Tropical plane curves with first order tangency

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The complete solution is given by Kontsevich and Manin [3] and independently by Ruan and Tian [6] by giving an explicit recursion formula.

$$n_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left\{ \binom{3d-4}{3d_1-2} d_1^2 d_2^2 - \binom{3d-4}{3d_1-2} d_1^3 d_2 \right\} n_{d_1} n_{d_2} \quad (1.1)$$

Base case for the recursion $n_1 = 1$.

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Gathmann-Markwig, [2] 2008

Rederived the same recursion formula 1.1 by modifying the WDVV equation entirely in tropical geometry.

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Question 1.3

What is the number of rational plane degree d curves that pass through $3d - 2$ generic points and, at the same time, are tangent to a generic line?

Tropical analogue for Question 1.3

We show that there is a Kontsevich-type recursion to compute the above characteristic numbers $N_d^{\mathsf{T}_1}$ using the procedure followed in Gathman-Markwig.

Theorem 1.4 (_____, A. Subramaniam, and B. Khan, [5])

The recursion for $N_d^{\mathsf{T}_1}$ is given by

$$N_d^{\mathsf{T}_1} = \sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} \binom{3d-4}{3d_1-2} d_1^2 d_2^2 \; N_{d_1}^{(\mathsf{T}_0)_{\mathrm{pt}}} \; N_{d_2}^{(\mathsf{T}_0)_{\mathrm{pt}}} - \binom{3d-4}{3d_2-3} d_1 d_2 \; N_{d_1}^{\mathsf{T}_0\mathsf{T}_0} \; n_{d_2}.$$

Main idea for Kontsevich's recursion (classical WDVV):

Let $\overline{M}_{0,n}(\mathbb{CP}^2, d)$ be the Kontsevich moduli space parametrizing plane rational curves of degree d with n distinct smooth marking.

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Let $D(ij \mid kl)$ be the divisor in $\overline{M}_{0,4}$ representing the two componented rational curves where the marked points (x_i, x_j) lie in one component and (x_k, x_l) lie on the other component. Thus, we have the following divisorial identity in $H^*(\overline{M}_{0,4},\ \mathbb{Z})$

$$D(ij|kl) \equiv D(ik|jl)$$

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Let us consider the forgetful map

$$\pi : \overline{M}_{0,3d}(\mathbb{CP}^2, d) \longrightarrow \overline{M}_{0,4}.$$

Then

$$\pi^* D(ij|kl) \equiv \pi^* D(ik|jl).$$

Let us define a 1 dimensional class $\mathcal{Z}$ in $\overline{M}_{0,3d}(\mathbb{CP}^2, d)$ as

$$\mathcal{Z} := ev_1^*([\mathrm{L}]) \cdot ev_2^*([\mathrm{L}]) \cdot ev_3^*([pt]) \cdot ev_4^*([pt]) \cdot \left(ev_5^*([pt]) \cdots ev_{3d}^*([pt])\right).$$

We now consider the intersection

$$\pi^* D(ij|kl) \cdot \mathcal{Z} = \pi^* D(ik|jl) \cdot \mathcal{Z}$$

inside the moduli space $\overline{M}_{0,3d}(\mathbb{CP}^2, d)$, giving us the recursion.

Tropical curves

Definition 2.1

An abstract tropical curve is a connected graph Γ *of genus-zero whose all vertices have valence at least* 3*. An* n*-marked tropical curve is a tuple* $(\Gamma, x_1, \ldots, x_n)$ *where* $x_1, \ldots, x_n \in \Gamma_0^\infty$*.*

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An abstract tropical curve is a connected graph Γ *of genus-zero whose all vertices have valence at least* 3. *An n-marked tropical curve is a tuple* $(\Gamma, x_1, \ldots, x_n)$ *where* $x_1, \ldots, x_n \in \Gamma_0^\infty$.

