Spectral Data for Real Regular KP2 solutions and tropicalization of M-curves

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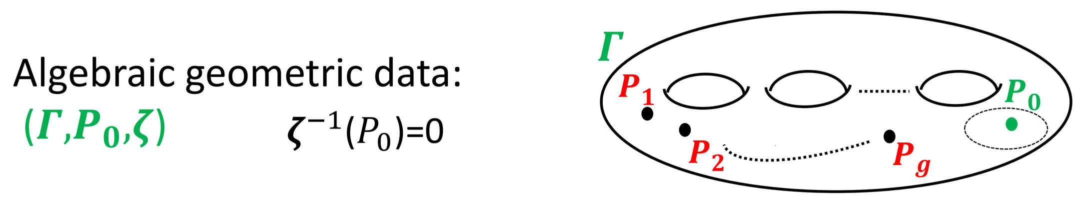
KP2 equation and algebraic geometry

KP2 equation : $(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$

is the first member of the most relevant $2+1$ integrable hierarchy and it has turned relevant in the solution to problems in algebraic geometry.
We focus on two classes of real KP2 solutions and their relations with real algebraic geometry to provide a combinatorial approach to tropicalization of M-curves:

- Real regular finite–gap KP2 solutions are parametrized by degree g real regular non–special divisors on genus g M-curves [DN-1988]
- Real regular multiline KP2 solitons are parametrized by points in totally non–negative Grassmannians $Gr^{\geq 0}(k,n)$ [KW-2013], where $Gr^{\geq 0}(k,n) \equiv GL_k^+ \backslash Mat^{\text{TNN}}_{k,n}$.

Finite-gap KP2 solutions and algebraic geometry



[Kr-1976] Given a non–singular genus g algebraic curve Γ with marked point P_0, families of KP2 smooth quasi–periodic solutions

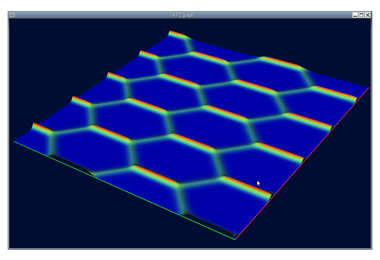
$$u(x,y,t) = 2\partial_x^2 \log \Theta(xU^{(1)} + yU^{(2)} + tU^{(3)}) + c$$

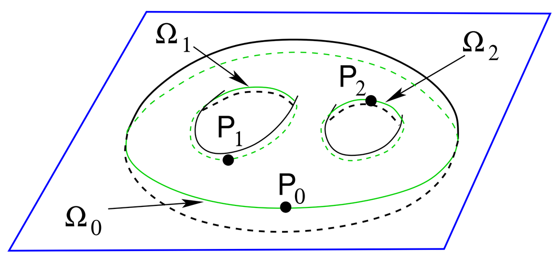
on (Γ, P_0) are parametrized by non special divisors $\mathcal{D} = (P_1, \dots, P_g)$.
There exists a unique normalized KP2 wave–function $\Psi(P,\vec{x})$, meromorphic on $\Gamma \backslash \{P_0\}$, with poles in $\mathcal{D}$ and asymptotics at P_0 $(\zeta^{-1}(P_0) = 0)$:

$$\Psi(\zeta,\vec{x}) = \left(1 - \frac{w_1(\vec{x})}{\zeta} + O(\zeta^{-2})\right) e^{\zeta x + \zeta^2 y + \zeta^3 t + \cdots} \quad (\zeta \to \infty).$$

[DN-1988]: Smooth, real (quasi–)periodic KP2 solutions $u(x,y,t)$ correspond to real and regular divisors on smooth M–curves:

- Γ possesses an antiholomorphic involution which fixes the maximum number $g+1$ of ovals, $\Omega_0, \dots, \Omega_g$;
- $P_0 \in \Omega_0$ (infinite oval) and the divisor points $P_j \in \Omega_j$, $j = 1, \dots, g$ (finite ovals).





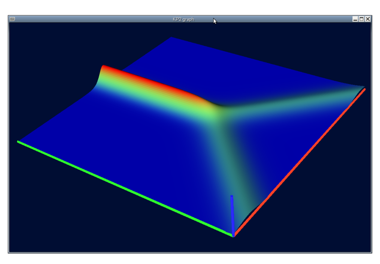
[Ich-23] has proven that 1-dimensional families of M–curves degenerate to rational curves and the corresponding regular finite-gap solutions degenerate to soliton solutions (not necessarily real and regular!). In this limit the theta-function becomes a finite sum. This is compatible with the form of multiline-soliton KP2 solutions.

