The Chirotropical Grassmannian

Dario Antolini,

Università di Trento

Joint work with Nick Early

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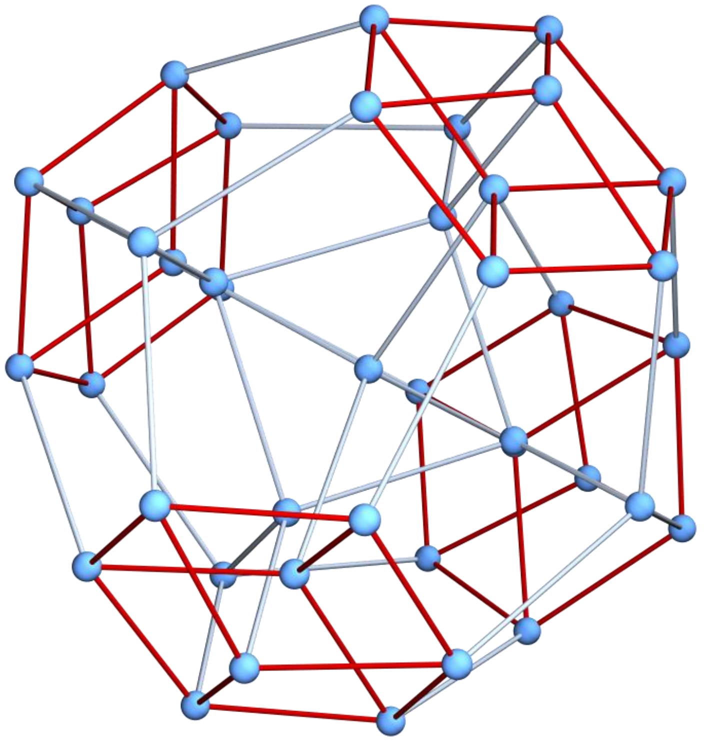
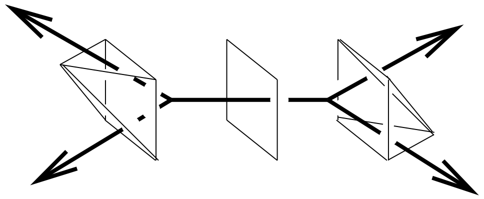
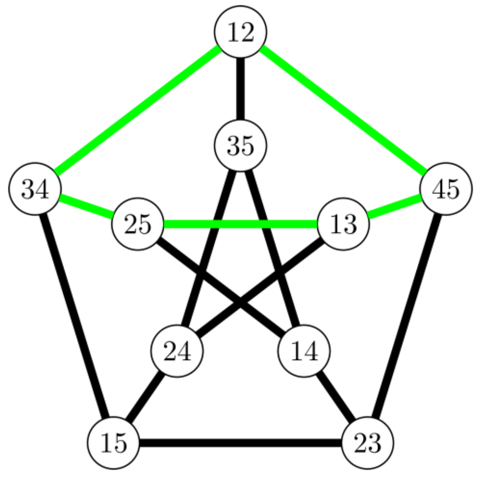




