Trieste Algebraic Geometry Summer School (TAGSS) 2024 - Tropical Geometry and Related Topics | (SMR 3965)

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P01

Activity:Trieste Algebraic Geometry Summer School (TAGSS) 2024 - Tropical Geometry and Related Topics | (smr 3965)

Title: Spectral Data for Real Regular KP solutions on Rational Degenerations of M-curves and tropical M-curves

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In recent papers D. Agostini et al. [1,2] and T. Ichikawa [3] have proven that real theta functions associated with periods of tropical curves have tropical limits as KP solutions.

In a series of papers in collaboration with P.G. Grinevich [4-8] we have used the combinatorial structure of the totally non-negative real Grassmannians to explicitly construct the spectral data for the family of real regular KP multi-line solitons on reducible rational M-curves and we have proven that the desingularization of such data leads to real regular quasi-periodic solutions to the KP equation on smooth M-curves.

More precisely, each real regular KP soliton family is represented by a positroid cell in the totally non-negative part of a real Grassmannian. Each planar bicolored graph representing this positroid cell in Postnikov's classification [9] is dual to the topological model of the reducible M-curve. The KP divisor for the soliton solution is then obtained solving a system of relations on such graph. Finally, Dubrovin-Natanzon theorem [10], which characterizes the reality and regularity of the desingularized KP solution, holds true if and only the soliton data are in the totally non-negative part of the Grassmannian.

Thus, we have proven that each graph in Postnikov classification can be used to provide the model of the tropical limit of a smooth M-curve.

[1] D. Agostini, T.O. Çelik, J. Struwe, B. Sturmfels, Vietnam J. Math. **49** (2021), no. 2, 319-347.
[2] D. Agostini, C. Fevola, Y. Mandelshtam, B. Sturmfels,, J. Symb. Comp **114** (2023), 282--301
[3] T. Ichikawa, Comm. Math. Phys.**402** (2023), no.2, 1707-1723.
[4] S. Abenda, Math. Phys. Anal. Geom. **24** (2021), Art. 35: 64 pp.
[5] S. Abenda, P.G. Grinevich, Commun. Math. Phys. **361** Issue 3 (2018) 1029--1081.
[6] S. Abenda, P.G. Grinevich, Proc. Steklov Inst. Math. **302** (2018), no. 1, 1-15.
[7] S. Abenda, P.G. Grinevich, Sel. Math. New Ser. **25**, no. 3 (2019) 25:43.
[8] S. Abenda, P.G. Grinevich, Lett. Math. Phys. **112** (2022), no. 6, Paper No. 115, 64 pp.
[9] A. Postnikov "Total positivity, Grassmannians, and networks**.", arXiv:math/0609764** [math.CO].
[10] B. Dubrovin and S. Natanzon. Math. USSR-Izv. **32** (1989), no. 2, 269–288.

P02

Poincaré and Picard Bundles on Moduli Spaces of Vector Bundles on Nodal Curves

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S. Ramanan proved that a universal family (also called a Poincaré bundle) exists for the moduli problem of vector bundles on a smooth curve if and only if the rank and degree are coprime [2]. One of the key elements in his proof is the computation of the Picard group of the moduli space. In this talk, we first discuss the non-existence of a Poincaré bundle for the moduli problem of vector bundles on nodal curves when the degree and rank are not coprime closely following [2].

When the degree is sufficiently high, the pushforward of a Poincaré bundle to the moduli space is a vector bundle, called the Picard bundle. Although the existence of Poincaré bundles (hence Picard bundles) depend on the rank and degree being relatively prime, there always exists a universal family of projective bundles; called the projective Poincaré bundle. Similarly, there is a projective Picard bundle. Next, we discuss the stability of these bundles.

On the way to achieve these goals, we compute the codimension of a few closed subsets of the moduli spaces. U.N. Bhosle proved that not all stable bundles arise from the irreducible unitary representations of the fundamental group of the nodal curve unlike that of smooth curves [1]. Using these results on codimension, we show that the stable vector bundles on nodal curves, which arise from representations, form a big open set of the moduli space. We also use them to compute Picard groups of the moduli spaces.

[1] Bhosle, Usha N., Representations of the fundamental group and vector bundles, Math. Ann. **302** (1995).
[2] Ramanan S., The moduli spaces of vector bundles over an algebraic curve, Math. Ann. **200** (1973).

P03

Tropical Tevelev degrees

Erin Dawson and **Renzo Cavalieri**

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Tropical Hurwitz spaces parameterize genus g, degree d covers of the tropical line with fixed branch profiles. Since tropical curves are metric graphs, this gives us a combinatorial way to study Hurwitz spaces. Tevelev degrees are the degrees of a natural finite map from the Hurwitz space to a product $M_{0,n} \times M_{g,n}$. In 2021, Cela, Pandharipande and Schmitt presented this interpretation of Tevelev degrees in terms of moduli spaces of Hurwitz covers. We define the *tropical Tevelev degrees*, $\mathsf{Tev}_g^{\mathtt{trop}}$, as the degree of a natural finite morphism between certain tropical moduli spaces, in analogy to the algebraic case. We exhibit combinatorial recursions among well-chosen tropical covers that compute $\mathsf{Tev}_g^{\mathtt{trop}}$.

[1] A. Cela, R. Pandharipande, J. Schmitt, J. Sci. Res. **13**, 1357 (2012).

P04

Patchworking in F_1 geometry for Trieste Algebraic Geometry Summer School (TAGSS) 2024: Tropical Geometry and related topics

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The field of one element, F_1, is an idea that was first proposed by Jaques Tits as a link between Chevalley groups and their Weyl groups, but it didn't garner serious interest until the late 20th century when its links to other areas, including arithmetic, combinatorics and tropical geometry, was unearthed.

In the last two decades, several approaches to F_1-geometry were developed that generalize algebraic geometry from different perspectives. What is common to most approaches is that F1-scheme is a space with a covering by affine patches.

In this talk, we explain this patchworking from a general and simplified perspective, and we comment on the topological realizations of F1-schemes. This is work in progress, in collaboration with Matt Baker and Oliver Lorscheid.

P05

Multi Symmetric Products and Higher Rank Divisors on Curves

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In this talk, we introduce the notion of the diagonal property and the weak point property for an ind-variety, i.e an inductive system of varieties. Following that, we check that these properties are being satisfied by some particular ind-varieties, i.e higher rank divisors on curves, which are really important in the context of studying a higher dimensional analogue of the classical Abel-Jacobi Map. To be specific, we observe that the ind-varieties of higher rank divisors of integral slopes on a smooth projective curve have the weak point property. Moreover, we show that the ind-variety of $(1, n)$-divisors has the diagonal property and is a locally complete linear ind-variety and calculate its Picard group. Furthermore, we obtain that the Hilbert schemes of a curve associated to the good partitions of a constant polynomial satisfy the diagonal property and count the exact number of such schemes by proving that the multi symmetric products associated to two distinct partitions of a positive integer are not isomorphic. This is a joint work with Prof. D. S. Nagaraj (cf. [1]).

[1] A. Mukherjee and D. S. Nagaraj, *Diagonal property and weak point property of higher rank divisors and certain Hilbert schemes*, (2024). (https://arxiv.org/abs/2401.00852)

P06

Cremona transformations of $\mathbb{P}^3$ stabilizing quartic surfaces

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We are interested in Gizatullin's problem which consists in the following question: *Given a smooth quartic surface $S \subset \mathbb{P}^3$, which automorphisms of S are induced by Cremona transformations of $\mathbb{P}^3$?*
Cremona transformations of $\mathbb{P}^3$ can be written as a composition of a finite sequence of elementary maps. This is an algorithmic process called the Sarkisov Program. In this talk, we will solve Gizatullin's problem when $S \subset \mathbb{P}^3$ has Picard number two by using the Sarkisov program. The results that will be presented are in collaboration with Ana Quedo, and with Carolina Araujo and Sokratis Zikas.

P07

TROPICAL CYCLES OF DISCRETE ADMISSIBLE COVERS.

