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# Multi Symmetric Products and Higher Rank Divisors on Curves

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## Introduction

Let $C$ be a smooth projective curve of genus $g$ over $\mathbb{C}$. Given any positive integer $d$, one can associate a smooth projective variety $\mathrm{Sym}^d(C) := \frac{C^d}{S^d}$, the $d$-th symmetric product of the curve $C$. Although $\mathrm{Sym}^d(C)$ is very well studied not only in the realm of Mathematics, but also in Physics, the following natural question remained unanswered : *Given any two distinct partitions $(m_1, m_2, \ldots, m_r)$ and $(n_1, n_2, \ldots, n_s)$ of a positive integer $n$, are the multi symmetric products $\mathrm{Sym}^{m_1}(C) \times \cdots \times \mathrm{Sym}^{m_r}(C)$ and $\mathrm{Sym}^{n_1}(C) \times \cdots \times \mathrm{Sym}^{n_s}(C)$ non-isomorphic?* We answer this question affirmatively first for genus 0 curves, and then use it to answer for higher genus curves.

In algebraic geometry, $\mathrm{Sym}^d(C)$'s, the parameter space of the degree $d$ effective divisors on $C$, were studied in the context of studying several properties of the Jacobian variety $J(C)$ using the classical Abel-Jacobi map. In [Bi-Gh-Le], the authors studied higher rank divisors on $C$ (higher dimensional analogue of $\mathrm{Sym}^d(C)$'s) to study several properties of the parameter space $N_C(n, d)$ of the stable bundles of rank $n$ and degree $d$ over $C$ (higher dimensional analogue of $J(C)$) via higher dimensional analogue of the Abel-Jacobi map. Here, we study a couple of very important properties, namely diagonal property and weak point property of the ind-variety of higher rank divisors and also calculate its Picard group. In [Bi], the author introduced the notion of a Hilbert schemes of a curve associated to the good partitions of a polynomial while providing some stratification of some Quot schemes. Here we study the Hilbert schemes of a curve associated to the good partitions of a constant polynomial as these provide a stratification of the constituent varieties of the ind-variety of the higher rank divisors. We prove that these mentioned Hilbert schemes also satisfy diagonal property. Furthermore, we provide the exact number of such Hilbert schemes up to isomorphism by proving that the multi symmetric products associated to two distinct partitions of a positive integer $n$ are not isomorphic.

## Background Material

### 2.1 Higher Rank Divisors on Curves

**Definition 2.1.** Let $\mathcal{O}_C$ be the structure sheaf of $C$ and $K$ be its field of rational functions. A divisor of rank $r$ and degree $n$ over $C$, denoted by $(r, n)$-divisor, is a coherent sub $\mathcal{O}_C$-module of $K^{\oplus r}$ having rank $r$ and degree $n$.

- For any two effective divisors with $D_2 \geq D_1$, we denote $D_2 - D_1$ as $D$. Then, we have the following structure map, denoted by $\mathcal{O}_C(-D)$, obtained by tensoring the submodules with $\mathcal{O}_C(-D)$.

$$\mathcal{O}_C(-D) : Q^{r,n}(D_1) := \mathrm{Quot}^{n+r\cdot\deg(D_1)}_{\mathcal{O}^r_C} \to \mathrm{Quot}^{n+r\cdot\deg(D_2)}_{\mathcal{O}^r_C} =: Q^{r,n}(D_2). \tag{1}$$

**Definition 2.2.** The ind-variety determined by the inductive system consisting of the varieties $Q^{r,n}(D)$ and the morphisms as in (1),denoted by $\mathbf{Q}^{r,n}$, is the ind-variety of $(r, n)$-divisors on $C$.

**Definition 2.3.** Let $\mathrm{Hilb}^P_C$ we denote the Hilbert scheme parametrizing all subschemes of $C$ having Hilbert polynomial $P(t)$. Let $\underline{P} = (P_i)^s_{i=1}$ be a family of polynomials with rational coefficients. Then $\underline{P}$ is said to be a good partition of $P$ if $\sum_{i=1}^s P_i = P$ and $\mathrm{Hilb}^{P_i}_C \neq \phi$ for all $i$.

**Definition 2.4.** The Hilbert scheme associated to a polynomial $P$ and its good partition $\underline{P}$, denoted by $\mathrm{Hilb}^{\underline{P}}_C$ , is defined as $\mathrm{Hilb}^{\underline{P}}_C := \mathrm{Hilb}^{P_1}_C \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} \mathrm{Hilb}^{P_s}_C$.