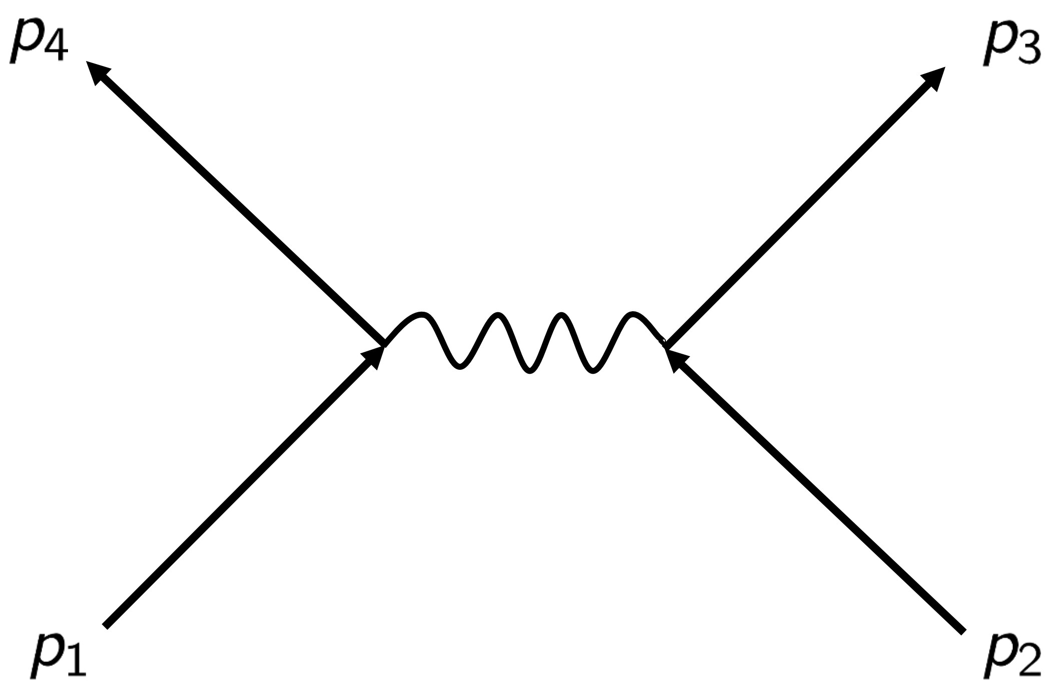


Summary

- Preliminaries and motivations:
  - Scattering amplitudes, Feynman Diagrams
  - The Tropical Grassmannian Trop $G(2, n)$ and phylogenetic trees
- Outline of our contribution:
  - Chirotopes
  - Chirotropical Grassmannians and chirotropical Dressians
  - $\mathrm{Trop}^{\chi}G(3, n)$ vs $\mathrm{Dr}^{\chi}(3, n)$
  - Computation of $\mathrm{Trop}^{\chi}G(3, n)$ for $n = 6, 7, 8$
  - $\mathrm{Trop}^{\chi}G(3, 6)$ is polytopal

Preliminaries and motivations

Scattering process

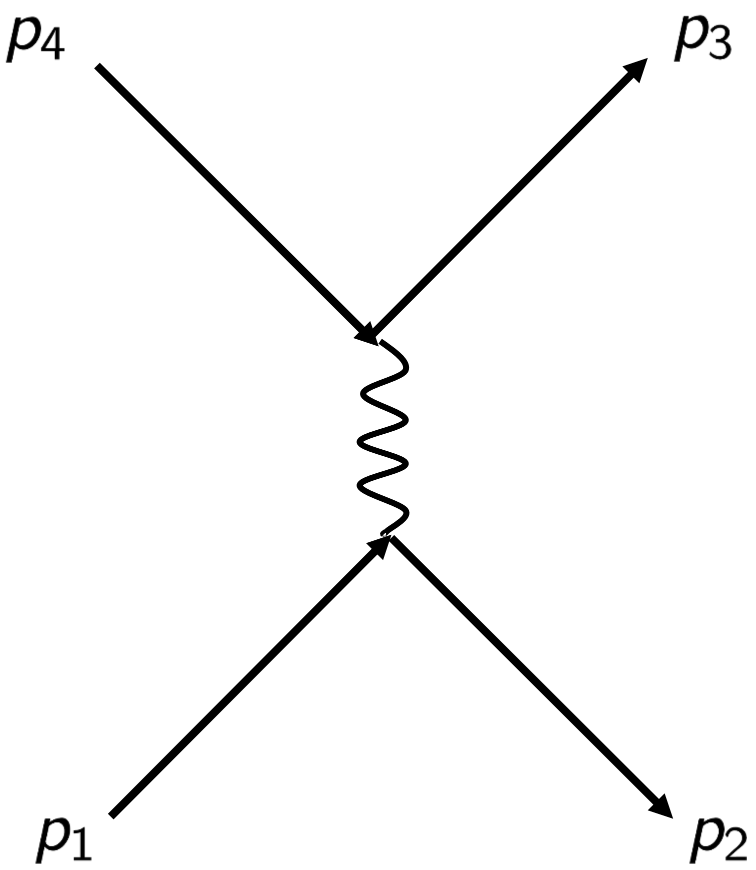


Scattering process

p_4 p_3

p_1 p_2

Scattering process



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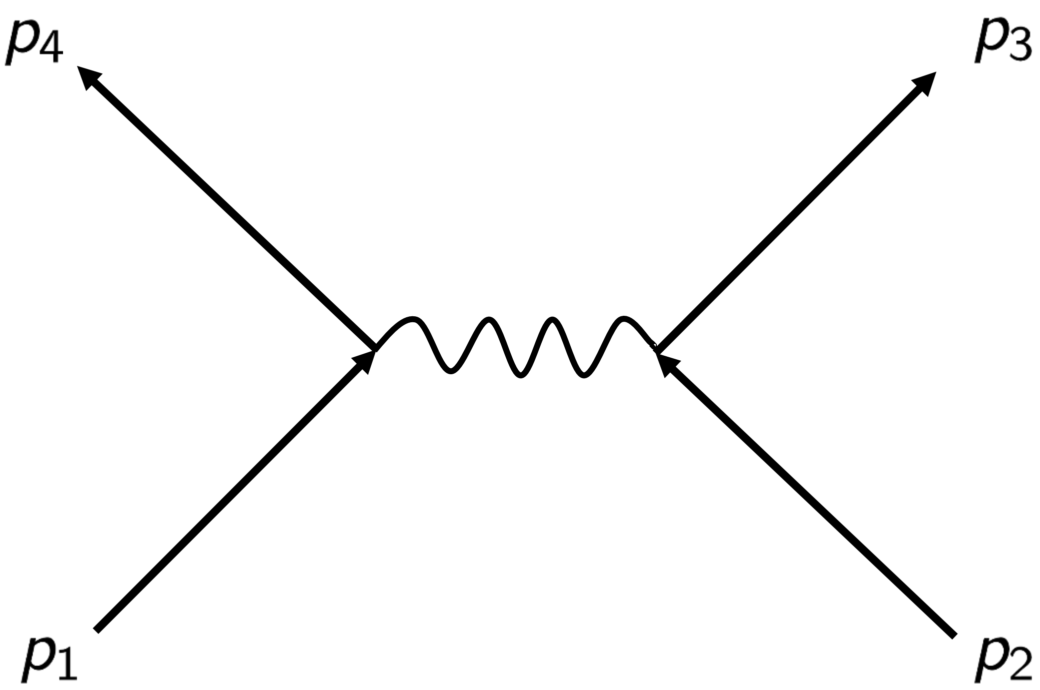

