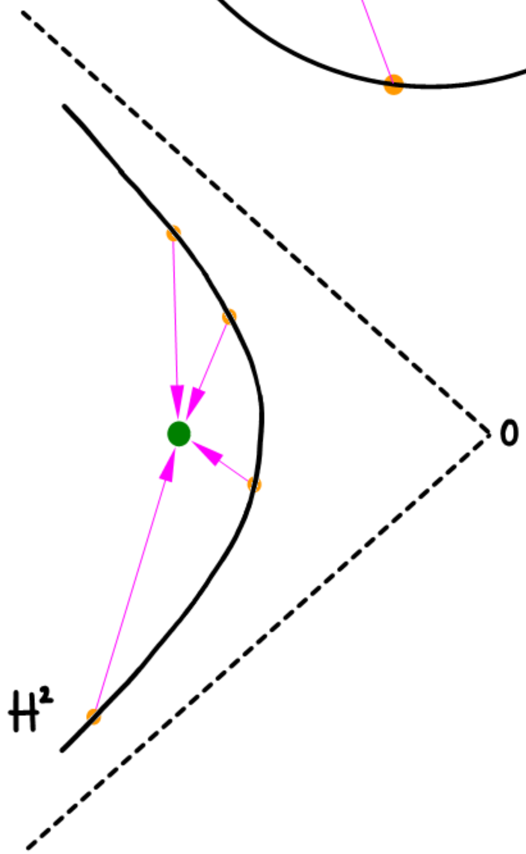
Real-space renormalization (non-pert.)

Kadanoff block spin RG transformation:

$\{a_s\} \curvearrowright \{A_b\}$:







$$1 = \int \mathcal{D}A \prod_{\text{blocks}} \delta(A_b - \mathcal{M}_b[a])$$

$$\int \mathcal{D}a \; 1 \; e^{-S_0[a]} =$$

$$\int \mathcal{D}A \int \mathcal{D}a \prod_{\text{blocks}} \delta(A_b - \mathcal{M}_b[a]) \, e^{-S_0[a]}$$

$$= \int \mathcal{D}A \; e^{-S_1[A]}.$$

WZW scenario (~Affleck & Haldane, 1987)

– Under renormalization, the spin stiffness β decreases.

– On entering the strong-coupling regime ($\beta \sim 1$), the coupling "constants" become dynamical variables as functions of an emergent field ψ:
$\beta(\psi) = \beta_0 \sin^2\psi$, $\Theta(\psi) = 2\psi - \sin 2\psi$ ($\leftarrow$ suspension $S^2 \hookrightarrow S^3 = SU(2)$).

$$\longrightarrow \quad Z = \int \mathcal{D}\psi \; e^{-\frac{k}{4\pi}\int d^2x\, \partial^\mu\psi\, \partial_\mu\psi \,-\, M^2 \int d^2x\, \cos^2\psi} \; Z^{O(3)}_{\beta(\psi),\,\Theta(\psi)} \,.$$

Heuristics. ① $O(3)_{\Theta=\pi}$ nonlinear σ-model renormalizes to a CFT. ✓

② 2D conformal symmetry requires conservation of Hodge-dual currents:
$$\partial_\mu j^\mu = 0 \implies \partial_\mu \epsilon^\mu{}_\nu\, j^\nu = 0.$$ ✓

③ In the non-Abelian setting, there exists but one solution to the CFT requirement ②: the WZW model (here for $SU(2)$ from $O(3)$). ?

④ Among all $SU(2)_k$ WZW models only the one for level $k=1$ is stable with respect to generic perturbations preserving criticality. ✓

Main issue: the mass term for the non-Goldstone field ψ renormalizes from $M^2 = \infty$ to $M^2 = 0$??

Fixed point of renormalization-group flow

$$\boxed{\begin{array}{c}O(3)\text{ nonlinear }\sigma\text{-model}\\ \theta=\pi\end{array}}$$