DIEGO A. ROBAYO BARGANS

Abstract: This project concerns itself with the moduli spaces of discrete admissible covers of tropical curves and their relationship with the moduli spaces of tropical curves. Its origin lies in the results on tree gonality (of tropical curves) of A. Vargas and J. Draisma. We introduce a systematic approach that allows us to describe and handle tropical cycles in the moduli space of tropical curves of genus g, and show that the loci of interest are tropical cycles therein. This involves the development of a framework that endows spaces concocted analogously to the moduli space of tropical curves of genus g with a tropical structure. Such spaces include the moduli space of n-marked tropical curves of genus g, and the moduli space of discrete admissible covers of a fixed degree to an m-marked tropical curve of genus h with prescribed ramification profiles over the marked ends. In the latter case, the usual weight assignment of admissible covers gives rise to a fundamental cycle, which can then be pushforwarded to a tropical cycle of the moduli space of tropical curves where the source curve lies. By subsequently forgetting the marking, we obtain the loci of curves that admit a tropical cover from a (tropical) modification onto a tropical curve of a given genus of a fixed degree and with the prescribed ramification. The aforementioned results on tree gonality, as well as further underlying tropical behavior of these gonality cycles, can then be recovered as a special case of this previous result.

P08

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009

ON SINGULAR VARIETIES ASSOCIATED TO A POLYNOMIAL MAPPING FROM $\mathbb{C}^n$ TO $\mathbb{C}^{n-1}$*

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Abstract. We construct singular varieties $\mathcal{V}_G$ associated to a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ where $n \geqslant 2$. Let $G : \mathbb{C}^3 \to \mathbb{C}^2$ be a local submersion, we prove that if the homology or the intersection homology with total perversity (with compact supports or closed supports) in dimension two of any variety $\mathcal{V}_G$ is trivial then G is a fibration. In the case of a local submersion $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ where $n \geqslant 4$, the result is still true with an additional condition.

Key words. Complex polynomial mappings, intersection homology, singularities at infinity.

Mathematics Subject Classification. 14P10, 14R15, 32S20, 55N33.

1. Introduction. Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a non-constant polynomial mapping ($n \geqslant 2$). It is well known [20] that G is a locally trivial fibration outside the bifurcation set $B(G)$ in $\mathbb{C}^{n-1}$. In a natural way appears a fundamental question: how to determine the set $B(G)$. In [12], Ha Huy Vui and Nguyen Tat Thang gave, for a generic class of $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ ($n \geqslant 2$), a necessary and sufficient condition for a point $z \in \mathbb{C}^{n-1}$ to be in the bifurcation set $B(G)$ in term of the Euler characteristic of the fibers at nearby points. The case n=2 was previously given in [11] .

In this paper, we want to construct singular varieties $\mathcal{V}_G$ associated to a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ ($n \geqslant 2$) such that the intersection homology of $\mathcal{V}_G$ can characterize the bifurcation set of G. The motivation for this paper comes from the paper [21], where Anna and Guillaume Valette constructed real pseudomanifolds, denoted V_F, associated to a given polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$, such that the singular part of the variety V_F is contained in $(S_F \times K_0(F)) \times \{0^p\}$ ($p > 0$), where $K_0(F)$ is the set of critical values and S_F is the set of non-proper points of F. In the case of dimension $n = 2$, the homology or intersection homology of V_F describes the geometry of the singularities at infinity of the mapping F. With Anna and Guillaume Valette, the first author generalized this result [18] for the general case $F : \mathbb{C}^n \to \mathbb{C}^n$ ($n \geqslant 2$). The idea to construct varieties V_F is the following: considering the polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ as a real one $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, then if we take a finite covering $\{V_i\}$ by smooth submanifolds of $\mathbb{R}^{2n}\backslash SingF$, the mapping F induces a diffeomorphism from V_i into its image $F(V_i)$. We use a technique in order to separate these $\{F(V_i)\}$ by embedding them in a higher dimensional space, then V_F is obtained by gluing $\{F(V_i)\}$ together along the set $S_F \cup K_0(F)$.

A natural question is that how can we apply this construction to the case of polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$, or, $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$. The main difficulty of this case is that if we take an open submanifold V in $\mathbb{R}^{2n}\backslash SingF$, then locally we do not have a diffeomorphism from V into its image $G(V)$. So we consider a generic $(2n-2)$- real dimensional submanifold in the source space $\mathbb{R}^{2n}$, denoted $\mathcal{M}_G$, which

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is called *the Milnor set of* G. Then we can apply the construction of singular varieties V_F in [21] for $F := G|_{\mathcal{M}_G}$ the restriction of G to the Milnor set $\mathcal{M}_G$.

We obtain the following result: let $G : \mathbb{C}^3 \to \mathbb{C}^2$ be a local submersion, then if the homology or the intersection homology with total perversity (with compact supports or closed supports) in dimension two of any among of the constructed varieties $\mathcal{V}_G$ is trivial then G is a fibration (Theorem 5.1). In the case of a local submersion $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ where $n \geqslant 4$, the result is still true with an additional condition (Theorem 5.2). Comparing with the paper [12], we obtain the Corollary 5.9.

2. Preliminaries and basic definitions. In this section we set-up our framework. All the varieties we consider in this article are semi-algebraic.

2.1. Notations and conventions. Given a topological space X, singular simplices of X will be semi-algebraic continuous mappings $\sigma : T_i \to X$, where T_i is the standard i-simplex in $\mathbb{R}^{i+1}$. Given a subset X of $\mathbb{R}^n$ we denote by $C_i(X)$ the group of i-dimensional singular chains (linear combinations of singular simplices with coefficients in $\mathbb{R}$); if c is an element of $C_i(X)$, we denote by $|c|$ its support. By $Reg(X)$ and $Sing(X)$ we denote respectively the regular and singular locus of the set X. Given $X \subset \mathbb{R}^n$, $\overline{X}$ will stand for the topological closure of X. The smoothness to be considered as the differentiable smoothness.

Notice that, when we refer to the homology of a variety, the notation $H^c_*(X)$ refers to the homology with compact supports, the notation $H^{cl}_*(X)$ refers to the homology with closed supports (see [1]).

2.2. Intersection homology. We briefly recall the definition of intersection homology; for details, we refer to the fundamental work of M. Goresky and R. MacPherson [6] (see also [1]).

Definition 2.1. Let X be a m-dimensional semi-algebraic set. A *semi-algebraic stratification of* X is the data of a finite semi-algebraic filtration

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing,$$

such that for every i, the set $S_i = X_i \backslash X_{i-1}$ is either empty or a manifold of dimension i. A connected component of S_i is called *a stratum* of X.

We denote by cL the open cone on the space L, the cone on the empty set being a point. Observe that if L is a stratified set then cL is stratified by the cones over the strata of L and a 0-dimensional stratum (the vertex of the cone).

Definition 2.2. A stratification of X is said to be *locally topologically trivial* if for every $x \in X_i \backslash X_{i-1}$, $i \geqslant 0$, there is an open neighborhood U_x of x in X, a stratified set L and a semi-algebraic homeomorphism

$$h : U_x \to (0;1)^i \times cL,$$

such that h maps the strata of U_x (induced stratification) onto the strata of $(0;1)^i \times cL$ (product stratification).

The definition of perversities as originally given by Goresky and MacPherson:

Definition 2.3. A *perversity* is an $(m+1)$-uple of integers $\bar{p} = (p_0, p_1, p_2, p_3, \dots, p_m)$ such that $p_0 = p_1 = p_2 = 0$ and $p_{k+1} \in \{p_k, p_k + 1\}$, for $k \geqslant 2$.

Traditionally we denote the zero perversity by $\overline{0} = (0,0,0,\dots,0)$, the maximal perversity by $\overline{t} = (0,0,0,1,\dots,m-2)$, and the middle perversities by $\overline{m} = (0,0,0,0,1,1,\dots,[\frac{m-2}{2}])$ (lower middle) and $\overline{n} = (0,0,0,1,1,2,2,\dots,[\frac{m-1}{2}])$ (upper middle). We say that the perversities $\overline{p}$ and $\overline{q}$ are *complementary* if $\overline{p}+\overline{q}=\overline{t}$.