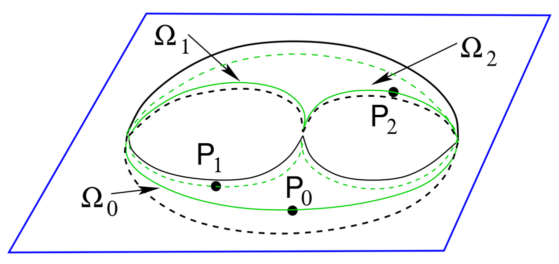
Real regular KP multi-solitons and $Gr^{\geq 0}(k,n)$

- Soliton data: $(\mathcal{K}, [A])$, with $\mathcal{K} = \{\kappa_1 < \cdots \kappa_n\}$, A real $k \times n$ matrix
- $\tau(x,y,t) = Wr_x(f^{(1)}, \dots f^{(k)})$, with $f^{(i)} = \sum_{j=1}^n A^i_j \exp(\kappa_j x + \kappa_j^2 y + \kappa_j^3 t)$
- $u(x,y,t) = 2\partial_x^2 \log(\tau)$ is regular for real (x,y,t) iff all maximal minors of A are non–negative [KW-2013]

Main results:

Question: is it possible to associate to real regular multiline KP2 solutions the rational degeneration of an M-curve and a divisor fulfilling the conditions in [DN-1988]?
Answer: Yes! we did it in [AG-2018, AG-2019, AG-2022b] using the combinatorial structure of $Gr^{\geq 0}(k,n)$, and thus verified that any real regular multi-line KP2 soliton is the tropical limit of a real regular finite-gap solution.

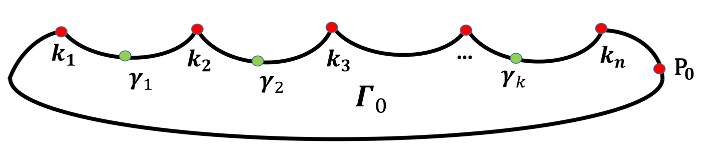




- In [AG-2022b], we start from soliton data $(\mathcal{K}, [A] \in \mathcal{S}_{\mathcal{M}}$. Then, any plabic graph representing $\mathcal{S}_{\mathcal{M}}$ is dual to the topological model of a reducible rational curve on which one can consistently assign spectral data (divisor) fulfilling the conditions of [DN-1988] by solving a system of relations on the graph.
- If one chooses the Le-graph, the M curve has genus g equal to the dimension of $\mathcal{S}_{\mathcal{M}}$ [AG-2019]
- Constructive approach! the divisor is computed solving a system of relations on the graph [AG-2022a, AG-2023] with a statistical mechanical interpretation [A-2021].
- In $Gr^{>0}(k,n)$ one can alternatively use classical total positivity [AG-2018a]

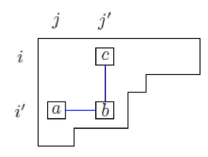
The Sato divisor on Γ_0

Soliton data: $(\mathcal{K}, [A]) \quad \mapsto \quad$ Sato algebraic geometric data Γ_0 rational curve, marked points $P_0, \kappa_1, \dots, \kappa_n$, k-point real non–special divisor
$\mathcal{D}^{(k)}_S = \{\kappa_1 < \gamma_1 < \cdots < \gamma_k \leq \kappa_n\}$ [Mal-1991]

