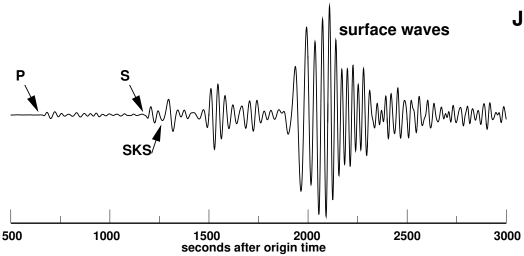
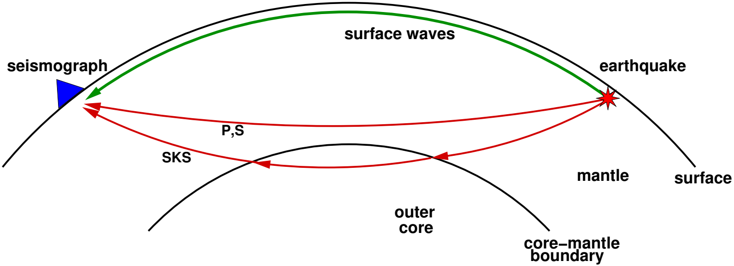
Concepts developed in this lecture

- The free surface boundary conditions give rise to interference waves which propagate parallel to the surface and whose amplitude decays with depth.
- There are two classes of surface waves: Rayleigh waves which are constructively-interfering P- and SV-waves, and Love waves which are constructively-interfering SH-waves.
- Rayleigh waves exist in a uniform half-space; Love waves exist only for a structure where the wave speed of the material increases with depth.
- Surface waves in the Earth are dispersive and their propagation is described by the phase velocity and the group velocity.

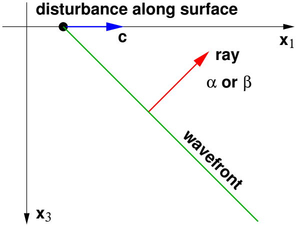
Rayleigh waves



J.W. Strutt (L. Rayleigh)
(1842–1919)



Rayleigh waves in a homogeneous half-space



For the first case we look at P- and SV-waves interacting at the surface of a uniform half-space. The displacements in terms of the potentials are

$$u_1 = \phi_{,1} - \psi_{2,3} \qquad u_2 = \psi_{1,3} - \psi_{3,1} \qquad u_3 = \phi_{,3} + \psi_{2,1}$$

For a monochromatic wave of frequency ω propagating in the x_1 direction with velocity c, the potentials ϕ and ψ, and the displacement u_2 are

$$\phi = f(x_3)e^{i(\omega t - kx_1)} \qquad \psi = g(x_3)e^{i(\omega t - kx_1)} \qquad u_2 = h(x_3)e^{i(\omega t - kx_1)}$$

Rayleigh waves in a homogeneous half-space

The amplitude dependence with depth is given by the terms $f(x_3)$, $g(x_3)$ and $h(x_3)$ and $k = \omega/c$ is the wavenumber. Substituting these in the wave equations for ϕ, ψ and u_2 gives

$$f'' + k^2 r_\alpha^2 f = 0 \qquad g'' + k^2 r_\beta^2 g = 0 \qquad h'' + k^2 r_\beta^2 h = 0$$

where

$$r_\alpha = \left[\frac{c^2}{\alpha^2} - 1\right]^{1/2} \quad r_\beta = \left[\frac{c^2}{\beta^2} - 1\right]^{1/2}$$

The solution for the first equation is

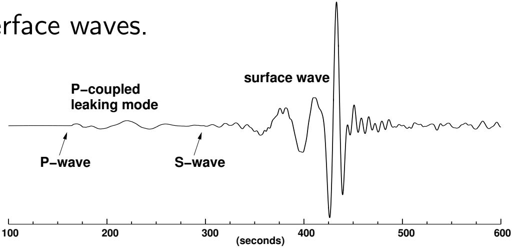
$$f(x_3) = Ae^{-ikr_\alpha x_3} + A'e^{ikr_\alpha x_3} = Ae^{-ikr_\alpha x_3}$$

with similar solutions for g and h but with r_α replaced with r_β. Surface waves have an amplitude which decreases with depth, so r_α must be imaginary and $A' = 0$.