Let X be a semi-algebraic variety such that X admits a locally topologically trivial stratification. We say that a semi-algebraic subset $Y \subset X$ is $(\bar{p}, i)$-*allowable* if

$$\dim(Y \cap X_{m-k}) \leqslant i - k + p_k \text{ for all } k.$$

Define $IC_i^{\overline{p}}(X)$ to be the $\mathbb{R}$-vector subspace of $C_i(X)$ consisting in those chains ξ such that $|\xi|$ is $(\overline{p}, i)$-allowable and $|\partial\xi|$ is $(\overline{p}, i-1)$-allowable.

Definition 2.4. The i^{th} *intersection homology group with perversity* $\overline{p}$, denoted by $IH_i^{\overline{p}}(X)$, is the i^{th} homology group of the chain complex $IC_*^{\overline{p}}(X)$.

Notice that, the notation $IH_*^{\overline{p},c}(X)$ refer to the intersection homology with compact supports, the notation $IH_*^{\overline{p},cl}(X)$ refer to the intersection homology with closed supports.

Goresky and MacPherson proved that the intersection homology is independent of the choice of the stratification [6, 7].

The Poincaré duality holds for the intersection homology of a (singular) variety:

Theorem 2.5 (Goresky, MacPherson [6]). *For any orientable compact stratified semi-algebraic m-dimensional variety X, generalized Poincaré duality holds:*

$$IH_k^{\overline{p}}(X) \simeq IH_{m-k}^{\overline{q}}(X),$$

where $\overline{p}$ and $\overline{q}$ are complementary perversities.

For the non-compact case, we have:

$$IH_k^{\overline{p},c}(X) \simeq IH_{m-k}^{\overline{q},cl}(X).$$

A relative version is also true in the case where X has boundary.

2.3. The bifurcation set, the set of asymptotic critical values and the asymptotic set. Let $G : \mathbb{C}^n \to \mathbb{C}^m$ where $n \geqslant m$ be a polynomial mapping.

i) The bifurcation set of G, denoted by $B(G)$ is the smallest set in $\mathbb{C}^m$ such that G is not C^∞ - fibration on this set (see, for example, [20]).

ii) The set of asymptotic critical values, denoted by $K_\infty(G)$, is the set

$$K_\infty(G) = \{\alpha \in \mathbb{C}^m : \exists \{z_k\} \subset \mathbb{C}^n, \text{ such that } |z_k| \to \infty, G(z_k) \to \alpha \text{ and } |z_k||dG(z_k)| \to 0\}.$$

The set $K_\infty(G)$ is an approximation of the set $B(G)$. More precisely, we have $B(G) \subset K_\infty(G)$ (see, for example, [14] or [3]).

iii) When $n = m$, we denote by S_G the set of points at which the mapping G is not proper, *i.e.*

$$S_G = \{\alpha \in \mathbb{C}^m : \exists \{z_k\} \subset \mathbb{C}^n, |z_k| \to \infty \text{ such that } G(z_k) \to \alpha\},$$

and call it the *asymptotic variety*. In the case of polynomial mappings $F : \mathbb{C}^n \to \mathbb{C}^n$, the following holds: $B(G) = S_G$ ([9]).

3. The variety $\mathcal{M}_G$. We consider polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ as real ones $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$. By $Sing(G)$ we mean the singular locus of G, that is the set of points for which the (complex) rank of the Jacobian matrix is less than $n-1$. We denote by $K_0(G)$ the set of critical values. From here, we assume always $K_0(G) = \varnothing$.

Definition 3.1. Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a polynomial mapping. Consider G as a real polynomial mapping $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$. Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a real function such that $\rho(z) \geqslant 0$ for any $z \in \mathbb{C}^n$. Let

$$\varphi = \frac{1}{1+\rho}.$$

Consider (G, φ) as a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n-1}$. Let us define

$$\mathcal{M}_G := Sing(G, \varphi) = \{x \in \mathbb{R}^{2n} \text{ such that } \mathrm{Rank} D_{\mathbb{R}}(G, \varphi)(x) \leqslant 2n-2\},$$

where $D_{\mathbb{R}}(G, \varphi)(x)$ is the Jacobian matrix of $(G, \varphi) : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ at x.

Remark 3.2. Since $K_0(G) = \varnothing$, then $\mathrm{Rank} D_{\mathbb{R}}(G) = 2n-2$, so we have

$$Sing(G, \varphi) = \{x \in \mathbb{R}^{2n} \text{ such that } \mathrm{Rank} D_{\mathbb{R}}(G, \varphi) = 2n-2\}.$$

Note that, from here, if we want to refer to the source space as a complex space, we will write $(G, \varphi) : \mathbb{C}^n \to \mathbb{R}^{2n-1}$, if we want to refer to the source space as a real space, we will write $(G, \varphi) : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$. Moreover, in general, we denote by z a complex element in $\mathbb{C}^n$ and by x a real element in $\mathbb{R}^{2n}$.

Lemma 3.3. *For any ρ, φ and (G, φ) as in the Definition 3.1 and for any $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, we have*

$$\mathrm{Rank} D_{\mathbb{R}}(G, \varphi)(x) = \mathrm{Rank} D_{\mathbb{R}}(G, \rho)(x),$$

so we have

$$\mathcal{M}_G = Sing(G, \varphi) = Sing(G, \rho).$$

Proof. For any $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, we have

$$D_{\mathbb{R}}(G, \rho)(x) = \begin{pmatrix} & D_{\mathbb{R}}(G) & \\ \rho_{x_1} & \cdots & \rho_{x_{2n}} \end{pmatrix},$$

$$D_{\mathbb{R}}(G, \varphi)(x) = \begin{pmatrix} & D_{\mathbb{R}}(G) & \\ \frac{-\rho_{x_1}}{(1+\rho)^2} & \cdots & \frac{-\rho_{x_{2n}}}{(1+\rho)^2} \end{pmatrix},$$

where ρ_{x_i} is the derivative of ρ with respect to x_i, for $i = 1, \ldots, 2n$. We have $\mathrm{Rank} D_{\mathbb{R}}(G, \varphi)(x) = \mathrm{Rank} D_{\mathbb{R}}(G, \rho)(x)$ for any $x \in \mathbb{R}^{2n}$ and $\mathcal{M}_G = Sing(G, \varphi) = Sing(G, \rho)$. ∎

Remark 3.4. From here, we consider the function ρ of the following form

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2,$$

where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0$, $a_i \geqslant 0$, and $a_i \in \mathbb{R}$ for $i = 1, \dots, n$.

PROPOSITION 3.5. *Let $G = (G_1, \dots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(n \geqslant 2)$ be a polynomial mapping such that $K_0(G) = \varnothing$ and $\rho : \mathbb{C}^n \to \mathbb{R}$ be such that $\rho = a_1|z_1|^2 + \dots + a_n|z_n|^2$, where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0$, $a_i \geqslant 0$ and $a_i \in \mathbb{R}$, for $i = 1, \dots, n$. Denote by $\mathbf{v}_i$ the determinant of the cofactor of $\frac{\partial}{\partial z_i}$ of the matrix*

$$\mathbf{V}(z) = \begin{pmatrix} \frac{\partial}{\partial z_1} & \dots & \frac{\partial}{\partial z_n} \\ \frac{\partial G_1}{\partial z_1} & \dots & \frac{\partial G_1}{\partial z_n} \\ & \dots & \\ \frac{\partial G_{n-1}}{\partial z_1} & \dots & \frac{\partial G_{n-1}}{\partial z_n} \end{pmatrix},$$

for $i = 1, \dots, n$. Then we have

$$\mathcal{M}_G = h^{-1}(0),$$

where

$$h : \mathbb{C}^n \to \mathbb{C}, \quad h(z) = 2\Sigma a_i \mathbf{v}_i(z)\overline{z_i}.$$