Scattering amplitudes are related to probability of interactions between particles in a scattering process

Mandelstam invariants

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such that:

- $s_{ij} = s_{ji}$ for all $i, j \in [n]$
- $s_{ii} = 0$ for all $i \in [n]$
- $\sum_{j \neq i} s_{ij} = 0$ for all $i \in [n]$

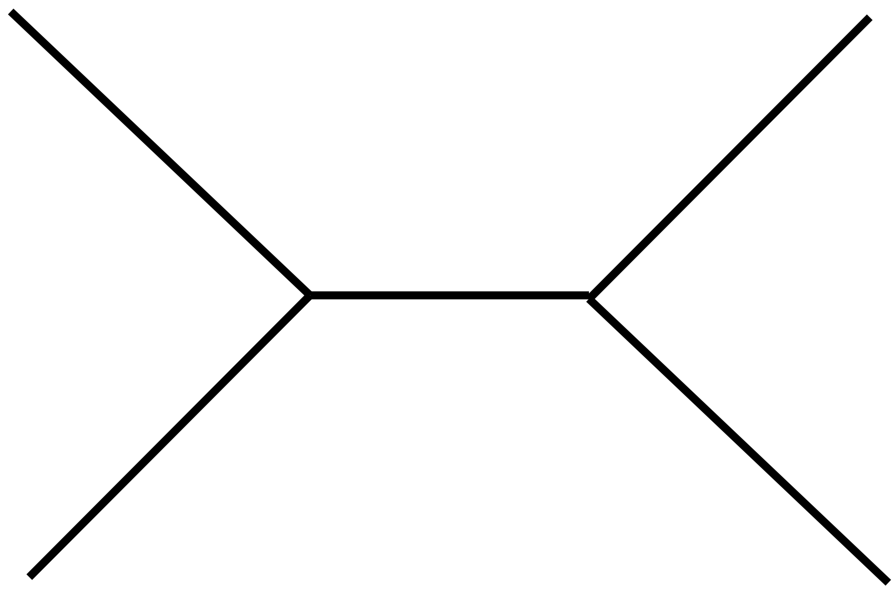
Feynman expansion

Perturbative expansion using Feynman diagrams:

$$m_n^{\text{tree}}(\mathfrak{s}) = \sum_{\substack{G \text{ Feynman diagram} \\ G \text{ tree}}} \prod_{e \in IE(G)} \frac{1}{X_e(\mathfrak{s})}$$

Feynman diagrams (ϕ^3 theory)

G Feynman diagram on n particles, G tree $\overset{\sim}{\longleftrightarrow}$ phylogenetic tree on n leaves



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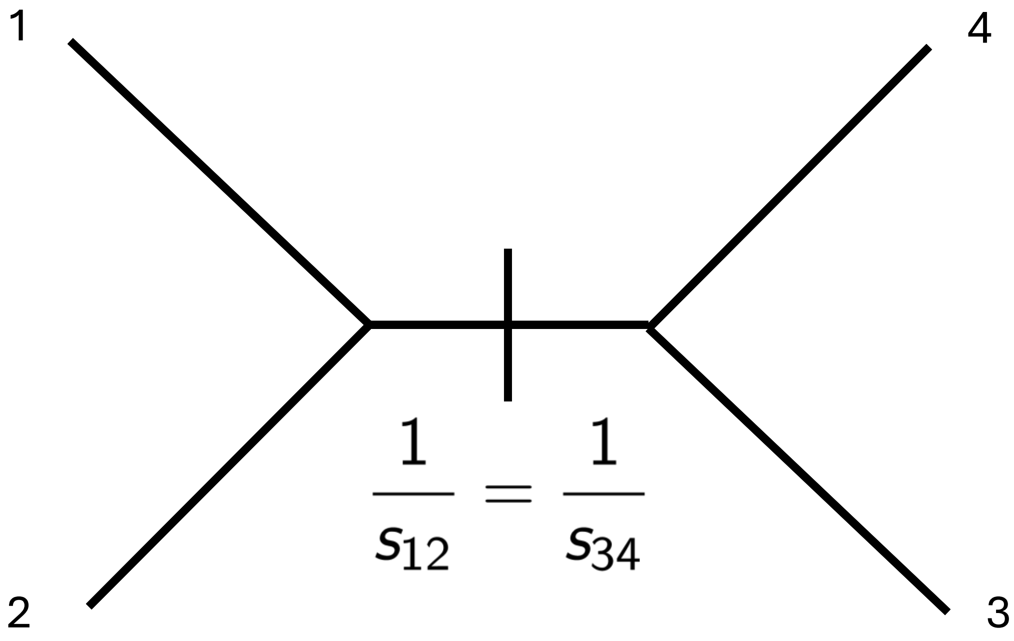