! RG $\Downarrow$

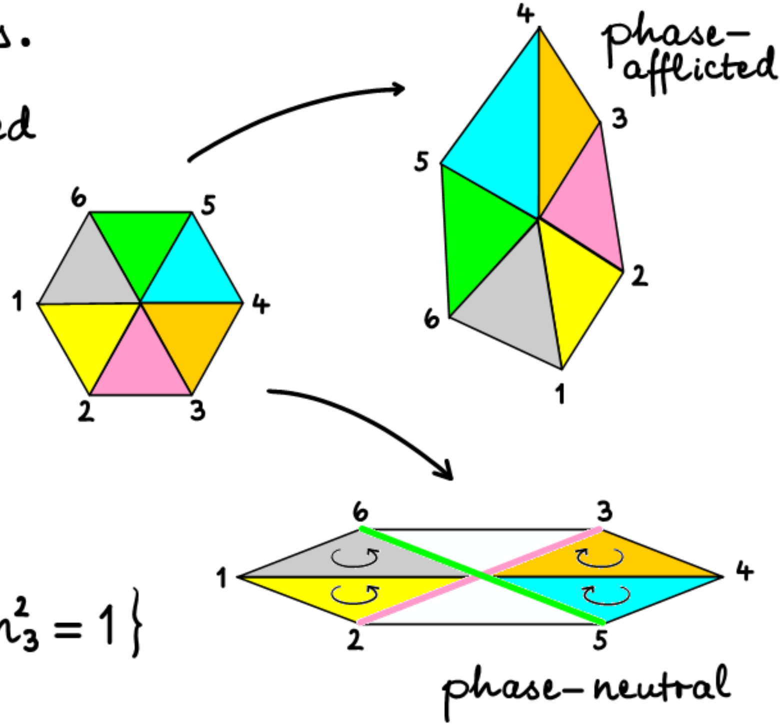
flip the script …

$$\boxed{\begin{array}{c}U(1)\text{ boson (xy-model)}\\ \text{radius } r=1/\sqrt{2}\end{array}}$$

$$\boxed{\begin{array}{c}SU(2)\text{ Wess-Zumino-Witten}\\ \text{model, level } k=1\end{array}} \cong$$

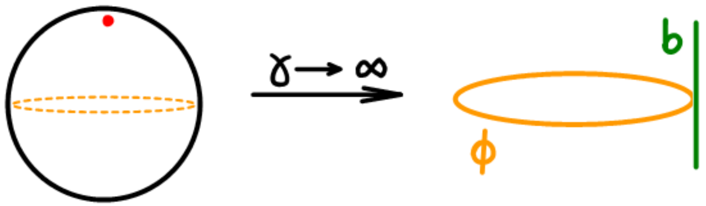
Heuristic scenario for RG flow to $U(1)_{r=1/\sqrt{2}}$

- Spin stiffness $\beta = |S| \gg 1$ (weak coupling) renormalizes toward zero (strong coupling).
- @ strong coupling: $O(3)$–orbit radius (hence β, θ) fluctuates.
  Then, $\mathbb{E}\left(\exp i\int d^2x\ \theta(x)\,\mathcal{L}_{top}(x)\right) \xrightarrow[\text{principle}]{\text{Fourier}} \mathcal{L}_{top}(x)$ suppressed
  $\Longrightarrow$ change of target-space topology $S^2 \to S^2\setminus\{p,-p\}$.



- Orbit–radius fluctuations die out in the final (IR) stage of the RG flow (ψ is massive).
- Cauchy deformation:
  $S^2\setminus\{p,-p\} \xrightarrow{\text{deform}} S^1\times\mathbb{R}$ inside $\mathcal{C} = \{\vec{n}\in\mathbb{C}^3 \mid n_1^2+n_2^2+n_3^2 = 1\}$

Fubini-Study



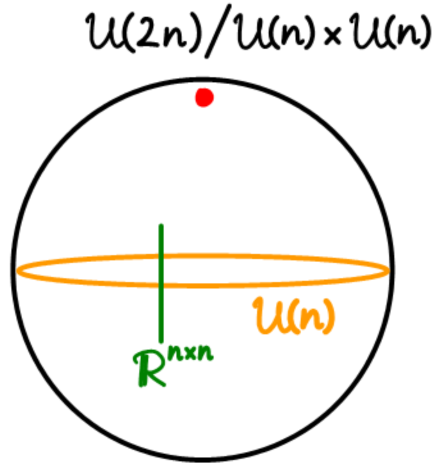
$d\phi^2 + (db + ib\,d\phi)^2$

- RG–fixed point: $Z_* = \int_{SO(3)} dR\ Z(R\cdot C_p)$, $\quad \nabla_\mu b = \partial_\mu b + ib\,\partial_\mu\phi$,

$$Z(C_p) = \int\mathcal{D}\phi\int\mathcal{D}b\ e^{-\int d^2x\ \nabla^\mu b\,\nabla_\mu b} \propto \int\mathcal{D}\phi\ e^{-\frac{1}{4\pi}\int d^2x\ \partial^\mu\phi\,\partial_\mu\phi} \quad \blacksquare$$

Conformal field theory for IQHT (finally "derived")

Start from the Pruisken model (here use replicas for pedagogy) and follow the blue print from $O(3)_{\theta=\pi}$ to $U(1)_{r=1/\sqrt{2}}$:



- Suppression of $\mathcal{L}_{top}$ (at $\theta=\pi$) effectively changes the target-space topology from $U(2n)/U(n)\times U(n)$ to $U(n)\times \mathbb{R}^{n\times n}$, $n\to 0$.