Proof. Let $G = (G_1, \dots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(n \geqslant 2)$ and $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $\rho = a_1|z_1|^2 + \dots + a_n|z_n|^2$, where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0$, $a_i \geqslant 0$ and $a_i \in \mathbb{R}$. Let us consider the vector field

$$\mathbf{V}(z) = \begin{pmatrix} \frac{\partial}{\partial z_1} & \dots & \frac{\partial}{\partial z_n} \\ \frac{\partial G_1}{\partial z_1} & \dots & \frac{\partial G_1}{\partial z_n} \\ & \dots & \\ \frac{\partial G_{n-1}}{\partial z_1} & \dots & \frac{\partial G_{n-1}}{\partial z_n} \end{pmatrix}.$$

We have

$$\mathbf{V}(z) = \mathbf{v}_1 \frac{\partial}{\partial z_1} + \dots + \mathbf{v}_n \frac{\partial}{\partial z_n},$$

where $\mathbf{v}_i$ is the determinant of the cofactor of $\frac{\partial}{\partial z_i}$, for $i = 1, \dots, n$. The vector field $\mathbf{V}(z)$ is tangent to the curve $G = c$. Let $R(z) = a_1 z_1^2 + \dots + a_n z_n^2$, then we have $\mathcal{M}_G = h^{-1}(0)$, where

$$h : \mathbb{C}^n \to \mathbb{C}, \quad h(z) = < \mathbf{V}(z), \operatorname{Grad} R(z) > .$$

More precisely, we have $h(z) = 2\Sigma a_i \mathbf{v}_i(z)\overline{z_i}$. □

PROPOSITION 3.6. *For an open and dense set of polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ such that $K_0(G) = \varnothing$, the variety $\mathcal{M}_G$ is a smooth manifold of dimension $2n - 2$.*

Proof. The question is of local nature. In a neighbourhood of a point z_0 in $\mathbb{C}^n$, we can choose coordinates such that the level curve $G = c$, where $c = G(z_0) \in \mathbb{C}^{n-1}$ is parametrized

$$\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, z_0)$$

$$s \mapsto (\gamma_1(s), \dots, \gamma_n(s)).$$

Since $\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2$, then $\rho \circ \gamma : (\mathbb{C}, 0) \to \mathbb{R}$ and

$$\rho \circ \gamma(s) = a_1|\gamma_1(s)|^2 + \cdots + a_n|\gamma_n(s)|^2.$$

If z_0 is a singular point of $\rho|_{G=c}$, then

$$\rho \circ \gamma(0) = \rho(\gamma(0)) = \rho(z_0),$$

$$(\rho \circ \gamma)'(0) = 0.$$

For an open and dense set of G, we have

$$(\rho \circ \gamma)''(0) \neq 0.$$

Hence, z_0 is a Morse singularity of $\rho|_{G=c}$. In particular, it is an isolated point of the level curve $G = c$. When c varies in $\mathbb{C}^{n-1}$, it follows that the set $\mathcal{M}_G$ has dimension $2n - 2$.

We prove now that $\mathcal{M}_G$ is smooth. By Proposition 3.5, the variety $\mathcal{M}_G$ is the set of solutions of $h = 0$, where

$$h(z) = 2\Sigma a_i \mathbf{v}_i(z)\overline{z_i},$$

and $\mathbf{v}_i$ is the determinant of the cofactor of $\frac{\partial}{\partial z_i}$ of $\mathbf{V}(z)$, for $i = 1, \dots, n$. Since $K_0(G) = \varnothing$ then $\mathbf{V}(z) = (\mathbf{v}_1(z), \dots, \mathbf{v}_n(z)) \neq 0$. We can assume that $\mathbf{V}(z_0) \neq 0$ for a fixed point z_0. For a generic polynomial mapping, we can solve the equation $h = 0$ in a neighbourhood of z_0. This shows that $h = 0$ is smooth in a neighbourhood of z_0. Then M_G is smooth. □

Remark 3.7. From here, we consider always generic polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ as in the Propostion 3.6.

4. The variety $\mathcal{V}_G$.

4.1. The construction of the variety $\mathcal{V}_G$. Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ and $\rho, \varphi : \mathbb{C}^n \to \mathbb{R}$ such that

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2, \quad \varphi = \frac{1}{1+\rho},$$

where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0,\ a_i \geqslant 0$ and $a_i \in \mathbb{R}$. Let us consider:

a) $F := G_{|\mathcal{M}_G}$ the restriction of G on $\mathcal{M}_G$,

b) $\mathcal{N}_G = \mathcal{M}_G \backslash F^{-1}(K_0(F))$.

Since the dimension of $\mathcal{M}_G$ is $2n-2$ (Proposition 3.6), then locally, in a neighbourhood of any point x_0 in $\mathcal{M}_G$, we get a mapping $F : \mathbb{R}^{2n-2} \to \mathbb{R}^{2n-2}$. Now, we can apply the construction of singular varieties V_F in [21] for $F := G_{|\mathcal{M}_G}$: there exists a covering $\{U_1, \dots, U_p\}$ of $\mathcal{N}_G$ by open semi-algebraic subsets (in $\mathbb{R}^{2n}$) such that on every element of this covering, the mapping F induces a diffeomorphism onto its image (see Lemma 2.1 of [21], see also [16]). We can find semi-algebraic closed subsets $V_i \subset U_i$ (in $\mathcal{N}_G$) which cover $\mathcal{N}_G$ as well. Thanks to Mostowski's Separation Lemma (see Separation Lemma in [15], page 246), for each $i = 1, \dots, p$, there exists a Nash function $\psi_i : \mathcal{N}_G \to \mathbb{R}$, such that ψ_i is positive on V_i and negative on $\mathcal{N}_G \backslash U_i$.

LEMMA 4.1. *We can choose the Nash functions ψ_i such that $\psi_i(x_k)$ tends to zero when $\{x_k\} \subset \mathcal{N}_G$ tends to infinity.*

Proof. If ψ_i is a Nash function, then with any $N_i \in (\mathbb{N}\backslash\{0\})$, the function

$$\psi_i'(x) = \frac{\psi_i(x)}{(1 + |x|^2)^{N_i}},$$

where $x \in \mathcal{N}_G$, is also a Nash function, for $i = 1, \dots, p$. The Nash function ψ_i' satisfies the property: ψ_i' is positive on V_i and negative on $\mathcal{N}_G \backslash U_i$. With N_i large enough, $\psi_i'(x_k)$ tends to zero when $\{x_k\} \subset \mathcal{N}_G$ tends to infinity, for $i = 1, \dots, p$. We replace the function ψ_i by ψ_i'. ◻

DEFINITION 4.2. Let the Nash functions ψ_i and ρ be such that $\psi_i(x_k)$ tends to zero and $\rho(x_k)$ tends to infinity when $x_k \subset \mathcal{N}_G$ tends to infinity. Define a variety $\mathcal{V}_G$ associated to (G, ρ) as

$$\mathcal{V}_G := \overline{(F, \psi_1, \dots, \psi_p)(\mathcal{N}_G)}.$$

REMARK 4.3. For a given polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$, the variety $\mathcal{V}_G$ is not unique. It depends on the choice of the function ρ and the Nash functions ψ_i.

