Trieste Algebraic Geometry Summer School 2024

Thuy Nguyen - Sao Paulo State University, Brazil

Polynomial maps via Intersection Homology

Jacobian Conjecture

"Let $F = (F_1, \dots, F_n) : \mathbb{K}^n \to \mathbb{K}^n$, where $n \geq 1$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, be a *polynomial map*.

$JF(p)$: Jacobian matrix of F at p.

Jacobian conjecture: If

$$\det JF(p) \neq 0, \quad \forall p \in \mathbb{K}^n,$$

(Jacobian condition)

then F is an automorphism."
(*Heinrich Keller, 1939*)

- $n = 1$: the conjecture is true for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.
- $n = 2$: The conjecture is false in the real case! (Pinchuk, 1994).

The conjecture is open in the complex case.

Properness and Jac. Conj.

If $\det JF(p) \neq 0, \forall p \in \mathbb{K}^n$, then F is proper. (*Hadamard Theorem, 1906*)

- The map F is proper if and only if its asymptotic set:.

$$S_F := \{a \in \mathbb{K}^n : \exists \xi_k \to \infty,\ F(\xi_k) \to a\}$$

is empty.

Inter. homol. approach

Valette Theorem (A. Valette, G. Valette, 2010).

$F : \mathbb{C}^2 \to \mathbb{C}^2$: satisfies Jacobian condition $\Rightarrow$ $\exists$ singular varieties V_F such that F is proper iff the intersection homology $IH_2^{\overline{p}}(V_F, \mathbb{R}) = 0$, for *any* perversity $\overline{p}$.

Article: *Geometry of polynomial mappings at infinity via intersection homology*, Ann. I. Fourier vol. 64, fascicule 5 (2014), 2147-2163.

Pinchuk counter-example

Given $(x, y) \in \mathbb{R}^2$, let: $t = xy - 1, h = t(xt+1), f = (xt+1)^2(t^2+y)$. Considere

$$p = f + h$$

$$q = -t^2 - 6th(h+1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - \tfrac{75}{4}h^4.$$

Then $\mathcal{P} = (p, q)$ satisfies the Jacobian condition but it is not injective/proper:

1) By changing variables $t := t(x, y)$, $h := h(x, y)$ and $f := f(x, y)$, the Pinchuk map is a polynomial map of variables t, h and f.

2) The sequence $\{z_k\} = \{(k, 1/k)\} \subset \mathbb{R}^2$ tends to infinity but $t(z_k) = h(z_k) = 0$ and $f(z_k) = 1/k$. Then the Pinchuk maps (p, q), being polynomial maps of new variables t, h and f, satisfies that $(p(z_k), q(z_k))$ does not tend to infinity, in other words (p, q) is non-proper. Then (p, q) is not an automorphism.

Results

1) **A generalization of Valettes' result**: The Valettes' Theorem is true for general case $F : \mathbb{C}^n \to \mathbb{C}^n$, $n \geq 2$, under an additional condition: "The parts at infinity of the fibres of F are complete intersections."
Article: Thuy N., Valette G. and Valette A., *On a singular variety associated to a polynomial mapping*, Journal of Singularities volume 7 (2013), 190-204.

2) **An "adapted" Valette variety in the case $F : \mathbb{C}^n \to \mathbb{C}^{n-1}$ can measure the bifurcation set of the map.**
Theorem (Thuy N., Ruas M.A.S., 2018). Let $F = (F_1, \dots, F_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n \geq 2$, be a generic polynomial mapping such that $K_0(F) = \emptyset$ (local submersion) and the parts at infinity of the fibres of F are complete intersections. Then there exist singular varieties $\mathcal{V}_F$ associated to F such that if $IH_2^{\bar{t}}(\mathcal{V}_F, \mathbb{R})$ is trivial then the bifurcation set $B(F)$ is empty, where $\bar{t}$ is the total perversity.
Article: *Thuy N. and Ruas M.A.S, On singular varieties associated to a polynomial mapping from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$*, Asian Journal of Mathematics, v.22, p.1157-1172, 2018.