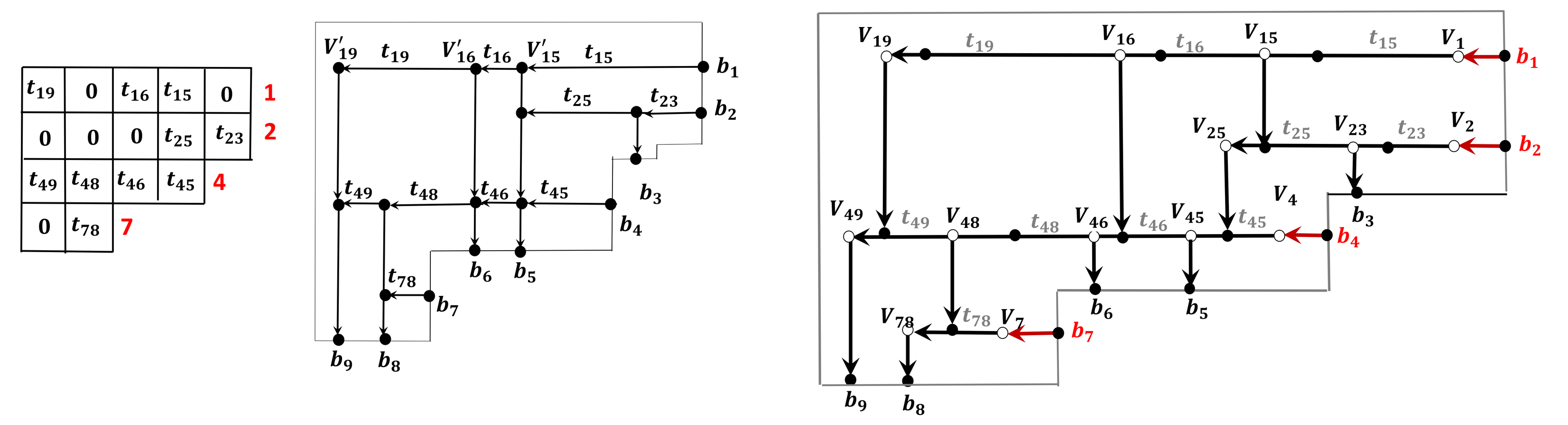


Intermezzo on $Gr^{\geq 0}(k,n)$

[Pos-06]: Each $[A] \in Gr^{\geq 0}(k,n)$ belongs to a unique positroid cell $\mathcal{S}_{\mathcal{M}} = \{[A] \in Gr^{\text{TNN}}(k,n) \,:\, \Delta_I(A) > 0$ for $I \in \mathcal{M}$, and $\Delta_I(A) = 0$ for $I \notin \mathcal{M}\}$. $\mathcal{S}_{\mathcal{M}}$ is represented by a Le-diagram (=Young diagram with a filling rule for 0s and 1s) and by an equivalence class of planar bicolored (=plabic) graphs in the disk.
There is a bijection between $\mathcal{S}_{\mathcal{M}} \subset Gr^{\text{TNN}}(k,n)$ and { Le–diagrams } in $k \times n$ boxes.
A Le–diagram is a filling of Young diagram with 0's and 1's s.t. for any 3 boxes (i',j), (i,j'), (i',j'), with $i < i'$, $j < j'$, $a, c = 1 \implies b = 1$:



Le diagram (tableau) $\iff$ perfectly oriented bipartite Le–graph (network) in the disk:



[Pos-2006]: Classification of planar networks in the disk representing $[A] \in \mathcal{S}_{\mathcal{M}}$

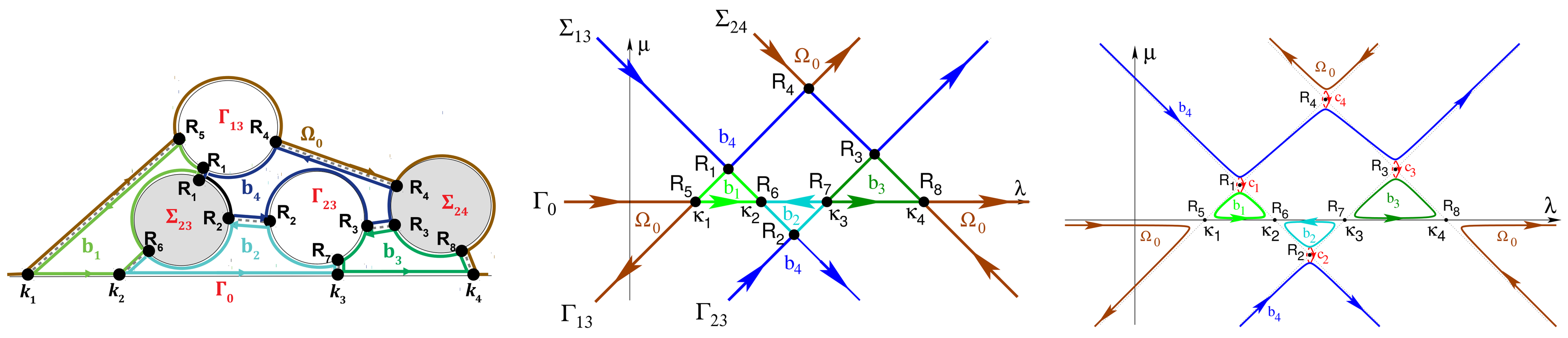
The construction of Γ rational degeneration of M–curve