PROPOSITION 4.4. *The real dimension of $\mathcal{V}_G$ is $2n - 2$.*

Proof. By Proposition 3.6, in the generic case, the real dimension of $\mathcal{M}_G$ is $2n-2$. Moreover, F is a local immersion in a neighbourhood of a point in $\mathcal{M}_G$. So, the real dimension of $F(\mathcal{M}_G)$ is also $2n - 2$. Since

$$F(\mathcal{N}_G) = F(\mathcal{M}_G) \backslash K_0(F),$$

so the real dimension of $F(\mathcal{N}_G)$ is $2n-2$. By Definition 4.2, the real dimension of $\mathcal{V}_G$ is $2n - 2$. ◻

DEFINITION 4.5 (see, for example, [4]). Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a polynomial mapping and $\rho : \mathbb{C}^n \to \mathbb{R}$ a real function such that $\rho \geqslant 0$. Define

$$\mathcal{S}_G := \{\alpha \in \mathbb{C}^{n-1} : \exists \{z_k\} \subset Sing(G, \rho), \text{ such that } z_k \text{ tends to infinity, } G(z_k) \text{ tends to } \alpha\}.$$

REMARK 4.6. a) For any real function $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $\rho \geqslant 0$, we have

$$B(G) \subset \mathcal{S}_G \subset K_\infty(G),$$

where $B(G)$ is the bifurcation set and $K_\infty(G)$ is the set of asymptotic critical values of G (see, for example, [4]).

b) By Lemma 3.3, we have $Sing(G, \rho) = \mathcal{M}_G$, so the set $\mathcal{S}_G$ can be written

$$\mathcal{S}_G := \{\alpha \in \mathbb{C}^{n-1} : \exists \{x_k\} \subset \mathcal{M}_G, \text{ such that } x_k \text{ tends to infinity}, G(x_k) \text{ tends to } \alpha\}.$$

Definition 4.7. The *singular set at infinity* of the variety $\mathcal{V}_G$ is the set

$$\{\beta \in \mathcal{V}_G : \exists \{x_k\} \subset \mathcal{N}_G, x_k \to \infty, (G, \psi_1, \dots, \psi_p)(x_k) \to \beta\}.$$

Proposition 4.8. *The singular set at infinity of the variety $\mathcal{V}_G$ is contained in the set $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$.*

Proof. At first, by Proposition 3.6, for the generic case, the real dimension of $\mathcal{V}_G$ associated to $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is $2n-2$. Moreover, we have the following facts:

a) $\mathcal{S}_G \subset K_\infty(G)$,

b) $\dim_{\mathbb{C}}(K_\infty(G)) \leqslant n-2$ (see [14]), so $\dim_{\mathbb{R}}(K_\infty(G)) \leqslant 2n-4$.

Hence, we have $\dim_{\mathbb{R}} \mathcal{S}_G \times \{0_{\mathbb{R}^p}\} \leqslant 2n-4$. Moreover, by Proposition 4.4, we have $\dim_{\mathbb{R}} \mathcal{V}_G = 2n-2$. Let β be a singular point at infinity of the variety $\mathcal{V}_G$, then there exists a sequence $\{x_k\}$ in $\mathcal{N}_G$ tending to infinity such that $(G, \psi_1, \dots, \psi_p)(x_k)$ tends to β. Assume that $G(x_k)$ tends to α, then α belongs to $\mathcal{S}_G$. Moreover, the Nash function $\psi_i(x_k)$ tends to 0, for any $i = 1, \dots, p$. So $\beta = (\alpha, 0_{\mathbb{R}^p})$ belongs to $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$. Notice that, by Definition of $\mathcal{V}_G$, the set $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$ is contained in $\mathcal{V}_G$. Then $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$ contains the singular set at infinity of the variety $\mathcal{V}_G$. □

Remark 4.9. The singular set at infinity of $\mathcal{V}_G$ depends on the choice of the function ρ, since when ρ changes, the set $\mathcal{S}_G$ also changes. But, the property $B(G) \subset \mathcal{S}_G$ does not depend on the choice of the function ρ (see, for example, [4]).

The previous results show the following Proposition:

Proposition 4.10. *Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a polynomial mapping such that $K_0(G) = \varnothing$ and let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a real function such that*

$$\rho = a_1|z_1|^2 + \cdots + a_n|z_n|^2,$$

where $\Sigma_{i=1,\dots,n} a_i^2 \neq 0$, $a_i \geqslant 0$ and $a_i \in \mathbb{R}$ for $i = 1, \dots, n$. Then, there exists a real variety $\mathcal{V}_G$ in $\mathbb{R}^{2n-2+p}$, where $p > 0$, such that:

1) *The real dimension of $\mathcal{V}_G$ is $2n-2$,*

2) *The singular set at infinity of the variety $\mathcal{V}_G$ is contained in $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$.*

Remark 4.11. The variety $\mathcal{V}_G$ depends on the choice of the function ρ and the functions ψ_i. From now, we denote by $\mathcal{V}_G(\rho)$ any variety $\mathcal{V}_G$ associated to (G, ρ). If we refer to $\mathcal{V}_G$, that means a variety $\mathcal{V}_G$ associated to (G, ρ) for any ρ.

Remark 4.12. 1) In the construction of singular varieties $\mathcal{V}_G$, we can put $F := (G, \varphi)_{|\mathcal{M}_G}$, that means F is the restriction of (G, φ) on $\mathcal{M}_G$. In this case, since the dimension of $\mathcal{M}_G$ is $2n-2$ then locally, in a neighbourhood of any point x_0 in $\mathcal{M}_G$, we get a mapping $F : \mathbb{R}^{2n-2} \to \mathbb{R}^{2n-1}$. The construction of singular varieties $\mathcal{V}_G$ can be applied also in this case.

2) The construction of singular varieties $\mathcal{V}_G$ can be applied for polynomial mappings $G : \mathbb{C}^n \to \mathbb{C}^p$ where $p < n-1$ if the Milnor set $\mathcal{M}_G$ is smooth is this case.

4.2. A variety $\mathcal{V}_G$ in the case of the Broughton's Example.

EXAMPLE 4.13. We compute in this example a variety $\mathcal{V}_G$ in the case of the Broughton's example [2]:

$$G : \mathbb{C}^2 \to \mathbb{C}, \qquad G(z, w) = z + z^2 w.$$

We see that $K_0(G) = \varnothing$ since the system of equations $G_z = G_w = 0$ has no solutions. Let us denote

$$z = x_1 + ix_2, \qquad w = x_3 + ix_4,$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Consider G as a real polynomial mapping, we have

$$G(x_1, x_2, x_3, x_4) = (x_1 + x_1^2 x_3 - x_2^2 x_3 - 2x_1x_2x_4, x_2 + 2x_1x_2x_3 + x_1^2 x_4 - x_2^2 x_4).$$

Let $\rho = |w|^2$, then

$$\varphi = \frac{1}{1+\rho} = \frac{1}{1+|w|^2} = \frac{1}{1+x_3^2+x_4^2}.$$

The Jacobian matrix of (G, ρ) is

$$D_{\mathbb{R}}(G,\rho) = \begin{pmatrix} 1 + 2x_1x_3 - 2x_2x_4 & -2x_2x_3 - 2x_1x_4 & x_1^2 - x_2^2 & -2x_1x_2 \\ 2x_2x_3 + 2x_1x_4 & 1 + 2x_1x_3 - 2x_2x_4 & 2x_1x_2 & x_1^2 - x_2^2 \\ 0 & 0 & 2x_3 & 2x_4 \end{pmatrix}.$$

By an easy computation, we have $\mathcal{M}_G = Sing(G, \rho) = M_1 \cup M_2$, where
$M_1 := \{(x_1, x_2, 0, 0) : x_1, x_2 \in \mathbb{R}\}$,
$M_2 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 1 + 2x_1x_3 - 2x_2x_4 = 2x_2x_3 + 2x_1x_4 = 0\right\}.$
Let us consider G as a real mapping from $\mathbb{R}^4_{(x_1,x_2,x_3,x_4)}$ to $\mathbb{R}^2_{(\alpha_1,\alpha_2)}$, then:

a) If $x = (x_1, x_2, 0, 0) \in M_1$, we have $G(x) = (x_1, x_2)$.

b) If $x = (x_1, x_2, x_3, x_4) \in M_2$, then we have $G(x) = (\alpha_1, \alpha_2)$, where

$$\alpha_1 = \frac{-x_3}{4(x_3^2 + x_4^2)}, \qquad \alpha_2 = \frac{x_4}{4(x_3^2 + x_4^2)}.$$

Let $F := G_{|\mathcal{M}_G}$. We can check easily that $K_0(F) = \varnothing$. Choosing $\mathcal{M}_G$ as a covering of $\mathcal{M}_G$ itself. We choose the Nash function $\psi = \varphi$, then ψ is positive on all $\mathcal{M}_G$. So, by Definition 4.2, we have

$$\mathcal{V}_G = \overline{(F,\varphi)(\mathcal{M}_G)} = \overline{(G,\varphi)(\mathcal{M}_G)} = (G,\varphi)(M_1) \cup (G,\varphi)(M_2) \cup (\mathcal{S}_G \times 0_{\mathbb{R}}),$$

where $(G, \varphi) : \mathbb{R}^4_{(x_1,x_2,x_3,x_4)} \to \mathbb{R}^3_{(\alpha_1,\alpha_2,\alpha_3)}$. Then

a) $(G, \varphi)(M_1)$ is the plane $\{\alpha_3 = 1\} \subset \mathbb{R}^3_{(\alpha_1,\alpha_2,\alpha_3,)}$.

b) Assume that $(\alpha_1, \alpha_2, \alpha_3) \in (G, \varphi)(M_2)$, and let

$$x_3 = r cos\theta, \qquad x_4 = r sin\theta,$$

where $r \in \mathbb{R}$, $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$, then

$$\alpha_1^2 + \alpha_2^2 = \frac{1}{16r^2}, \qquad \alpha_3 = \frac{1}{1+r^2}.$$

So $(G,\varphi)(M_2)$ is a 2-dimensional open cone. In fact, when r tends to infinity, then α_1, α_2 and α_3 tend to 0, but the origin does not belong to this cone.