Rayleigh waves in a homogeneous half-space

There are three cases:

- If $\beta < \alpha < \mathrm{c}$, then both r_α and r_β are real and both P-and S-waves reflect from the surface and propagate back into the half-space as body-waves.
- If $\beta < \mathrm{c} < \alpha$, then r_α is imaginary, r_β is real, then P-waves are trapped at the surface but S-waves reflect from the surface and propagate back into the half-space.
- If $\mathrm{c} < \beta < \alpha$, then both r_α and r_β are imaginary, then both P- and S-waves are trapped at the surface, giving rise to interface waves.



Rayleigh waves in a homogeneous half-space

Substituting the amplitude terms $f(x_3)$ *etc.*, the potentials ϕ and ψ and displacement u_2 are

$$\phi = Ae^{[-ikr_\alpha x_3 + ik(x_1 - ct)]} \quad \psi = Be^{[-ikr_\beta x_3 + ik(x_1 - ct)]} \quad u_2 = Ce^{[-ikr_\beta x_3 + ik(x_1 - ct)]}$$

To evaluate A, B and C we apply the free surface boundary condition $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. Substituting the potentials for the displacements in the free surface boundary condition at $x_3 = 0$

$$2\phi_{,31} + \psi_{,11} - \psi_{,33} = 0 \quad (\lambda + 2\mu)\phi_{,33} + \lambda\phi_{,11} + 2\mu\psi_{,13} = 0 \quad u_{2,3} = 0$$

Substituting u_2 in the equation for $u_{2,3}$ gives $C = 0$. Therefore, the surface wave in a half-space has no transverse (SH) component of motion.

Rayleigh waves in a homogeneous half-space

Substituting the potentials in the remaining boundary conditions

$$2r_\alpha A - (1 - r_\beta^2)B = 0$$

$$\alpha^2(r_\alpha^2 + 1) - 2\beta^2]A - 2\beta^2 r_\beta B = 0$$

This 2 x 2 system of homogeneous linear equations has a non-trivial solution when its determinate is zero, giving

$$[\alpha^2(r_\alpha^2 + 1) - 2\beta^2]\,(1 - r_\beta^2) - 4r_\alpha r_\beta \beta^2 = 0$$

and substituting for the values of r_α and r_β gives

$$\left[2 - \frac{c^2}{\beta^2}\right]^2 = 4\left[1 - \frac{c^2}{\beta^2}\right]^{1/2}\left[1 - \frac{c^2}{\alpha^2}\right]^{1/2}$$

Rayleigh waves in a homogeneous half-space

For the case of a Poisson solid ($\lambda = \mu, \text{ and } \alpha^2/\beta^2 = 3$)

$$\frac{c^2}{\beta^2}\left[\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + \frac{56c^2}{3\beta^2} - \frac{32}{3}\right] = 0$$

This equation has four real roots:

- $c^2/\beta^2 = 0$ – no physical significance,
- $c^2/\beta^2 = 4$ – $\beta < \alpha < c$ – reflected wave,
- $c^2/\beta^2 = 2 + 2/\sqrt{3}$ – $\beta < \alpha < c$ – reflected waves,
- $c^2/\beta^2 = 2 - 2/\sqrt{3} = 0.8453$ – evanescent wave.

For this case

$$c = 0.9194\beta$$

The apparent velocity of the Rayleigh wave in a homogeneous Poisson solid half-space is independent of frequency and is $\sim$0.92 the shear wave speed of the medium.

Rayleigh waves in a homogeneous half-space

For a Poisson solid $r_\alpha = 0.85i$ and $r_\beta = 0.39i$ and substituting in the expressions for the wave amplitude and taking the real part of the displacement gives

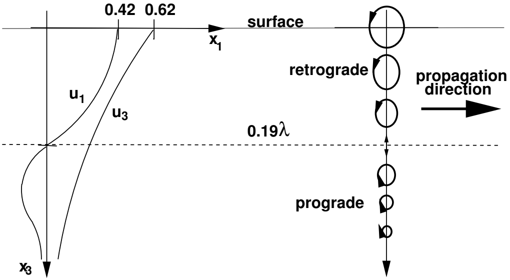
$$\begin{aligned} u_1 &= -Ak\sin(\omega t - kx_1)(e^{0.85kx_3} - 0.58e^{0.39kx_3}) \\ u_3 &= -Ak\cos(\omega t - kx_1)(-0.85e^{0.85kx_3} + 1.47e^{0.39kx_3}) \end{aligned}$$

At the surface $x_3 = 0$ and these become

$$\begin{aligned} u_1 &= 0.42a\sin(\omega t - kx_1) \\ u_3 &= 0.62a\cos(\omega t - kx_1) \end{aligned}$$

where $a = -Ak$. Then the Rayleigh wave is polarized in the vertical–radial plane; the horizontal component leads the vertical component of motion by $\pi/2$ and the amplitude of the vertical component is about 1.5 times the amplitude of the horizontal component. Therefore, the Rayleigh wave has retrograde elliptical particle motion.