$$\frac{1}{s_{12}} = \frac{1}{s_{34}}$$

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The connection with Tropical Geometry lies in the **Tropical Grassmannian** Trop G(2, n).

Tropical Grassmannians

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This is done by considering the Grassmannian in its Plücker embedding:

$$\begin{aligned} \mathrm{G}(k, n) &\hookrightarrow \mathbb{P}^{\binom{n}{k}-1} \\ g &\mapsto [p_{i_1 \cdots i_k}(g)]_{i_1 \cdots i_k \in \binom{[n]}{k}} \end{aligned}$$

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The ideal $I(\mathrm{G}(k,n))$ is generated by quadrics, the **Plücker relations**.
These include the **3-term Plücker relations**.
Example. $I(\mathrm{G}(2,4)) = (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})$.

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The ideal $I(\mathrm{G}(k,n))$ is generated by quadrics, the **Plücker relations**.
These include the **3-term Plücker relations**.
We will always work in the constant coefficients case (trivial valuation)
$\rightsquigarrow$ Trop $\mathrm{G}(k,n)$ is a polyhedral fan.

Tropical Grassmannians

[SS03] Speyer and Sturmfels $\rightsquigarrow$ The tropical Grassmannian Trop $\mathrm{G}(k, n)$

$$\text{Trop } \mathrm{G}(2, n) \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \text{Phylogenetic trees on} \\ n \text{ leaves} \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \text{Feynman diagrams on} \\ n \text{ particles in } \phi^3 \text{ theory} \end{matrix} \right\}$$

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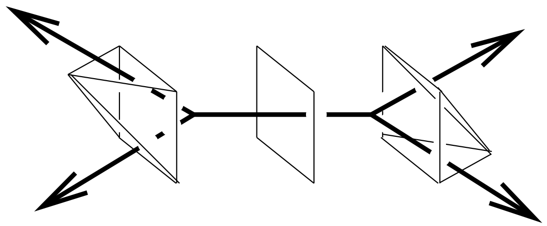

Credit: David Speyer, *Tropical Linear Spaces* (2004)

Tropical Grassmannians

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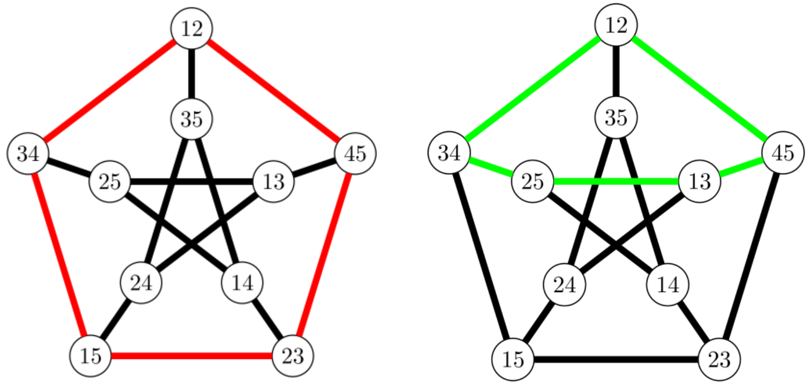
Trop $G(2,n)$ is covered by relabelings of $\text{Trop}^+G(2,n)$ $\overset{\sim}{\longleftrightarrow}$ partial decompositions of amplitudes



Tropical Grassmannians

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$$\begin{array}{c} \text{Tropical Plücker relations} \\ \& \\ \text{compatible tropical Plücker vectors} \end{array} \xleftrightarrow{\sim} \text{Feynman rules}$$

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Question. What happens for the next family of Grassmannians Trop $G(3, n)$? What is the correspondence on the Physics side?

Physics side: CEGM theory

The generalized Mandelstam invariants are symmetric order 3 tensors $\mathfrak{s} = (s_{ijk})_{i,j,k=1}^{n}$ satisfying linear equations:

$$\sum_{j,k} s_{ijk} = 0, \;\; s_{iij} = 0 \quad \text{for all } i, j \in [n]$$

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CEGM theory $\rightsquigarrow$ analogue of the tree-level amplitude: $m_n^{3,\text{ tree}}(\chi; \mathfrak{s})$

Feynman rules for Trop $\mathrm{G}(3, n)$

Trop $\mathrm{G}(3, n)$ is **not** covered by relabelings of $\mathrm{Trop}^+\mathrm{G}(3, n)$ (for $n \geq 6$).

