Poincaré and Picard bundles on the Moduli Spaces of Vector Bundles

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Setup and Notations

Let Y be an irreducible projective curve of genus $g_Y \geq 3$ over $\mathbb{C}$ with only nodes as singularities and $p : X \to Y$ be its normalisation. Let $U_Y(n,d)$ (resp. $U_Y^s(n,d)$) denote the coarse moduli space of slope semistable (resp. stable) torsionfree sheaves of rank n and degree d.

- If $(n,d) = 1$, then $U_Y^s(n,d)$ is a fine moduli space, i.e., there exists a universal family $\mathcal{U}$ on $U_Y^s(n,d) \times Y$. The universal family $\mathcal{U}$ is also called the *Poincaré sheaf.*
- The restriction of $\mathcal{U}$ to $\{\mathcal{E}\} \times Y$ is isomorphic to $\mathcal{E}$ for all $\mathcal{E} \in U_Y^s(n,d)$.
- $U_Y'^s(n,d)$ denote the subset of $U_Y^s(n,d)$ consisting of stable locally free sheaves.
- $\mathcal{U}$ restricted $U_Y'^s(n,d) \times Y$ is a vector bundle, called the Poincaré bundle.
- When $d > 2n(g-1)$, $H^1(E) = 0$ for all $E \in U_Y'^s(n,d)$. Hence the direct image sheaf $p_{1*}\mathcal{U}$ over $U_Y'^s(n,d)$ is locally free where p_1 is the projection $U_Y'^s(n,d) \times Y \to U_Y'^s(n,d)$.
- The associated vector bundle, denoted by $\mathcal{W}$, is called the Picard bundle over $U_Y'^s(n,d)$.
- $U_L'(n,d)$ denote the moduli space of stable vector bundles of rank n, degree d and determinant L and $U_L(n,d)$ denote its closure in $U_Y(n,d)$.

Is there a universal family when $(n,d) \neq 1$?

[Ramanan, 1973]: Let X be a non-singular algebraic curve of genus $g \geq 2$. If n and d are not coprime, there does not exist a Poincaré bundle on any Zariski open subset of $U_L(n,d)$.

Theorem 1

Let Y be an integral nodal curve of geometric genus $g(X) \geq 2$. If n and d are not coprime, then there does not exist a Poincaré family on any Zariski open subset V of $U_L(n,d)$.

A Brauer group argument for non existence ($V = U_L'(n,d)$)

- If there exists a Poincaré bundle $\mathcal{V}$ on $U_L'(n,d) \times Y$, then by uniqueness of projective Poincaré bundles, we have $\mathcal{PU} \cong P(\mathcal{V})$ and $\mathcal{PU}_x \cong P(\mathcal{V}_x)$ for a nonsingular point $x \in Y$.
- This will imply that the Brauer class of $\mathcal{PU}_x$ is trivial as it is the projectivisation of a vector bundle.
- As a consequence we get $\mathrm{Br}(U_L'(n,d)) = \{0\}$. But $\mathrm{Br}(U_L'(n,d)) \cong \mathbb{Z}/h\mathbb{Z}$ where $h = gcd(n,d)$ and is generated by the Brauer class of $\mathcal{PU}_x$ [Bhosle and Biswas, 2014].

Idea of proof of Theorem 1

- If we assume the existence of a Poincaré bundle on $V \times Y$, it gives rise to a vector bundle on the moduli space (taking direct image), say E.
- We show that $P(E)$ parametrises a family of vector bundles on Y, say $\mathcal{V}$.
- Next, we show that there is a morphism from the open set corresponding to the stable vector bundles in the family $\mathcal{V}$ to another projective bundle.
- This morphism gives rise to a surjective map between their Picard groups.
- Further, we show that the Picard group in the codomain is $\mathbb{Z}$ and the image is generated by $gcd(n,d)$.
- The surjectivity forces the rank and degree to be coprime.

How do we handle the node?

Observations:
1) The subset $\bar{U}_L'^s(n,d)$ of $U_L'(n,d)$ consisting of vector bundles whose pullback to the normalisation of Y is stable behaves nicely.
2) If $\bar{U}_L'^s(n,d)$ forms a big open set of the moduli space, some arguments used for the case of smooth curves transcends to nodal curves.

Theorem 2

Let $p : X \to Y$ denote the normalisation. Denote by $U_L'^{ss}(n,d)$ (resp. $U_L'^s(n,d)$) the subset of $U_L(n,d)$ consisting of vector bundles F such that p^*F is semistable (resp. stable). Then

- $\mathrm{codim}(U_L'(n,d) - U_L'^s(n,d), U_L'(n,d)) \geq 2g(X) - 2$ (resp. $g(X) - 1$)
- $\mathrm{codim}(U_L'^{ss}(n,d) - U_L'^s(n,d), U_L'^{ss}(n,d)) \geq 2g(X) - 2$ (resp. $g(X) - 1$)

Another application of Theorem 2

[U.N. Bhosle, 1995]: The Narasimhan-Seshadri theorem is not true for nodal curves.