The tropical moduli space of n-marked tropical curves will be denoted by $\mathcal{M}_n$ (Analogous to $\overline{M}_{0,n}$ in the classical setting). Then an element of $\mathcal{M}_4$ has four possible combinatorial types:

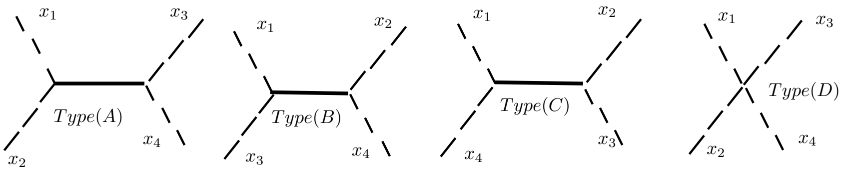

Figure: Tropical curves of types (A), (B), (C), and (D)

Tropical plane curves

Definition 2.2

An n-marked plane tropical curve will be a tuple $(\Gamma, x_1, \ldots, x_n, h)$*, where* Γ *is an abstract tropical curve,* $x_1, \ldots, x_n$ *are distinct unbounded ends of* Γ*, and* $h : \Gamma \longrightarrow \mathbb{R}^2$ *satisfying the following:*

1. *On each edge of* Γ*, the map* $h(t) = a + t \cdot v$ *for some* $a \in \mathbb{R}^2$*,* $v \in \mathbb{Z}^2$*.*
2. *Balancing condition on each vertex.*
3. *Every marked point is contracted to a point in* $\mathbb{R}^2$ *by* h*.*

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- Let us denote $\mathcal{M}_{d,n}$ (tropical analogue of $\overline{M}_{0,n}(\mathbb{CP}^2, d)$) to be the tropical moduli spaces of plane curves of degree d.
- We have well-defined evaluation and forgetful morphisms.

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1. *On each edge of* Γ, *the map* $h(t) = a + t \cdot v$ *for some* $a \in \mathbb{R}^2$, $v \in \mathbb{Z}^2$.
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Tropical WDVV algorithm : Compute the numbers of tropical plane curves satisfying given point conditions and mapping to two curves in $\mathcal{M}_4$ that are of type (A) and type (B), respectively.

Fix $d \geq 2$, and let $n = 3d$. We set

$$\pi : ev_1^1 \times ev_2^2 \times ev_3 \times \ldots \times ev_n \times ft_4 : \mathcal{M}_{d,n} \longrightarrow \mathbb{R}^{2n-2} \times \mathcal{M}_4$$

- This does not depend on the choice of a point in $\mathcal{M}_4$. Hence, we get the tropical version of Kontsevich's recursion.

Tropical WDVV

Let $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{2n-2} \times \mathcal{M}_4$ be two π general points with large $\mathcal{M}_4$ coordinates of type (A) and type (B), respectively. Then, the recursion is obtained from

$$deg_{\mathcal{P}}(\pi) = deg_{\mathcal{Q}}(\pi).$$

Main idea for the first order tangency to a line

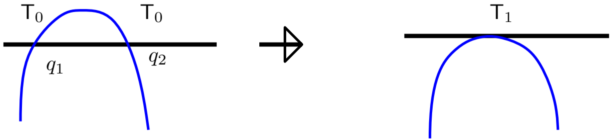

Figure: Two transverse intersections limiting to a first order tangency

- The right hand side of the above shows a rational curve tangent to a line at some point.
- The left hand side represents a rational curve intersecting a line at two distinct points transversely.

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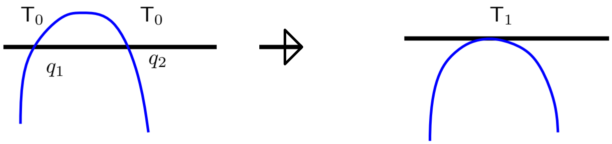

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We apply the WDVV equation against the left hand side and, by the expected degeneration in this process, produce rational curves with first-order tangency.

Let $n = 3d - 1$. We consider the map

$$\tilde{\pi} : \mathcal{M}_{d,n} \longrightarrow \mathbb{R}^{2n-1} \times \mathcal{M}_4,$$

where the image of the first two marked points lie on a tropical line. The image of the rest describes both the coordinates in $\mathbb{R}^2$ of the other markings (since other markings meet distinct general points).

Tropical WDVV for tangency

Let $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{2n-1} \times \mathcal{M}_4$ be two $\tilde{\pi}$ general points with large $\mathcal{M}_4$ coordinates of type (A) and type (B), respectively. Then, the recursion is obtained from

$$deg_{\mathcal{P}}(\tilde{\pi}) = deg_{\mathcal{Q}}(\tilde{\pi}).$$

Thank You.

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