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These were introduced by **[CEZ22]**.

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Moreover, for $n = 6, 7, 8$*:

$$m_n^{3,\text{ tree}}(\chi; \mathfrak{s}) = \sum_{\substack{F \text{ maximal cone} \\ \text{of } \text{Trop}^\chi G(3,n)}} \frac{N_F(\mathfrak{s})}{\prod_{X \text{ ray of } F} X(\mathfrak{s})},$$

* in the case $(3, 8)$, **[CEZ22]** verified it numerically with very high precision.

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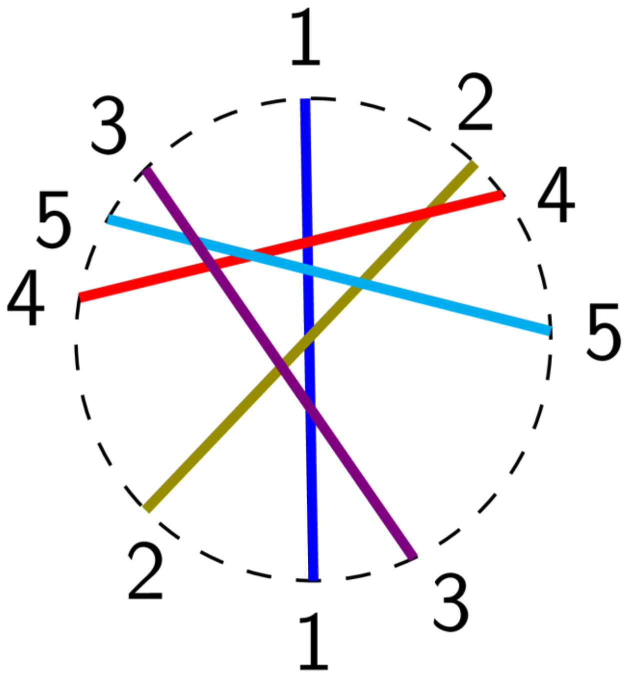
$\rightsquigarrow$ the CEGM bi-adjoint amplitude at tree-level for $k = 3$ can be computed from the maximal cones of each $\mathrm{Trop}^{\chi}\mathrm{G}(3,n)$, $n = 6, 7, 8$.

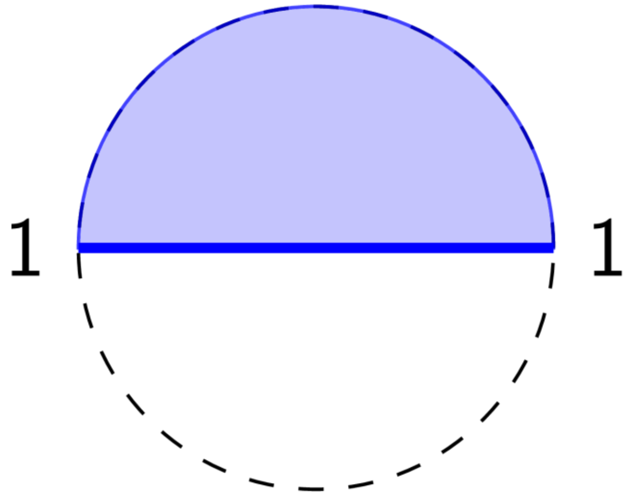
Outline of our contribution

Realizable matroids as hyperplane arrangements



Example. Line arrangement in $\mathbb{P}^2$

Realizable chirotopes (oriented matroids) as signed hyperplane arrangements



Example. Signed line in $\mathbb{P}^2$

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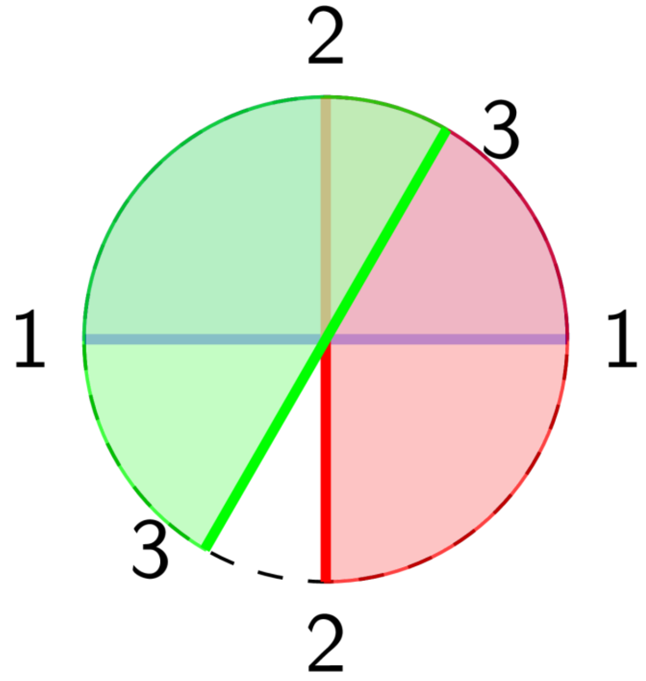