Theorem 3: Let Y be a complex nodal curve with $g(X) \geq 2$. The subset of $U_Y'(n,d)$ (respectively of $U_L'(n,d)$) consisting of vector bundles which come from representations $\pi_1(Y)$ has complement of codimension at least 2 for $g(X) \geq 2$ except possibly when $n = g(X) = 2, d$ even.

Projective Poincaré and Picard bundles

- Although there does not exist a universal vector bundle in the non-coprime case, there always exist a universal projective bundle.
- There exists a projective bundle $\mathcal{PU}$ whose restriction to $Y \times \{E\}$ is isomorphic to $P(E)$ for all $E \in U_L'^s(n,d)$ and we call it the projective Poincaré bundle.
- When $d > 2n(g-1)$, there exists a projective bundle $\mathcal{PW}$ on $U_L'^s(n,d)$ called the projective Picard bundle.

Are $\mathcal{PU}$ and $\mathcal{PW}$ stable?

[2009]: I. Biswas, L. Brambila-Paz and P. Newstead proved that the projective Poincaré bundle and the projective Picard bundle are stable when $(n,d) \neq 1$ for smooth curves.

How is stability defined for projective bundles?

Let U be an open subset of a projective variety W such that $\mathrm{codim}(W - U, W) \geq 2$. Let $P \xrightarrow{p} U$ be a projective bundle on U and let $P' \xrightarrow{p'} Z$ be a projective subbundle of $P|_Z$ where Z is a Zariski open subset of U with $\mathrm{codim}(U - Z, U) \geq 2$.

$$0 \to \mathcal{T}_{P'/Z} \to \mathcal{T}_{P|_Z/Z} \to N_{P'/P} \to 0$$

where $\mathcal{T}_{P'/Z}$ and $\mathcal{T}_{P|_Z/Z}$ are the relative tangent bundles Then $N = p'_*(N_{P'/P})$ is a vector bundle on Z.
The projective bundle P is stable (semistable) if for every subbundle P', $\deg N > 0$ ($\deg N \geq 0$).

Remarks:
1) This definition is equivalent to the standard notion of stability for principal $PGL(N)$-bundles.
2) The varieties $U_L'(n,d)$ and $U_L'^s(n,d)$ are quasiprojective varieties.

Codimension of $U_L'(n,d)^c$ and $U_L'^s(n,d)^c$ in $U_L(n,d)$?

[U.N. Bhosle, 2020]: Let Y be an integral nodal curve of arithmetic genus $g \geq 2$ with m nodes, $m \geq 1$. For $n \geq 2$, $\mathrm{codim}(U_L(n,d) - U_L'(n,d), U_L(n,d)) \geq 2$.

Theorem 4: Let Y be an integral nodal curve of arithmetic genus $g \geq 3$. Then $\mathrm{codim}(U_L'(n,d) - U_L^s(n,d), U_L'(n,d)) \geq 2$.

Theorem 5: $\mathcal{PU}$ and $\mathcal{PW}$ are stable

Define $\mathcal{PU}_x = \mathcal{PU}|_{\{x\} \times U_L'^s(n,d)}$.

1. $\mathcal{PU}_x$ is stable for all $x \in Y_{reg}$ where Y_{reg} is the set of all nonsingular points of Y.
2. Let η and θ_L be divisors defining the polarisation on Y and $U_L'^s(n,d)$ respectively. Then $\mathcal{PU}$ is stable with respect to $a\eta + b\theta_L$, $a, b > 0$.
3. Suppose further that $d > 2n(g-1)$. Then the projective Picard bundle $\mathcal{PW}|_{U_L'^s(n,d)}$ is stable.

Remark: The proof uses Hecke cycles to obtain a projective space P, an injective morphism $\psi : P \to U_L'^s(n,d)$. We show that $\deg \psi^* N > 0$.

Another Application of Theorem 4

Let Y be an integral nodal curve of arithmetic genus $g \geq 2$. Assume that if $n = 2$ and $g = 2$, then d is odd. Then

1. $\mathrm{Pic}\ U_L'^s(n,d) \cong \mathbb{Z}$.
2. $\mathrm{Pic}\ U_L'(n,d) \cong \mathbb{Z}$.
3. The class group $\mathrm{Cl}(U_L(n,d)) \cong \mathbb{Z}$. The class group $\mathrm{Cl}(U_L'(n,d)) \cong \mathbb{Z}$.

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