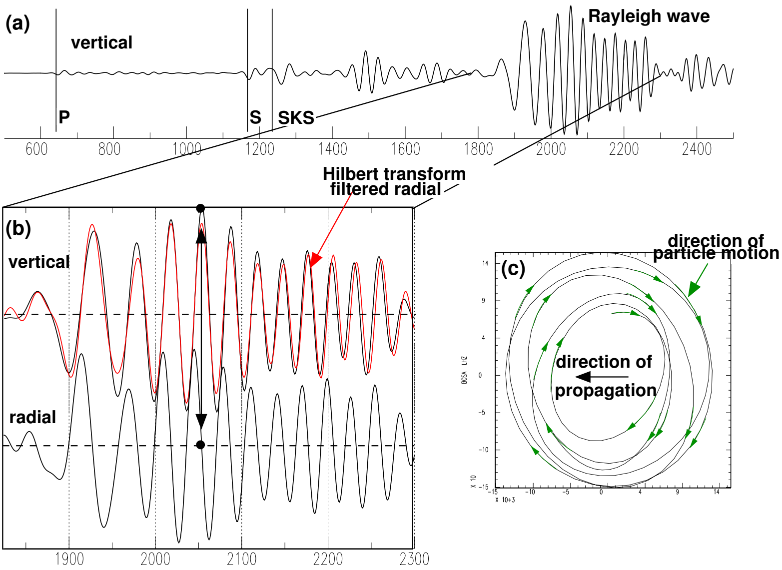
Rayleigh waves in a homogeneous half-space



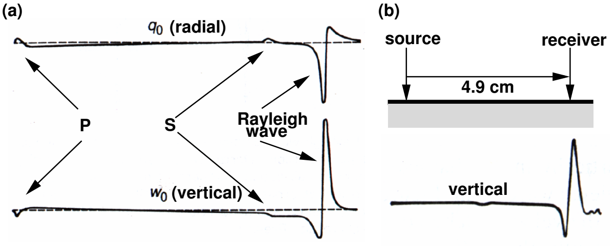
The particle displacement is the same for all frequencies, but the absolute values depend on ω or λ.

Since the decay of displacement with depth is controlled by factors like $\exp(-k_1 x_3) = \exp(-2\pi x_3/\lambda)$, long wavelength Rayleigh waves penetrate deeper than shorter wavelength Rayleigh waves.

Observed Rayleigh waves



Lamb (1904) extended Rayleigh's results by finding the complete response of a homogeneous half-space to a vertical point force acting on the surface of the half-space.



Horace Lamb (1849–1934)

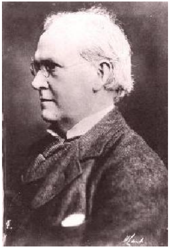
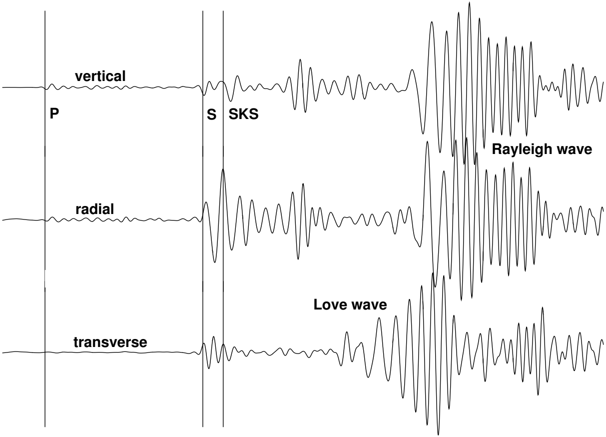


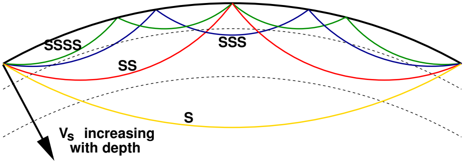
Figure: Lamb's Problem. (a) Lamb's transient solution to an impulsive vertical point force applied to the surface of a half-space; (b) recording of the vertical motion from a vertical point force on the surface of a half-space. (adapted from Ewing *et al*, 1957)