Moreover, by an easy computation, we can verify that the set $\mathcal{S}_G$ is $0 = (0,0) \in \mathbb{R}^2_{(\alpha_1,\alpha_2)}$. So the origin 0 of $\mathbb{R}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ belongs to $\mathcal{V}_G$. In conclusion, the variety $\mathcal{V}_G$ is the union of the plane $\alpha_3 = 1$ and a 2-dimensional cone $\mathcal{C}$ with vertex 0, where the cone $\mathcal{C}$ tends to infinity and asymptotic to the plane $\alpha_3 = 1$ in $\mathbb{R}^3_{(\alpha_1,\alpha_2,\alpha_3)}$ (see Figure 1).

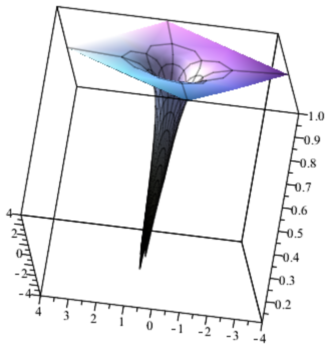

FIG. 1. *A variety $\mathcal{V}_G$ in the case of the Broughton's Example $G(z,w) = z + z^2w$.*

REMARK 4.14. We can use the Proposition 3.5 with the view of mixed functions (see [19]) to determine the variety $\mathcal{M}_G$. Let us return to the Example 4.13. In this case $\rho = |w|^2$, then

$$\mathcal{M}_G = \left\{(z,w) \in \mathbb{C}^2 : \frac{\partial G}{\partial z}\overline{w} = 0\right\}.$$

Hence we have $(1+2zw)\overline{w} = 0$, that implies the following two cases:

i) $\overline{w} = 0$: We have $x_3 = x_4 = 0$, where $w = x_3 + ix_4$.

ii) $\overline{w} \neq 0$ and $z = -\frac{1}{2w} = -\frac{\overline{w}}{2|w|^2}$: We have

$$x_1 = \frac{-x_3}{2(x_3^2+x_4^2)}, \qquad x_2 = \frac{x_4}{2(x_3^2+x_4^2)},$$

where $z = x_1 + ix_2$.

So we get $\mathcal{M}_G = M_1 \cup M_2$ as the computations and notations in the Example 4.13.

5. Results.

THEOREM 5.1. *Let $G = (G_1, G_2) : \mathbb{C}^3 \to \mathbb{C}^2$ be a polynomial mapping such that $K_0(G) = \emptyset$. If one the groups $IH_2^{\overline{t},c}(\mathcal{V}_G,\mathbb{R})$, $IH_2^{\overline{t},cl}(\mathcal{V}_G,\mathbb{R})$, $H_2^c(\mathcal{V}_G,\mathbb{R})$ and $H_2^{cl}(\mathcal{V}_G,\mathbb{R})$ is trivial then the bifurcation set $B(G)$ is empty.*

THEOREM 5.2. *Let $G = (G_1,\ldots,G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ $(n \geqslant 4)$ be a polynomial mapping such that $K_0(G) = \emptyset$ and $\mathrm{Rank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\ldots,n-1} > n-3$, where $\hat{G}_i$*

is the leading form of G_i, that is the homogenous part of highest degree of G_i, for $i = 1, \dots, n-1$. Then if one the groups $IH_2^{\bar{t},c}(\mathcal{V}_G, \mathbb{R})$, $IH_2^{\bar{t},cl}(\mathcal{V}_G, \mathbb{R})$, $H_2^c(\mathcal{V}_G, \mathbb{R})$, $H_2^{cl}(\mathcal{V}_G, \mathbb{R})$, $H_{2n-4}^c(\mathcal{V}_G, \mathbb{R})$ and $H_{2n-4}^{cl}(\mathcal{V}_G, \mathbb{R})$ is trivial then the bifurcation set $B(G)$ is empty.

Before proving these Theorems, we recall some necessary Definitions and Lemmas.

DEFINITION 5.3. A *semi-algebraic family of sets (parametrized by $\mathbb{R}$)* is a semi-algebraic set $A \subset \mathbb{R}^n \times \mathbb{R}$, the last variable being considered as parameter.

REMARK 5.4. A semi-algebraic set $A \subset \mathbb{R}^n \times \mathbb{R}$ will be considered as a family parametrized by $t \in \mathbb{R}$. We write A_t, for "the fiber of A at t", *i.e.*:

$$A_t := \{x \in \mathbb{R}^n : (x, t) \in A\}.$$

LEMMA 5.5 ([21]). *Let β be a j-cycle and let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a compact semi-algebraic family of sets with $|\beta| \subset A_t$ for any t. Assume that $|\beta|$ bounds a $(j+1)$-chain in each A_t, $t > 0$ small enough. Then β bounds a chain in A_0.*

DEFINITION 5.6 ([21]). Given a subset $X \subset \mathbb{R}^n$, we define the *"tangent cone at infinity"*, called "*contour apparent à l'infini*" in [16] by:

$$C_\infty(X) := \{\lambda \in \mathbb{S}^{n-1}(0,1) \text{ such that } \exists \eta : (t_0, t_0 + \varepsilon] \to X \text{ semi-algebraic,}$$
$$\lim_{t \to t_0} \eta(t) = \infty, \lim_{t \to t_0} \frac{\eta(t)}{|\eta(t)|} = \lambda\}.$$

LEMMA 5.7 ([18]). *Let $G = (G_1, \dots, G_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial mapping and V the zero locus of $\hat{G} := (\hat{G}_1, \dots, \hat{G}_m)$, where $\hat{G}_i$ is the leading form of G_i. If X is a subset of $\mathbb{R}^n$ such that $G(X)$ is bounded, then $C_\infty(X)$ is a subset of $\mathbb{S}^{n-1}(0,1) \cap V$, where $V = \hat{G}^{-1}(0)$.*

Proof of the Theorem 5.1. Recall that in this case, $\dim_{\mathbb{R}} \mathcal{V}_G = 4$ (Proposition 4.4) and $\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})$ is not smooth in general. Consider a stratification of $\mathcal{V}_G$, the strata of which are the strata of $\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}$ union the strata of the stratification of $K_0(F)$ defined by the rank, according to Thom [20]. Assume that $B(G) \neq \varnothing$, then by Remark 4.6, the set $\mathcal{S}_G$ is not empty. This means that there exists a complex Puiseux arc in $\mathcal{M}_G$

$$\gamma : D(0, \eta) \to \mathbb{R}^6, \quad \gamma = uz^\alpha + \dots,$$

(with α negative integer and u is an unit vector of $\mathbb{R}^6$) tending to infinity such a way that $G(\gamma)$ converges to a generic point $x_0 \in \mathcal{S}_G$. Then, the mapping $h_F \circ \gamma$, where $h_F = (F, \varphi_1, \dots, \varphi_p)$ and F is the restriction of G on $\mathcal{M}_G$ provides a singular 2-simplex in $\mathcal{V}_G$ that we will denote by c. We prove now the simplex c is $(\bar{t}, 2)$-allowable for total perversity $\bar{t}$. In fact, by [14], in this case we have $\dim_{\mathbb{C}} \mathcal{S}_G \leqslant 1$, then $\alpha = \operatorname{codim}_{\mathbb{R}} \mathcal{S}_G \geqslant 2$. The condition

$$0 = \dim_{\mathbb{R}}\{x_0\} = \dim_{\mathbb{R}}((\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}) \cap |c|) \leqslant 2 - \alpha + t_\alpha,$$

implies $t_\alpha \geqslant \alpha - 2$, with $\alpha \geqslant 2$, which is true for total perversity $\bar{t}$. Take now a stratum V_i of $\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})$. Denote by $\beta = \operatorname{codim}_{\mathbb{R}} V_i$. If $\beta \geqslant 2$, we can choose the Puiseux arc γ such that c lies in the regular part of $\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})$. In fact, this comes from the generic position of transversality. So c is $(\bar{t}, 2)$-allowable in this case. We need to consider only the cases $\beta = 0$ and $\beta = 1$. We have the following two cases:

1) If V_i intersects c, again by the generic position of transversality, we can choose the Puiseux arc γ such that $0 \leqslant \dim_{\mathbb{R}}(V_i \cap |c|) \leqslant 1$. The condition

$$\dim_{\mathbb{R}}(V_i \cap |c|) \leqslant 2 - \beta + t_\beta$$

holds since $2 - \beta + t_\beta \geqslant 1$, for $\beta = 0$ and $\beta = 1$.

2) If V_i does not meet c, then the condition

$$-\infty = \dim_{\mathbb{R}} \varnothing = \dim_{\mathbb{R}}(V_i \cap |c|) \leqslant 2 - \beta + t_\beta$$

holds always.

So the simplex c is $(\bar{t}, 2)$-allowable for total perversity $\bar{t}$.

From here, the proof of the Theorem follows the ideas of [21]: We can always choose the Puiseux arc such that the support of ∂c lies in the regular part of $\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})$. We have

$$H_1(\mathrm{Reg}(\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}))) = 0,$$

then the chain ∂c bounds a singular chain $e \in C^2(\mathrm{Reg}(\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\})))$, where e is a chain with compact supports or closed supports. So $\sigma = c - e$ is a $(\bar{t}, 2)$-allowable cycle of $\mathcal{V}_G$, with compact supports or closed supports.

We claim that σ may not bound a 3-chain in $\mathcal{V}_G$. Assume otherwise, *i.e.* assume that there is a chain $\tau \in C_3(\mathcal{V}_G)$, satisfying $\partial\tau = \sigma$. Let

$$A := h_F^{-1}(|\sigma| \cap (\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}))),$$

$$B := h_F^{-1}(|\tau| \cap (\mathcal{V}_G \backslash (\mathcal{S}_G \times \{0_{\mathbb{R}^p}\}))).$$

By definition, $C_\infty(A)$ and $C_\infty(B)$ are subsets of $\mathbb{S}^5(0,1)$. Observe that, in a neighborhood of infinity, A coincides with the support of the Puiseux arc γ. The set $C_\infty(A)$ is equal to $\mathbb{S}^1.a$ (denoting the orbit of $a \in \mathbb{C}^3$ under the action of $\mathbb{S}^1$ on $\mathbb{C}^3$, $(e^{i\eta}, z) \mapsto e^{i\eta}z$). Let V be the zero locus of the leading forms $\hat{G} := (\hat{G}_1, \hat{G}_2)$. Since $G(A)$ and $G(B)$ are bounded, by Lemma 5.7, $C_\infty(A)$ and $C_\infty(B)$ are subsets of $V \cap \mathbb{S}^5(0,1)$.

For R large enough, the sphere $\mathbb{S}^5(0,R)$ with center 0 and radius R in $\mathbb{R}^6$ is transverse to A and B (at regular points). Let

$$\sigma_R := \mathbb{S}^5(0,R) \cap A, \qquad \tau_R := \mathbb{S}^5(0,R) \cap B.$$

Then σ_R is a chain bounding the chain τ_R. Consider a semi-algebraic strong deformation retraction $\Phi : W \times [0;1] \to \mathbb{S}^1.a$, where W is a neighborhood of $\mathbb{S}^1.a$ in $\mathbb{S}^5(0,1)$ onto $\mathbb{S}^1.a$. Considering R as a parameter, we have the following semi-algebraic families of chains:

1) $\tilde{\sigma}_R := \frac{\sigma_R}{R}$, for R large enough, then $\tilde{\sigma}_R$ is contained in W,
2) ${\sigma'}_R = \Phi_1(\tilde{\sigma}_R)$, where $\Phi_1(x) := \Phi(x,1), \qquad x \in W$,
3) $\theta_R = \Phi(\tilde{\sigma}_R)$, we have $\partial\theta_R = \sigma'_R - \tilde{\sigma}_R$,
4) ${\theta'}_R = \tau_R + \theta_R$, we have $\partial\theta'_R = \sigma'_R$.

As, near infinity, σ_R coincides with the intersection of the support of the arc γ with $\mathbb{S}^5(0,R)$, for R large enough the class of σ'_R in $\mathbb{S}^1.a$ is nonzero.

Let $r = 1/R$, consider r as a parameter, and let $\{\tilde{\sigma}_r\}$, $\{\sigma'_r\}$, $\{\theta_r\}$ as well as $\{\theta'_r\}$ the corresponding semi-algebraic families of chains.

Denote by $E_r \subset \mathbb{R}^6 \times \mathbb{R}$ the closure of $|\theta_r|$, and set $E_0 := (\mathbb{R}^6 \times \{0\}) \cap E$. Since the strong deformation retraction Φ is the identity on $C_\infty(A) \times [0,1]$, we see that

$$E_0 \subset \Phi(C_\infty(A) \times [0,1]) = \mathbb{S}^1.a \subset V \cap \mathbb{S}^5(0,1).$$

Denote by $E'_r \subset \mathbb{R}^6 \times \mathbb{R}$ the closure of $|\theta'_r|$, and set $E'_0 := (\mathbb{R}^6 \times \{0\}) \cap E'$. Since A bounds B, then $C_\infty(A)$ is contained in $C_\infty(B)$. We have

$$E'_0 \subset E_0 \cup C_\infty(B) \subset V \cap \mathbb{S}^5(0,1).$$

The class of σ'_r in $\mathbb{S}^1.a$ is, up to a product with a nonzero constant, equal to the generator of $\mathbb{S}^1.a$. Therefore, since σ'_r bounds the chain θ'_r, the cycle $\mathbb{S}^1.a$ must bound a chain in $|\theta'_r|$ as well. By Lemma 5.5, this implies that $\mathbb{S}^1.a$ bounds a chain in E'_0 which is included in $V \cap \mathbb{S}^5(0,1)$.

The set V is a projective variety which is an union of cones in $\mathbb{R}^6$. Since $\dim_{\mathbb{C}} V \leqslant 1$, so $\dim_{\mathbb{R}} V \leqslant 2$ and $\dim_{\mathbb{R}} V \cap \mathbb{S}^5(0,1) \leqslant 1$. The cycle $\mathbb{S}^1.a$ thus bounds a chain in $E'_0 \subseteq V \cap \mathbb{S}^5(0,1)$, which is a finite union of circles, that provides a contradiction. □

Now we provides the proof of the Theorem 5.2.

Proof. [Proof of the Theorem 5.2]

The proof of this Theorem follows the idea of [18] and the proof of Theorem 5.1.