- Take soliton data in $\mathcal{S}_{\mathcal{M}}$ and choose a graph in the disk $\mathcal{G}$ representing $\mathcal{S}_{\mathcal{M}}$ in Postnikov classification.
- $\mathcal{G}$ is dual to the reducible rational curve Γ:

$\mathcal{G}$	Γ
Boundary of disk	Sato component Γ_0
Boundary vertex b_l	Marked point κ_l on Γ_0
Internal vertex Σ_s	Copy of $\mathbb{CP}^1$ denoted Σ_s
Internal white vertex Γ_l	Copy of $\mathbb{CP}^1$ denoted Γ_l
Edge e	Intersection/node
Face f	Oval

- Perturb Γ to Γ_ϵ opening gaps so that Γ_ϵ is an M–curve of genus $g = F - 1$, where F is the number of faces of the graph ($g \geq$ dim $\mathcal{S}_{\mathcal{M}}$, [AG-2019]: have = for the Le–graph)

Example: $g = 4$ M-curve for soliton data in $Gr^{>0}(2,4)$

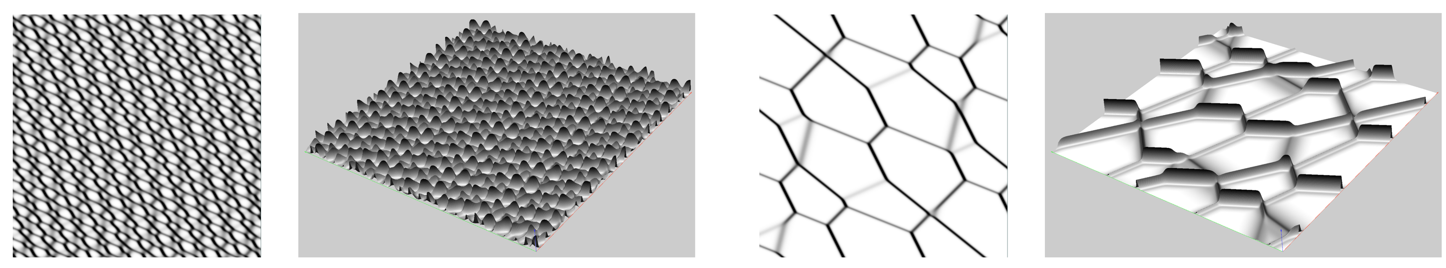


$$0 = P_0(\lambda,\mu) = \mu \cdot (\mu - (\lambda - \kappa_1)) \cdot (\mu + (\lambda - \kappa_2)) \cdot (\mu - (\lambda - \kappa_3)) \cdot (\mu + (\lambda - \kappa_4)).$$

Genus 4 M–curve after desingularization:

$$\Gamma(\varepsilon) \;:\quad P(\lambda,\mu) = P_0(\lambda,\mu) + \varepsilon(\beta^2 - \mu^2) = 0, \qquad 0 < \varepsilon \ll 1,$$

$\beta = \frac{\kappa_4 - \kappa_1}{4} + \frac{1}{4}\max\{\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \kappa_4 - \kappa_3\}$,
$\kappa_1 = -1.5, \quad \kappa_2 = -0.75, \quad \kappa_3 = 0.5, \quad \kappa_4 = 2.$



Level plots for KP-2 finite gap solutions: $\epsilon = 10^{-2}$ [left], $\epsilon = 10^{-18}$ [right].
Horizontal axis is $-60 \leq x \leq 60$, vertical axis is $0 \leq y \leq 120$, $t = 0$.
White (black) = lowest (highest) value of u.

Alternative approaches

In [AFMS-2023,FM-2024] they start from a reducible rational curve and associate a KP2-soliton solution in the finite dimensional reduction of the Sato Grassmannian generalizing [Nak-2019]. In [ACFM-xxx] we build a dictionary using both [AG-2019,AG-2022,A-2021] and [AFMS-2023,FM-2024].

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