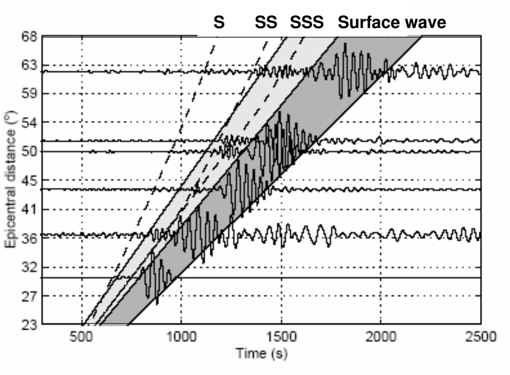
Love waves



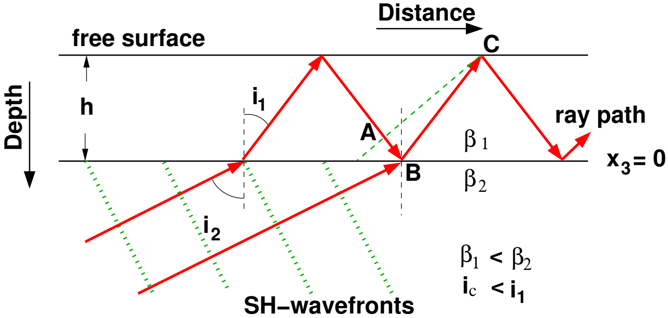
A.E.H. Love (1863–1940)

Love waves – Multiple surface reflecting S–waves





Love wave period equation



There is no phase shift in the wavefront when it reflects from the free surface. The phase shift on reflecting from the interface at B is $\Theta = 2\tan^{-1}(\mu_2 r^*_{\beta_2}/\mu_1 r_{\beta_1})$. The phase delay along the path ABC is $-k_{\beta_1}$ times the path length

$$-k_{\beta_1}(BC\cos 2i_1 + h/\cos i_1) = -k_{\beta_1}(h\cos 2i_1/\cos i_1 + h/\cos i_1)$$

$$= -2hk_{\beta_1}\cos i_1$$

Love wave period equation

For constructive interference to occur

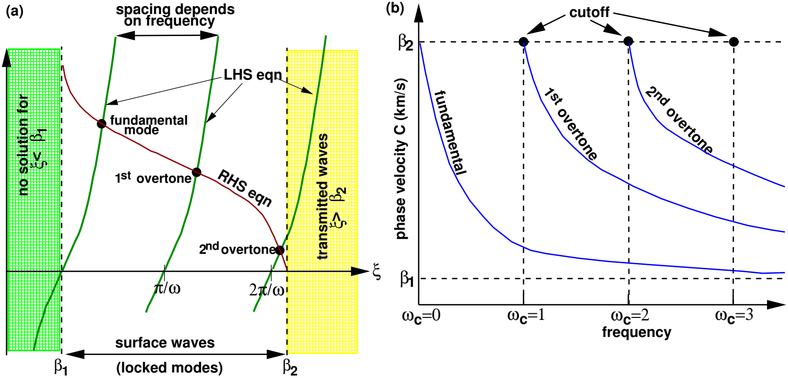
$$2\tan^{-1}\left(\frac{\mu_2\, r^*_{\beta_2}}{\mu_1\, r_{\beta_1}}\right) \;-\; 2hk_{\beta_1}\cos i_1 \;=\; 2n\pi$$

or

$$\tan(hk_{\beta_1}\cos i_1) \;=\; \tan(hk_{\beta_1} r_{\beta_1}) \;=\; \frac{\mu_2\, r^*_{\beta_2}}{\mu_1\, r_{\beta_1}}$$

which is the Love wave period equation. For this case the waves are dispersive, $c(\omega)$. The tangent has positive values between zero and ∞ for various intervals of $kr_{\beta_1}h$, the first for $0 \rightarrow \pi/2$ which corresponds to the fundamental mode, the second for $\pi \rightarrow 3\pi/2$ which corresponds to the first higher mode, and so on. **Both the dispersion and the modes come from the finite dimension of the layer**.

Love wave period equation



Love wave period equation

The period equation has real roots only for $\beta_1 < c < \beta_2$.