Assume that $B(G) \neq \emptyset$. Similarly to the proof of the Theorem 5.1 and with the same notations in this proof but for the general case, we have: since

$$\mathrm{Rank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\dots,n-1} > n-3$$

then

$$\mathrm{corank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\dots,n-1} = \dim_{\mathbb{C}} V \leqslant 1,$$

so $\dim_{\mathbb{R}} V \leqslant 2$ and $\dim_{\mathbb{R}} V \cap \mathbb{S}^{2n-1}(0,1) \leqslant 1$. The cycle $\mathbb{S}^1.a$ bounds a chain in $E'_0 \subseteq V \cap \mathbb{S}^{2n-1}(0,1)$, which is a finite union of circles, that provides a contradiction. So we have

$$IH_2^{\bar{t},c}(\mathcal{V}_G,\mathbb{R}) \neq 0, \quad IH_2^{\bar{t},cl}(\mathcal{V}_G,\mathbb{R}) \neq 0, \quad H_2^{c}(\mathcal{V}_G,\mathbb{R}) \neq 0 \text{ and } H_2^{cl}(\mathcal{V}_G,\mathbb{R}) \neq 0.$$

From the Goresky-MacPherson Poincaré duality Theorem, we have

$$IH_2^{\bar{t},c}(\mathcal{V}_G,\mathbb{R}) = IH_{2n-4}^{\bar{0},cl}(\mathcal{V}_G,\mathbb{R}) \text{ and } IH_2^{\bar{t},cl}(\mathcal{V}_G,\mathbb{R}) = IH_{2n-4}^{\bar{0},c}(\mathcal{V}_G,\mathbb{R}),$$

that implies $H_{2n-4}^{c}(\mathcal{V}_G,\mathbb{R}) \neq 0$ and $H_{2n-4}^{cl}(\mathcal{V}_G,\mathbb{R}) \neq 0$. □

Remark 5.8. The variety $\mathcal{V}_G$ associated to a polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is not unique, but the result of the theorems 5.1 and 5.2 hold for any variety $\mathcal{V}_G$ among the constructed varieties $\mathcal{V}_G$ associated to G.

With the conditions of Theorem 5.2, the result of [12] also holds, hence as a consequence of Theorem 5.2 in this paper and Theorems 2.1 and 2.6 in [12], we obtain the following corollary.

Corollary 5.9. *Let $G = (G_1, \dots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n \geqslant 4$, be a polynomial mapping such that $K_0(G) = \emptyset$. Assume that the zero set $\{z \in \mathbb{C}^n :$*

$\hat{G}_i(z) = 0, i = 1, \ldots, n-1\}$ *has complex dimension one, where* $\hat{G}_i$ *is the leading form of* G_i*. If the Euler characteristic of* $G^{-1}(z^0)$ *is bigger than that of the generic fiber, where* $z^0 \in \mathbb{C}^{n-1}$*, then*

1) $H_2(\mathcal{V}_G(\rho), \mathbb{R}) \neq 0$*, for any* ρ*,*
2) $H_{2n-4}(\mathcal{V}_G(\rho), \mathbb{R}) \neq 0$*, for any* ρ*,*
3) $IH_2^{\bar{t}}(\mathcal{V}_G(\rho), \mathbb{R}) \neq 0$*, for any* ρ*, where* $\bar{t}$ *is the total perversity.*

Proof. At first, since the zero set $\{z \in \mathbb{C}^n : \hat{G}_i(z) = 0, i = 1, \ldots, n-1\}$ has complex dimension one, then by the Theorem 2.6 in [12], any generic linear mapping L is a very good projection with respect to any regular value z^0 of G. Then if the Euler characteristic of $G^{-1}(z^0)$ is bigger than that of the generic fiber, where $z^0 \in \mathbb{C}^{n-1}$, then by the Theorem 2.1 of [12], the set $B(G) \neq \emptyset$. Moreover, the complex dimension of the set $\{z \in \mathbb{C}^n : \hat{G}_i(z) = 0, i = 1, \ldots, n-1\}$ is the complex *corank* of $(D\hat{G}_i)_{i=1,\ldots,n-1}$. Hence $\mathrm{Rank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\ldots,n-1} = n-2$, and by the Theorem 5.2 we finish the proof. □

EXAMPLE 5.10. Consider the suspension of the Broughton's example:

$$G : \mathbb{C}^3 \to \mathbb{C}^2, \quad G(z, w, \zeta) = (z + z^2 w, \zeta),$$

or, more general $G(z, w, \zeta) = (z + z^2 w, g(\zeta))$ where $g(\zeta)$ is any polynomial of variable ζ and $g'(\zeta) \neq 0$. We can check that, for any function ρ, we have always $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) \neq 0$.

REMARK 5.11. The condition $B(G) = \emptyset$ does not imply $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) = 0$, since in this case $\mathcal{S}_G$ maybe not empty.

EXAMPLE 5.12. Let

$$G : \mathbb{C}^3 \to \mathbb{C}^2, \quad G(z, w, \zeta) = (z, z\zeta^2 + w).$$

1) If we choose the function $\rho = |\zeta|^2$, then $\mathcal{S}_G = \emptyset$ and $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) = 0$.
2) If we choose the function $\rho = |w|^2$, then $\mathcal{S}_G \neq \emptyset$ and $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) \neq 0$.

REFERENCES

[1] J-P. BRASSELET, *Introduction to intersection homology and perverse sheaves*, Publicações Matemáticas do IMPA, 27o Colóquio Brasileiro de Matemática, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 2009.

[2] A. BROUGHTON, *On the topology of polynomial hypersurfaces*, In "Singularities, Part 1 (Arcata, Calif., 1981)", volume 40 of Proc. Sympos. Pure Math., pp. 167–178.

[3] L. R. G. DIAS, *Regularity at infinity and global fibrations of real algebraic maps*, Tese de doutorado.

[4] L. R. G. DIAS, M. A. S. RUAS, AND M. TIBAR, *Regularity at infinity of real mappings and a Morse-Sard Theorem*, J. Topology, 5:2 (2012), pp. 323–340.

[5] M. CHAPERON, *Jets, transversalité, singularités : petite introduction aux grandes idées de René Thom*, Exposé du 29 semtembre 2001.

[6] M. GORESKY AND R. MACPHERSON, *Intersection homology.* II, Invent. Math., 72 (1983), pp. 77–129.

[7] M. GORESKY AND R. MACPHERSON, *Intersection homology theory,* Topology, 19 (1980), pp. 135–162.

[8] R. HARDT, *Semi-algebraic local-triviality in semi-algebraic mappings*, Amer. J. Math., 102:2 (1980), pp. 291–302.

[9] Z. JELONEK, *Testing sets for properness of polynomial mappings*, Math. Ann., 315:1 (1999), pp. 1–35.

[10] V. H. HA, *Nombres de Lojasiewicz et singularités à l'infini des polynômes de deux variables complexes*, C. R. Acad. Sci. Paris Ser. I, 311 (1990), pp. 429–432.

[11] V. H. Ha and L. D. Tráng, *Sur la topologie des polynomes complexes*, Acta Math. Vietnamica, 9 (1984), pp. 21–32.
[12] V. H. Ha and T. T. Nguyen, *On the topology of polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$*, International Journal of Mathematics, 22:3 (2011), pp. 435–448.
[13] O. H. Keller, *Ganze Cremonatransformationen Monatschr*, Math. Phys., 47 (1939), pp. 229–306.
[14] K. Kurdyka, P. Orro, and S. Simon, *Semialgebraic Sard Theorem for generalized critical values*, J. Differential Geometry, 56 (2000), pp. 67–92.
[15] T. Mostowski, *Some properties of the ring of Nash functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 3:2 (1976), pp. 245–266.
[16] T. B. T. Nguyen, *Étude de certains ensembles singuliers associés à une application polynomiale*, Thèse, Université d'Aix Marseille, http://tel.archives-ouvertes.fr/ ID : tel-00875930.
[17] T. B. T. Nguyen, *La méthode des façons*, ArXiv: 1407.5239.
[18] T. B. T. Nguyen, A. Valette, and G. Valette, *On a singular variety associated to a polynomial mapping*, Journal of Singularities, 7 (2013), pp. 190–204.
[19] M. Oka, *A Non-degenerate mixed function*, Kodai Math. J., 33 (2010), pp. 1–62.
[20] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc., 75:2 (1969), pp. 240–284.
[21] A. Valette and G. Valette, *Geometry of polynomial mappings at infinity via intersection homology*, Ann. I. Fourier, 64:5 (2014), pp. 2147–2163.