Example. Signed line arrangement in $\mathbb{P}^2$

Realizable chirotopes

Consider n lines in $\mathbb{P}^2$ cut out by $\boldsymbol{\ell}_i \cdot (x_0, x_1, x_2) = 0$, $\boldsymbol{\ell}_i \in \mathbb{R}^3$

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$$\rightsquigarrow g_{\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_n} = \begin{pmatrix} | & & | \\ \boldsymbol{\ell}_1 & \cdots & \boldsymbol{\ell}_n \\ | & & | \end{pmatrix} \in \mathrm{G}(3, n)$$

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The oriented matroid structure is stored into the signs of the Plücker coordinates:

$$\text{sign } p_{ijk}(g_{\ell_1,\dots,\ell_n}), \quad ijk \in \binom{[n]}{3}.$$

(Uniform, realizable) chirotopes

Def. A **uniform, realizable chirotope** is a vector $\chi \in \{-1, 1\}^{\binom{n}{3}}$ of the form:

$$\chi_{ijk} = \operatorname{sign} p_{ijk}(g), \quad g \in \mathrm{G}(3, n)^\circ$$

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Up to simultaneous sign change by $GL_3(\mathbb{R})$
$\rightsquigarrow$ we can assume $\chi_{123} = 1$

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Example. $\chi = + = (1, \ldots, 1) \in \binom{n}{3}$ will give us the positive tropicalization $\mathrm{Trop}^{+}\mathrm{G}(3, n)$.

Isomorphism classes

We define the configuration space:

$$X(3, n) = \mathrm{G}(3, n)^{\circ}/(\mathbb{R}^{\times})^{n}$$

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$$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \\ z_1 & \cdots & z_n \end{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \\ \lambda_1 y_1 & \cdots & \lambda_n y_n \\ \lambda_1 z_1 & \cdots & \lambda_n z_n \end{pmatrix}$$

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$$\rightsquigarrow \{\text{columns}\} \overset{\sim}{\longleftrightarrow} \{\text{points in } \mathbb{P}^2\}$$

$\rightsquigarrow X(3, n) =$ the configuration space of n distinct points in $\mathbb{P}^2$ modulo $\mathrm{Aut}(\mathbb{P}^2)$

Isomorphism classes

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Two uniform, realizable chirotopes $\chi, \chi' \in \{-1,1\}^{\binom{n}{3}}$ are **isomorphic** if they are related by exchanging and scaling columns.

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Remark. There are finitely many isomorphism classes of uniform, realizable chirotopes $\chi \in \{-1, 1\}^{\binom{n}{3}}$.

Theorem [Finschi01]. For $n = 6, 7, 8$, we have $4, 11, 135$ isomorphism classes of uniform, realizable chirotopes respectively.

Chirotopes

Def. A **chirotope** is a vector $\chi \in \{-1,1\}^{\binom{n}{3}}$ of the form:

$$\chi_{ijk} = \text{sign } p_{ijk}(g), \quad g \in \mathrm{G}(3,n)^{\circ}$$

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Chirotropicalization

Tropicalization:

$$f = \sum_{\alpha \in S} c_\alpha x^\alpha \rightsquigarrow \mathrm{Trop}(f)(x) = \min_{\alpha \in S} \{\alpha \cdot x\}$$

$$\rightsquigarrow \mathrm{Trop}\, V(f) = \left\{ x \in \mathbb{R}^{\binom{n}{3}} : \begin{array}{c}\text{the minimum in } \mathrm{Trop}(f)(x) \\ \text{is attained at least twice}\end{array} \right\}$$

Chirotropicalization

Given a chirotope $\chi \in \{-1,1\}^{\binom{n}{3}}$ and f a polynomial in $\binom{n}{3}$ variables, we have:

$$f = \sum_{\alpha \in S} c_\alpha x^\alpha = f_\chi^+ + f_\chi^-, \quad f_\chi^+ = \sum_{\substack{\alpha \in S \\ c_\alpha \chi^\alpha > 0}} c_\alpha x^\alpha, \; f_\chi^- = \sum_{\substack{\alpha \in S \\ c_\alpha \chi^\alpha < 0}} c_\alpha x^\alpha$$

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$$\rightsquigarrow \mathrm{Trop}^\chi V(f) = \left\{ x \in \mathbb{R}^{\binom{n}{3}} : \begin{array}{c} \text{the minimum in } \mathrm{Trop}(f)(x) \text{ is attained at least twice and} \\ \text{at least once in both } \mathrm{Trop}(f_\chi^+)(x) \text{ and } \mathrm{Trop}(f_\chi^-)(x) \end{array} \right\}$$