The right side is independent of ω; the left side is periodic and has zeros at $\xi = n\pi/\omega$. The $n = 1$ tangent curve enters the range when $\pi/\omega_{c1} = (h/\beta_1)(1 - \beta_1^2/\beta_2^2)^{1/2}$. For ω higher than ω_{c1}, two Love waves exist, both having the same frequency but with different velocities.

The $\mathrm{n^{th}}$ curve enters from the right when

$$\omega = \frac{n\pi\beta_1}{h} \left(1 - \frac{\beta_1^2}{\beta_2^2}\right)^{-1/2} = \omega_{cn}$$

Love wave particle motion with depth

The displacements for the Love wave are

$$u_2^L = (B_1 e^{ikr_{\beta_1} x_3} + B_1' e^{-ikr_{\beta_1} x_3})\, e^{ik(x_1 - ct)}$$

$$u_2^{HS} = B_2 e^{-ikr_{\beta_2} x_3}\, e^{ik(x_1 - ct)}$$

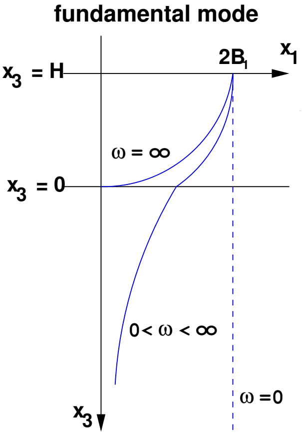
Using the boundary conditions at the surface and interface, the displacements are

$$u_2^L = 2B_1 \cos\left[kr_{\beta_1} h \left(1 - \frac{x_3}{h}\right)\right] \cos[k(r_{\beta_1} h + x_1 - ct)]$$

$$u_2^{HS} = 2B_1 \cos(kr_{\beta_1} h)\, e^{ikr^*_{\beta_2} x_3} \cos[k(r_{\beta_1} h + x_1 - ct)]$$

where $r^*_{\beta_2} = i(1 - c^2/\beta^2)^{1/2}$. In the half-space the displacement decreases exponentially with depth; in the layer the displacement varies with depth depending on $kr_{\beta_1} h$ with a different variation for each mode.

Love wave particle motion with depth



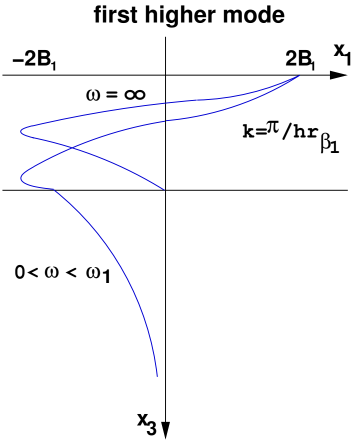
For the fundamental mode

$$k = 0 \quad |u_2| = 2B_1 \quad \text{for all values of } x_3$$

$$k = \infty \quad |u_2| = \begin{cases} 2B_1 & \text{for } x_3 = h \\ 0 & \text{for } x_3 = 0 \end{cases}$$

At $x_3 = h$, $u_2 = 2B_1$ for all ω, at $x_3 = 0$, $0 \leq u_2 \leq 2B_1$ for $\infty \geq \omega \geq 0$, and in the half-space u_2 decays exponentially with depth.

Love wave particle motion with depth



For the first higher mode
$\pi \leq k r_{\beta_1} h \leq 3\pi/2$

$$k = \pi / r_{\beta_1} h \quad |u_2| = \begin{cases} 2B_1 & \text{for } x_3 = h \\ 0 & \text{for } x_3 = h/2 \\ -2B_1 & \text{for } x_3 = 0 \end{cases}$$

$$k = \infty \qquad |u_2| = \begin{cases} 2B_1 & \text{for } x_3 = h \\ 0 & \text{for } x_3 = 2h/3 \\ -2B_1 & \text{for } x_3 = h/3 \\ 0 & \text{for } x_3 = 0 \end{cases}$$

Group velocity

The displacement of a dispersive wave with a continuous distribution of frequencies is given by

$$u(x, t) \;=\; \int_{-\infty}^{\infty} A(k)\, e^{i(kx\,-\,\omega t)}\, dk$$

The amplitude $A(k)$ of the integrand varies slowly compared to the phase $(kx - \omega t)$ and there is only a contribution to the amplitude of the wave for values of x and t when the phase is stationary. In this case

$$\frac{d\Phi}{dk} \;=\; \frac{d}{dk}(kx - \omega t) \;=\; x - \frac{d\omega}{dk} t \;=\; x - Ut = 0 \qquad \longrightarrow \qquad U \;=\; \frac{x}{t}$$

This is the group velocity corresponding to the frequency ω_0 or wavenumber k_0 which makes the phase stationary. At this point and time there is a contribution to the seismogram $u(x, t)$.

