TAGSS 2024 summer school

Tropicalizing Moduli Spaces

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1. Stable graphs and tropical curves

(1) Show that the genus and stability of weighted graphs with legs is preserved by weighted contractions.

(2) Draw a picture of $\mathcal{G}_{1,3}$.

(3) Prove that the stability condition implies that for fixed g and n, there are only finitely many isomorphim classes of (g,n)-stable graphs and, for fixed g and n, compute the maximal number of edges and vertices of a stable graph of type (g,n). Conclude that $M_{g,n}^{\mathrm{trop}}$ is pure dimensional of real dimension $3g-3+n$.

(4) Show that a nodal curve X with n marked points is stable (of genus g) if and only if its dual graph G_X is stable (of genus g). Show that $\mathcal{G}_{g,n}$ is non empty if and only if $2g-2+n>0$.

(5) Describe the boundary divisors δ_0 and $\delta_{i,I}$ of $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$.

(6) Show that the locus of pure tropical curves (i.e., curves for which $w \equiv 0$) is open and dense in $M_{g,n}^{\mathrm{trop}}$.

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2. Divisors on nodal and tropical curves and stability conditions

(1) Let Γ be a tropical curve. Show that Principal divisors are a subgroup of $\mathrm{Div}^0(\Gamma)$.

(2) Show that every divisor D is linearly equivalent to an admissible divisor.

(3) Show that there are finitely many quasistable graphs of fixed genus.

(4) Describe the canonical divisor of a weighted graph and check that it has total degree $2g-2$.

(5) Consider the tropical curve Γ indicated in Figure 1. For which values of $l(e_1)$ and $l(e_2)$ is Γ hyperelliptic?

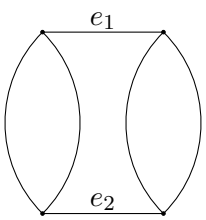

FIGURE 1

(6) Indicate all combinatorial types of pure tropical curves of genus 3 which are hyperelliptic. Instead, which ones are trigonal?

(1) Consider then the canonical polarization in degrees g and $g-1$ and discuss in which case it is non-degenerate.

(2) Theta characteristics are square roots of the canonical divisor, so for group theoretical reasons we should have $2^{b_1(\Gamma)}$ theta characteristics on a tropical curve Γ. Find all theta characteristics on binary curves and try to guess what the picture is in the general situation. Same question for square roots of the trivial divisor.

(3) Suppose that ϕ is a nondegenerate polarization of a tropical curve Γ. Then the support of a ϕ-semistable divisor is a quasistable model of Γ. Moreover, by removing from the curve the interior points on edges which support the divisor the curve remains connected. Conclude that in this case the dimension of $\mathrm{Jac}^{\phi}(\Gamma)$ is equal to $b_1(\Gamma)(= g$ if Γ is pure).

(4) Try to get the the stratified picture of $\mathrm{Jac}(\Gamma)$ for the theta graph by considering the canonical polarization in degree g. Check the connection with break divisors.