TAGSS 2024 SUMMER SCHOOL

Tropicalizing Moduli Spaces

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1. Stable graphs and tropical curves

(1) Show that the genus and stability of weighted graphs with legs is preserved by weighted contractions.

(2) Draw a picture of $\mathcal{G}_{1,3}$.

(3) Prove that the stability condition implies that for fixed g and n, there are only finitely many isomorphim classes of (g,n)-stable graphs and, for fixed g and n, compute the maximal number of edges and vertices of a stable graph of type (g,n). Conclude that $M_{g,n}^{\mathrm{trop}}$ is pure dimensional of real dimension $3g-3+n$.

(4) Show that a nodal curve X with n marked points is stable (of genus g) if and only if its dual graph G_X is stable (of genus g). Show that $\mathcal{G}_{g,n}$ is non empty if and only if $2g-2+n>0$.

(5) Describe the boundary divisors δ_0 and $\delta_{i,I}$ of $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$.

(6) Show that the locus of pure tropical curves (i.e., curves for which $w \equiv 0$) is open and dense in $M_{g,n}^{\mathrm{trop}}$.

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2. Divisors on nodal curves and stability conditions

(1) Let Γ be a tropical curve. Show that Principal divisors are a subgroup of $\mathrm{Div}^0(\Gamma)$.

(2) Show that every divisor D is linearly equivalent to an admissible divisor.

(3) Show that there are finitely many quasistable graphs of fixed genus.

(4) Describe the canonical divisor of a weighted graph and check that it has total degree $2g-2$.

3. Moduli of covers and special tropical curves

(1) Consider the tropical curve Γ indicated in Figure 1. For which values of $l(e_1)$ and $l(e_2)$ is Γ hyperelliptic?

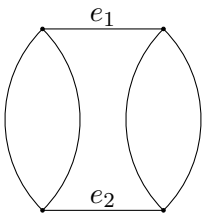

Figure 1

(2) Indicate all combinatorial types of pure tropical curves of genus 3 which are hyperelliptic. Instead, which ones are trigonal?