Group velocity

To determine the amplitude we expand the argument in a Taylor series about ω_0

$$kx - \omega t = (k_0 x - \omega_0 t) + (k - k_0)\frac{d}{dk}[kx - \omega t]_{k=k_0} + \frac{1}{2}(k - k_0)^2 \frac{d^2}{dk^2}[kx - \omega t]_{k=k_0} + ...$$

For a point where the phase is stationary, the first derivative is zero and using $\frac{d}{dk}(kx - \omega t) = x - Ut$, the integral becomes

$$u(x,t) = A(k_0) e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \exp\left\{ i\frac{1}{2}(k - k_0)^2 \frac{dU}{dk} t \right\} dk$$

Making the change of variable $\xi^2 = (1/2)(k - k_0)^2 (dU/dk) t$

$$u(x,t) = A(k_0) e^{i(k_0 x - \omega_0 t)} \left[\frac{t}{2}\frac{dU}{dk}\right]_{k_0}^{-1/2} \int_{-\infty}^{\infty} e^{-i\xi^2} d\xi$$

$$= A(k_0) e^{i(k_0 x - \omega_0 t)} \left[\frac{t}{2}\frac{dU}{dk}\right]_{k_0}^{-1/2} (i\pi)^{1/2}$$

Group velocity

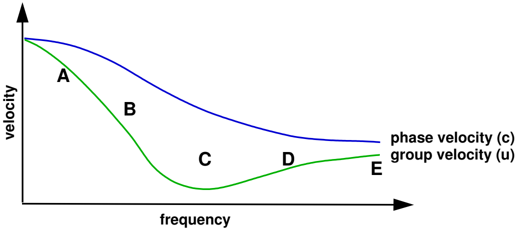
Taking the real part we obtain

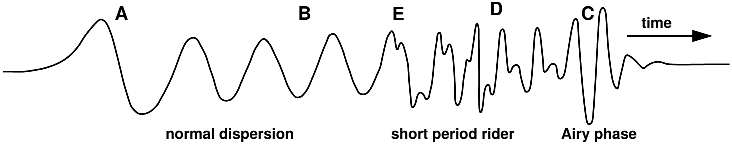
$$u(x,t) \;=\; A(k_0)\left[\frac{2\pi}{(x/U)\,(dU/dk)}\right]^{1/2}\cos(k_0x \;-\; \omega_0 t \;\pm\; \pi/4)$$

Then for a given x and t, energy is contained in a cosine wave of frequency k_0 or ω_0 corresponding to $d\Phi/d\omega \;=\; 0$. The largest amplitude corresponds to $dU/dk \;=\; 0$ and is called the Airy phase. When the second derivative is zero, we need the next term in the Taylor expansion. The Airy phase amplitude is

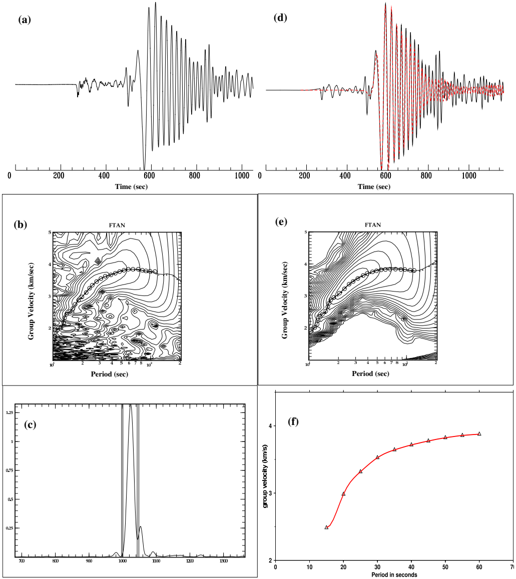
$$u(x,t) \;=\; A(k_0)\left[\frac{2\pi}{(x/U)\,(d^2U/dk^2)}\right]^{1/3}\cos(k_0x \;-\; \omega_0 t \;\pm\; \pi/4)$$