Chirotropical Grassmannian and Dressian

$$\mathrm{Trop}^{\chi}\mathrm{G}(3,n) = \bigcap_{f \in I(\mathrm{G}(3,n))} \mathrm{Trop}^{\chi} V(f)$$

$$\mathrm{Dr}^{\chi}(3,n) = \bigcap_{\substack{g \text{ 3-term} \\ \text{Plücker relation}}} \mathrm{Trop}^{\chi} V(g)$$

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Theorem 1 [A-E]. For $n = 6, 7, 8$, $\mathrm{Trop}^{\chi}\mathrm{G}(3,n) = \mathrm{Dr}^{\chi}(3,n)$ set-theoretically and $\mathrm{Trop}\ \mathrm{G}(3,n)$ is covered by relabelings of chirotropical Grassmannians.

Feynman rules for Trop G(3, n)

- χ-tropical Plücker relation $\mathrm{Trop}^{\chi}(F)$, F Plücker relation;
- χ-compatible pairs $v_1, v_2 \in \mathrm{Dr}^{\chi}(3, n)$ such that $v_1 + v_2 \in \mathrm{Dr}^{\chi}(3, n)$.

Algorithm for computing $\mathrm{Dr}^{\mathcal{X}}(3, n)$

Input : - R, the list of rays of $\mathrm{Dr}(3, n)$;

Algorithm for computing $\mathrm{Dr}^{\chi}(3, n)$

$R^{\chi} \leftarrow \{r \in R \mid r \in \mathrm{Dr}^{\chi}(3, n)\}$; / rays of $\mathrm{Dr}^{\chi}(3, n)$