- In [Bi], the author proved the following decomposition of the Quot scheme $\mathrm{Quot}^P_{\mathcal{O}^r_C}$ of all torsion quotients of $\mathcal{O}^r_C$ having Hilbert polynomial $P(t)$, whenever $\mathrm{Quot}^P_{\mathcal{O}^r_C}$ is smooth.

$$\mathrm{Quot}^P_{\mathcal{O}^r_C} = \coprod_{\substack{\underline{P} \text{ such that } \underline{P} \\ \text{is a good partition of } P}} \mathcal{S}_{\underline{P}},$$

where each $\mathcal{S}_{\underline{P}}$ is smooth, the torus $\mathbb{G}^r_m$-invariant, locally closed and isomorphic to a vector bundle over the scheme $\mathrm{Hilb}^{\underline{P}}_C$.

### 2.2 Diagonal Property and Weak Point Property of a Variety

**Definition 2.5.** Let $X$ be a variety over $\mathbb{C}$. Then $X$ is said to have the diagonal property if there exists a vector bundle $E \to X \times X$ of rank equal to the dimension of $X$, and a global section $s$ of $E$ such that the zero scheme $Z(s)$ of $s$ coincides with the diagonal $\Delta_X$ in $X \times X$.

**Definition 2.6.** Let $X$ be a variety over $\mathbb{C}$. Then $X$ is said to have the weak point property if there exists a vector bundle $F \to X$ of rank equal to the dimension of $X$, and a global section $t$ of $F$ such that the zero scheme $Z(s)$ of $s$ is a reduced point of $X$.

**Theorem 2.7.** *( [Bi-Si]) The $d$-th symmetric product $\mathrm{Sym}^d(C)$ of the curve $C$ has the diagonal property for any $d > 0$.*

**Theorem 2.8.** *( [Bi-Si]) Let $d$ and $n$ be two given positive integer such that $n|d$. Then the Quot scheme $\mathrm{Quot}^d_{\mathcal{O}^n_C}$ parametrizing the torsion quotients of $\mathcal{O}^n_C$ of degree $d$ has the weak point property.*

**Lemma 2.9.** *Let $X_1$ and $X_2$ be two varieties over $\mathbb{C}$ satisfying the the diagonal property. Then the product variety $X_1 \times X_2$ also have the diagonal property.*

## Main Results

**Definition 3.1.** Let $\Lambda$ be a filtered ordered set. Let $X = \{X_\lambda, f_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$ be an ind-variety. Then $X$ is said to have the diagonal property (respectively weak point property) if there exists some $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, the varieties $X_\lambda$'s have the diagonal property (respectively weak point property).

**Definition 3.2.** For a given $(r, n)$-divisor, the rational number $\frac{n}{r}$ is said to its slope.

**Theorem 3.3.** *( [Mu-Ng]) Let $C$ be a smooth projective curve over $\mathbb{C}$. Also let $r \geq 1$ and $n$ be two integers. Then the ind-variety of $(r, n)$-divisors having integral slope on $C$ has the weak point property.*

**Theorem 3.4.** *( [Mu-Ng]) Let $C$ be a smooth projective curve over $\mathbb{C}$ and $n$ any given integer. Then the ind-variety of $(1, n)$-divisors on $C$ has the diagonal property.*

**Theorem 3.5.** *( [Mu-Ng]) Let $C$ be a smooth projective curve over $\mathbb{C}$ and $n$ any given integer. Then the Picard group of the ind-variety of $(1, n)$-divisors on $C$ is $\mathrm{Pic}(J(C)) \oplus \mathbb{Z}$.*

**Proposition 3.6.** *( [Mu-Ng]) Let $n \geq 1$. Let $(m_1, m_2, \ldots, m_s)$ and $(n_1, n_2, \ldots, n_t)$ be two distinct partitions of $n$. Then $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_s}$ is not isomorphic to $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$.*

- *Proof.* We have $H^*(\mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_s}, \mathbb{Z}) = \dfrac{\mathbb{Z}[x_1, x_2, \cdots, x_s]}{\langle x_1^{m_1+1}, x_2^{m_2+1}, \cdots, x_s^{m_s+1}\rangle} = M$ (say).

The proof now follows from the following observation : The extremal rays of the nef cones of $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_s}$ are the one-dimensional sub cones generated by $x_i$'s and these are the primitive generators as well. □

**Proposition 3.7.** *( [Mu-Ng]) Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g$ with $g \geq 1$. Let $n$ be a positive integer, and $(n_1, n_2, \ldots, n_r)$ and $(m_1, m_2, \ldots, m_s)$ two distinct partitions of $n$ of different lengths. Then the multi symmetric product of $C$ of type $[(n_1, n_2, \ldots, n_r), n]$ and $[(m_1, m_2, \ldots, m_s), n]$ are not isomorphic.*

*Proof.* Follows from the fact that the first Betti number $B_1$ of the multi symmetric product of $C$ of type $[(n_1, n_2, \ldots, n_r), n]$ is $2rg$, obtained using Betti-number description given by [Mc]. □