Group velocity

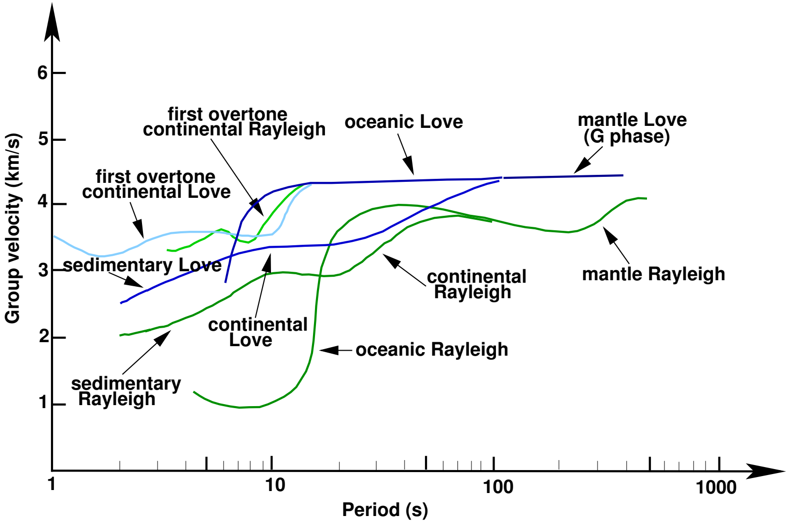




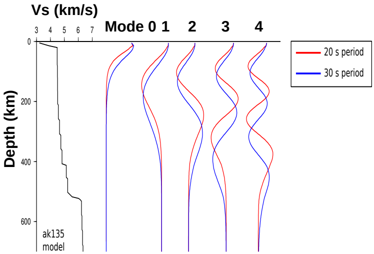
Observed group velocity

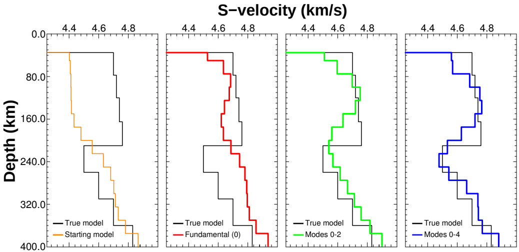


Observed group velocity

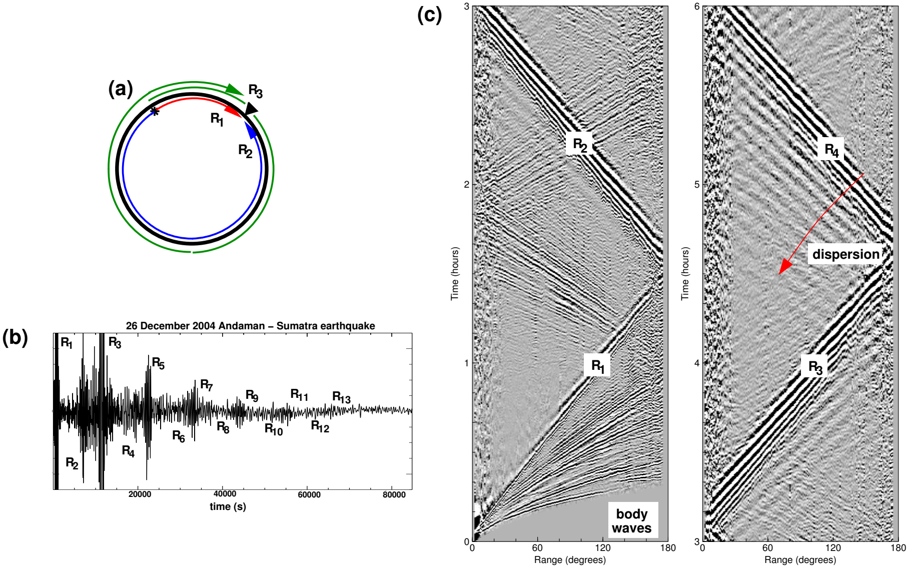


Depth resolution

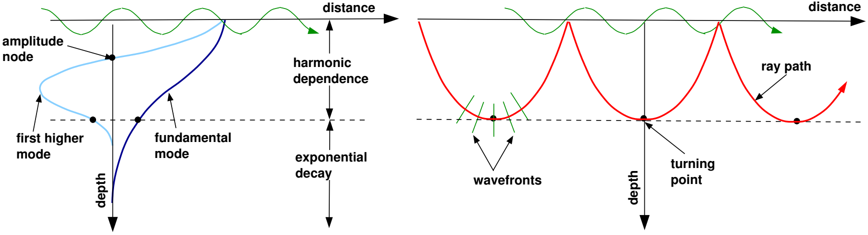




Spherical Earth model



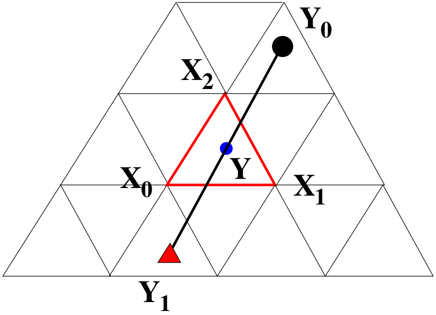
Correspondence between rays and modes

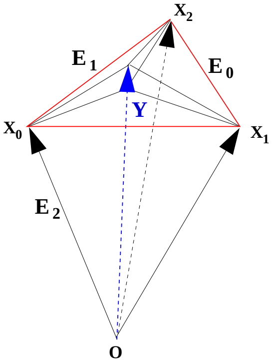


Surface wave summary

- The free surface boundary conditions give rise to interference waves which propagate parallel to the surface and whose amplitude decays with depth.
- There are two classes of surface waves: Rayleigh waves which are constructively-interfering P- and SV-waves and Love waves which are constructively-interfering SH-waves.
- Rayleigh waves exist in a uniform half-space; Love waves exist only for a structure where the wave speed of the material increases with depth.
- Surface waves in the Earth are dispersive and their propagation is described by the phase velocity and the group velocity.

The region of interest is gridded by a two-dimensional finite element mesh of triangular elements with sides of length 1° on a spherical surface. Each node point of the triangular element is defined by a position vector from the center of the Earth.





Surface wave tomography

The velocity/slowness is calculated at the nodes of these triangles from intersecting earthquake-receiver paths for which Rayleigh wave group velocity dispersion measurements were made.

$$t = \int_{path} s \; dx \tag{1}$$

where the integral is over the source-receiver path and s is the slowness (inverse velocity). Summing over all triangles along the ray path