Algorithm for computing $\mathrm{Dr}^{\chi}(3, n)$

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R^χ ← {r ∈ R | r ∈ Dr^χ(3, n)} ;                          / rays of Dr^χ(3, n)
CompatibleRays^χ ← {(r_1, r_2) ∈ R^χ × R^χ | r_1 + r_2 ∈ Dr^χ(3, n)} ;
```

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$\mathrm{CompatibleRays}^{\chi} \leftarrow \{(r_1, r_2) \in R^{\chi} \times R^{\chi} \mid r_1 + r_2 \in \mathrm{Dr}^{\chi}(3, n)\}$;

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We implement the algorithm in SageMath and we compute $\mathrm{Trop}^{\chi}\mathrm{G}(3, n) = \mathrm{Dr}^{\chi}(3, n)$ for $n = 6, 7, 8$ and for all $4, 11, 135$ isomorphism classes of chirotopes $\chi \in \{-1, 1\}^{\binom{n}{3}}$ (our results will be available on a Zenodo page).

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They are **pure** $2(n-4)$-dimensional polyhedral fans. Moreover, when computing lower dimensional faces, **only pairwise intersections of maximal cones are needed**.

$\mathrm{Trop}^{x} G(3,6)$ is polytopal

We found **positive parameterizations** for all four isomorphism types of $\mathrm{Trop}^{x} G(3,6)$: each is the normal fan to a Newton polytope $\mathrm{Newt}(f^{x})$, where f^{x} is a polynomial in 4 variables with positive coefficients.

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$\rightsquigarrow$ $X(3,6)$ is tiled by 372 **chirotopal configuration spaces**:

$$X^{\chi}(3,6) = \{g \in X(3,6) : \exists g' \in \mathrm{G}^{\circ}(3,6) \text{ s.t. } g = [g']_{(\mathbb{R}^{\times})^n} \\ \text{and sign } p_{ijk}(g') = \chi_{ijk} \; \forall ijk\}.$$

and each of these tiles is a positive geometry with an explict canonical form.

Example

For the chirotope:

$$\chi = (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1,-1) \in \{-1,1\}^{\binom{6}{3}}$$

Step 1. Starting from the positive parametrization of $X^+(3,6)$, we constructed the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & \frac{1}{y_1+1} & \frac{y_2y_3+y_1y_2y_4y_3+y_2y_4y_3+y_4y_3+y_3+y_1y_2y_4+y_2y_4+y_4+1}{y_1y_2y_3+y_2y_3+y_1y_2y_4y_3+y_2y_4y_3+y_4y_3+y_3+y_1y_2y_4+y_2y_4+y_4+1} \\ 0 & 1 & 0 & 1 & -\frac{1}{(y_1+1)y_2} & \frac{y_2y_3+y_1y_2y_4y_3+y_2y_4y_3+y_4y_3+y_3+y_1y_2y_4+y_2y_4+y_4+1}{(y_2y_3+y_3+1)(y_1y_2y_3+y_2y_3+y_1y_2y_4y_3+y_2y_4y_3+y_4y_3+y_3+y_1y_2y_4+y_2y_4+y_4+1)} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Example

Step 2. Clear denominators:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & y_2 & \dots \\ 0 & 1 & 0 & 1 & -1 & \dots \\ 0 & 0 & 1 & 1 & (y_1+1)y_2 & \dots \end{bmatrix}$$

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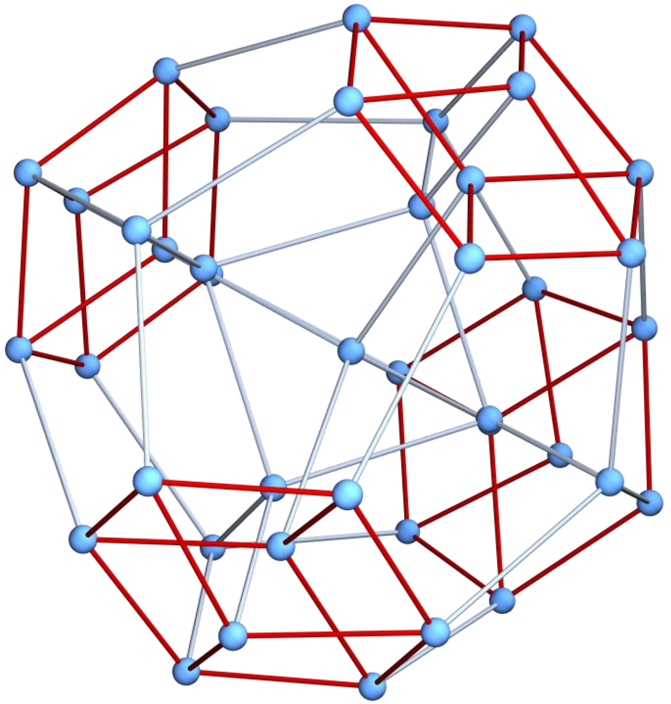
Step 3. Take 3×3 minors and their irreducible factors
Step 4. $f^X =$ product of all these irreducible factors.

Example

$$f^x(y_1, y_2, y_3, y_4) = \begin{array}{l} (y_1 y_2 y_4 + y_2 y_3 y_4 + y_2 y_3 + y_2 y_4 + y_3 y_4 + y_3 + y_4 + 1) \cdot (y_1 y_2 y_4 + y_2 y_4 + y_4 + 1) \cdot (y_1 y_2 + y_2 + 1) \cdot \\ (y_1 y_2 y_3 y_4 + y_1 y_2 y_3 + y_1 y_2 y_4 + y_2 y_3 y_4 + y_2 y_3 + y_2 y_4 + y_3 y_4 + y_3 + y_4 + 1) \cdot \\ (y_2 + 1) \cdot (y_1 y_2 y_3 y_4 + y_1 y_2 y_4 + y_2 y_3 y_4 + y_2 y_3 + y_2 y_4 + y_3 y_4 + y_3 + y_4 + 1) \cdot \\ (y_2 y_3 + y_3 + 1) \cdot (y_1 + 1) \cdot (y_3 + 1) \cdot (y_2 y_3 y_4 + y_2 y_3 + y_2 y_4 + y_3 y_4 + y_3 + y_4 + 1) \end{array}$$

Example

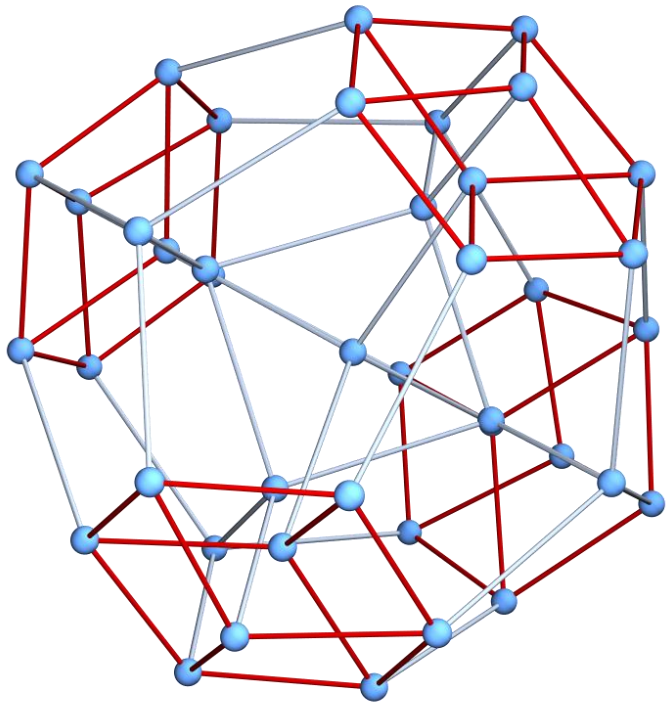
$$\mathrm{Newt}(f^{x}) =$$



Future directions

- exploring positive parametrizations of $X^{\chi}(k, n)$
- implementing the algorithm for the cases in which this parametrization exists.

Thanks for your attention!



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