**Proposition 3.8.** *( [Mu-Ng]) Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 1$. Let $n$ be a positive integer, and $(n_1, n_2, \ldots, n_r)$ and $(m_1, m_2, \ldots, m_r)$ two distinct partitions of $n$ of same length. Then the multi symmetric product of $C$ of type $[(n_1, n_2, \ldots, n_r), n]$ and $[(m_1, m_2, \ldots, m_r), n]$ are not isomorphic whenever $\min_{i=1}^r\{n_i, m_i\} \leq 2g - 1$.*

*Proof.* Follows from the observation that the (smallest part + 1)-th Betti number $B_{\text{smallest part}+1}$ are different for the multi symmetric product of $C$ of type $[(n_1, n_2, \ldots, n_r), n]$ and $[(m_1, m_2, \ldots, m_r), n]$, obtained using Betti-number description given by [Mc]. □

**Proposition 3.9.** *( [Mu-Ng]) Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g$ with $g \geq 1$. Let $n$ be a positive integer, and $(n_1, n_2, \ldots, n_r)$ and $(m_1, m_2, \ldots, m_r)$ two distinct partitions of $n$ of same length. Then the multi symmetric product of $C$ of type $[(n_1, n_2, \ldots, n_r), n]$ and $[(m_1, m_2, \ldots, m_r), n]$ are not isomorphic whenever $\min_{i=1}^r\{n_i, m_i\} \geq 2g - 1$.*

*Proof.* Consider the following map:

$$\alpha_{d_1,\cdots,d_r,P} : \mathrm{Sym}^{d_1}(C) \times \cdots \times \mathrm{Sym}^{d_r}(C) \to J(C) \times \cdots \times J(C)$$
$$(D_1, \ldots, D_r) \mapsto \mathcal{O}_C(D_1 + \cdots + D_r - (d_1 + \cdots + d_r)P).$$

The fibre of the map $\alpha_{d_1,\cdots,d_r,P}$ is isomorphic to $\mathbb{P}^{d_1-g} \times \cdots \times \mathbb{P}^{d_r-g}$.

$$\begin{array}{ccc} \mathrm{Sym}^{n_1}(C) \times \cdots \times \mathrm{Sym}^{n_r}(C) & \xrightarrow[\simeq]{\psi \text{ (if exists)}} & \mathrm{Sym}^{m_1}(C) \times \cdots \times \mathrm{Sym}^{m_r}(C) \\ & {\scriptstyle \alpha_{n_1,\cdots,n_r,P}} \searrow \quad \swarrow {\scriptstyle \alpha_{m_1,\cdots,m_r,P}} & \\ & J(C) \times \cdots \times J(C) & \end{array}$$

For any $(\mathcal{L}_1, \ldots, \mathcal{L}_r) \in J(C) \times \cdots \times J(C)$, the morphism $\alpha_{m_1,\cdots,m_r,P} \circ \psi|_{\alpha^{-1}_{n_1,\cdots,n_r,P}(\mathcal{L}_1,\ldots,\mathcal{L}_r)}$ is a morphism from $\mathbb{P}^{n_1-g} \times \cdots \times \mathbb{P}^{n_r-g}$ to the abelian variety $J(C)^r$ and therefore is constant, say $(\mathcal{M}_1, \ldots, \mathcal{M}_r)$. So,

$$\alpha^{-1}_{n_1,\cdots,n_r,P}(\mathcal{L}_1, \ldots, \mathcal{L}_r) \xrightarrow{\psi|_{\alpha^{-1}_{n_1,\cdots,n_r,P}(\mathcal{L}_1,\ldots,\mathcal{L}_r)}} \alpha^{-1}_{m_1,\cdots,m_r,P}(\mathcal{M}_1, \ldots, \mathcal{M}_r)$$
$$\alpha^{-1}_{m_1,\cdots,m_r,P}(\mathcal{M}_1, \ldots, \mathcal{M}_r) \downarrow \{(\mathcal{M}_1, \ldots, \mathcal{M}_r)\} \subseteq J(C)^r$$

Then by Proposition 3.6, $m_i = n_i$ for all $1 \leq i \leq r$. □

**Theorem 3.10.** *( [Mu-Ng]) Let $C$ be a smooth projective curve over $\mathbb{C}$ and $n$ a positive integer. Let $p(n)$ denote the number of partitions of $n$. Then the following hold:*

1. *There are at most $p(n)$ many Hilbert schemes $\mathrm{Hilb}^{\underline{n}}_C$ (up to isomorphism) associated to the constant polynomial $n$ and its good partitions $\underline{n}$ satisfying diagonal property.*
2. *Moreover, this upper bound is is attained by any genus 0 curve $C$ and hence is sharp.*
3. *Furthermore, for $n = 1, 2, 3$, the upper bound is attained by any curve $C$.*
4. *For any $n$, the upper bound is attained by any curve $C$ of any genus $g$.*

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