- Cauchy deformation inside the complex space $GL(2n)/GL(n)\times GL(n)$ and integration over the free fields ($\mathbb{R}^{n\times n}$) yields the effective action

$$S = \frac{\sigma_{xx}}{4}\int d^2x \, \mathrm{Tr}\,(\partial^\mu M^{-1}\partial_\mu M) - \frac{1}{4\pi}\int d^2x \, \mathrm{Tr}\,(M^{-1}\partial^\mu M)\,\mathrm{Tr}\,(M^{-1}\partial_\mu M).$$

$M(x)\in U(n)$ "Gade–Wegner"

NEW: in order for the residual curvature of $U(n)$, $n\to 0$, to be neutralized, the effect of torsion due to a WZW anomaly is needed.

Why/how does the WZW term come about?

WHY? By the unitarity of the disordered-electron quantum dynamics, one has a conserved current : $\boxed{\partial_\mu j^\mu = 0}$. At IQHT critical point, this current must acquire the additional property of being irrotational (on disorder average): $\boxed{\mathbb{E}(\bullet\, \partial_\mu \epsilon^\mu{}_\nu j^\nu) = 0}$.
The field theory must reproduce both conservation laws $\Longrightarrow$ a WZW term must appear (in the non-Abelian setting at hand).

HOW? The mechanism is totally non-perturbative (no replica version, sorry)!

Super space target GL(1|1) : $M = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} e^\tau & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$.

$$S_*[M] = \frac{1}{2\pi} \int d^2x \left(\partial^\mu \tau\, \partial_\mu \tau + \partial^\mu \phi\, \partial_\mu \phi + 2\, e^{\tau - i\phi}\, \partial^\mu \xi\, \partial_\mu \eta \right) \longleftarrow \frac{\sigma_{xx}}{8} \mathrm{Tr}\, \partial^\mu Q\, \partial_\mu Q$$

$$+ \frac{i}{\pi} \int d^2x\; e^{\tau - i\phi}\, \epsilon^{\mu\nu}\, \partial_\mu \xi\, \partial_\nu \eta \longleftarrow \frac{\sigma_{xy}}{8}\, \epsilon^{\mu\nu}\, \mathrm{Tr}\, Q\, \partial_\mu Q\, \partial_\nu Q$$

$$- \frac{1}{8\pi} \int d^2x \left(\partial^\mu \tau - i \partial^\mu \phi \right)\left(\partial_\mu \tau - i \partial_\mu \phi \right) \longleftarrow \text{effective from integration over Gaussian fields}$$

from NPB 941 (2019) 458

Folklore

- Anderson transitions are described by nonlinear sigma models (as effective field theories).
- Numerical observation of Weyl symmetry relations is a proof of that.
- Nonlinear sigma models are renormalizable (for all couplings, in any dimension).
- The upper critical dimension for Anderson transitions is $d_c = \infty$.
- There exists plenty of evidence that conformal invariance is violated at Anderson transitions.
- There exists little evidence supporting MZ´s proposal for the conformal field theory of the Integer Quantum Hall Transition.

Folklore revised

Anderson transitions are $\underbrace{}^{\text{not}}$ described by nonlinear sigma models (as effective field theories).

not

- Numerical observation of Weyl symmetry relations is ~~a~~ no proof of that.

  NLσM $\stackrel{\checkmark}{\Rightarrow}$ Weyl sym, but Weyl sym $\not\Rightarrow$ NLσM

- Nonlinear sigma models are renormalizable ~~(for all couplings, in any dimension)~~. in one space dimension and in weak-coupling perturbation theory

- The upper critical dimension for Anderson transitions is $d_c = \infty$.

  We expect a third phase (NEE) to appear in finite dimension.

- There exists $\boxed{\text{plenty of}}$ evidence that conformal invariance is violated at Anderson transitions.

- There exists $\boxed{\text{little}}$ evidence supporting MZ´s proposal for the conformal field theory of the Integer Quantum Hall Transition.

Outlook: why should we care?

- There exists a variety of (generalized) AF quantum spin chains with ground-state degeneracy (due to an anomaly of Lieb–Schultz–Mattis type). Field-theory target spaces: $\mathbb{CP}^n$, $U(m+n)/U(m)\times U(n)$, $SO(2n)/U(n)$, $Sp(2n)/U(n)$

→ Is their criticality of first order or second order (i.e. with massless excitations)?

cf. Pruisken et al., *Phys. Rev.* B 105, 15111 (2022)

- Among the symmetry classes of disordered free fermions ("Tenfold Way") there exist five classes (namely: A, C, D, AII, DIII) with topological phase transitions in 2D. In the weak-disorder regime all of these map to some nonlinear σ-model with topological term ($\theta = \pi$).

→ What are the conformal field theories describing their infrared behavior?

- What are the consequences (from the vantage point of non-perturbative RG) for universality and scaling at Anderson transitions in dimension $D \geqslant 3$?

- Can measurement-induced entanglement phase transitions be described by nonlinear σ-models?

Kudos and Cheers

to

Volodya