3) **The "real version" of Valettes' Theorem is no longer true.**
Theorem (-, 2023). There exists a Valette variety $V_{\mathcal{P}}$ associated to the Pinchuk map $\mathcal{P}$ such that

$$IH_1^{\bar{0},c}(V_{\mathcal{P}}) = IH_1^{\bar{0},cl}(V_{\mathcal{P}}) = 0.$$

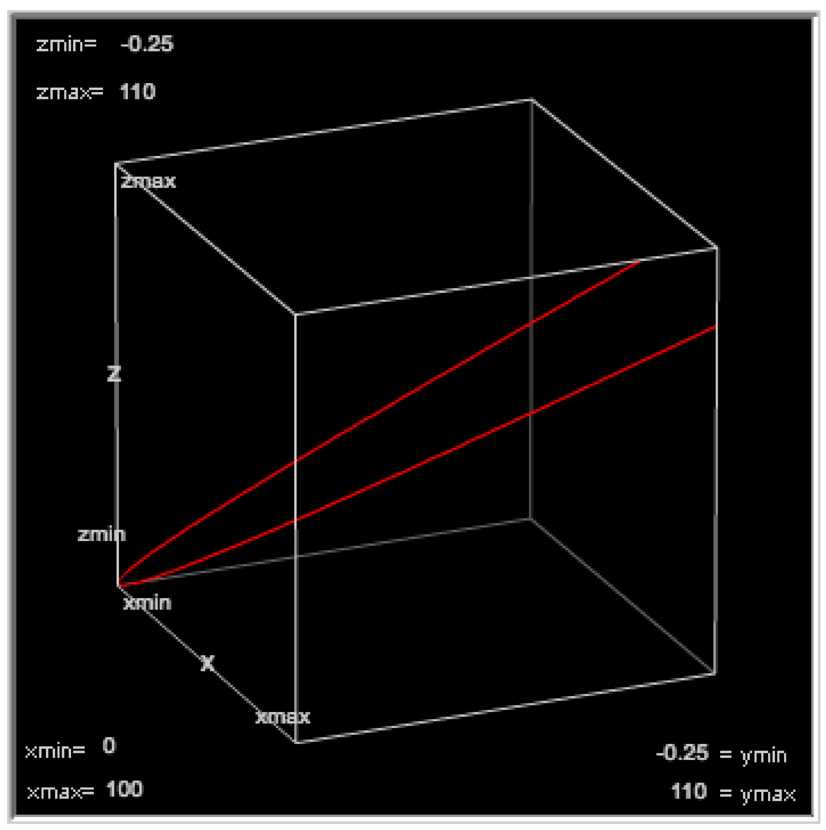
Article: *Thuy Nguyen, A singular variety associated to the smallest degree Pinchuk map, Matemática Contemporânea, Sociedade Brasileira de Matemática, 2023.*

Construction of Valette variety through an example

Example: Let $F : \mathbb{R} \to \mathbb{R}$ be a function defined by $F(x) = x^2$.

1. $\mathrm{Sing}(F) = \{0\}$.
2. $\mathrm{Sing}(F)$ divide $\mathbb{R}$ into two subsets:
$$U_1 = \{x \in \mathbb{R} : x > 0\} \quad , \quad U_2 = \{x \in \mathbb{R} : x < 0\}$$
3. $F(U_1) = F(U_2) = \{\alpha \in \mathbb{R} : \alpha > 0\}$.
4. Put $U := \mathbb{R} \setminus \{0\}$. There exist Nash functions:
$$\psi_1 : U \to \mathbb{R}, \quad \psi_1(x) := \frac{x}{1+x^2}$$
$$\psi_2 : U \to \mathbb{R}, \quad \psi_2(x) := \frac{-x}{1+x^2}.$$
satisfying the Mostowski's Seperation Lemma: ψ_i is positive on U_i and negative on U_j, $i \neq j, i, j = 1, 2$.
5. Define
$$V_F := \overline{(F, \psi_1, \psi_2)(U)} = \overline{\left\{\left(x^2, \frac{x}{1+x^2}, \frac{-x}{1+x^2}\right) : x \neq 0\right\}} \subset \mathbb{R}^3.$$

The Valette variety associated to $F : \mathbb{R} \to \mathbb{R}$, $\quad F(x) = x^2$:



In general, the singular locus of Valette variety V_F is contained in $(S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}$, where $K_0(F) = F(\mathrm{Sing}(F))$ (critical values) and p is the number of elements of the covering $\{U_1, \dots, U_p\}$ that covers $\mathbb{R}^{2n} \setminus \mathrm{Sing}(F)$ (considering $F : \mathbb{C}^n \to \mathbb{C}^n$ as the one $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$).