$$t = \sum_{i=1}^{n} \int_{0}^{l_i} s \; dx \tag{2}$$

where l_i denotes the length of the ray path segment within the ith triangle.

Surface wave tomography

Within each triangle slowness at any given point **X** along the ray path is given by:

$$s = \varepsilon_0 \cdot s_0(\mathrm{X}_0) + \varepsilon_1 \cdot s_1(\mathrm{X}_1) + \varepsilon_2 \cdot slow(\mathrm{X}_2) \tag{3}$$

where $\mathrm{X}_0, \mathrm{X}_1$ and X_2 are the position vectors of node points of the triangle (labeled anti-clockwise) with the center of the Earth taken as the origin.
Substituting equation 3 in equation 2

$$t = \sum_{i=1}^{n} \left[s_0(\mathrm{X}_0) \int_0^{l_i} \varepsilon_0 dx + s_1(\mathrm{X}_1) \int_0^{l_i} \varepsilon_1 dx + s_2(\mathrm{X}_2) \int_0^{l_i} \varepsilon_2 dx \right] \tag{4}$$

where ε_0, ε_1, ε_2 are the weights assigned to the three nodes and $\varepsilon_0 + \varepsilon_1 + \varepsilon_2 = 1$

Surface wave tomography

Summing over the node points of the triangles, equation 4 can be re-written as:

$$t = \sum_{points} s(X) \sum_{triangles} \int \varepsilon dx \tag{5}$$

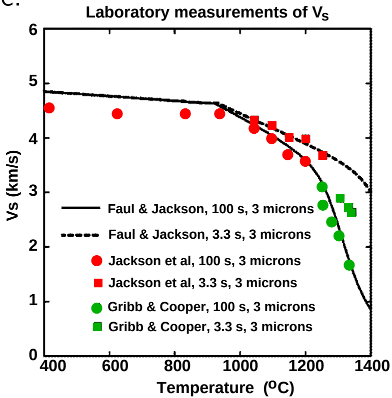
The forward problem can be expressed as:

$$\mathrm{d} = [\mathrm{A}] \ \mathrm{x} \ \mathrm{m} \tag{6}$$

where **d** is an N-dimensional vector of travel time data (t), **m** is an M-dimensional vector that describes the model (slowness values at nodes) and [**A**] is the operator that maps vectors in the model space into vectors in the data space. [**A**] represents the numerical computation of distances for the formulation of the operator [**A**]). We invert the matrix [**A**] to obtain the model vector **m**.

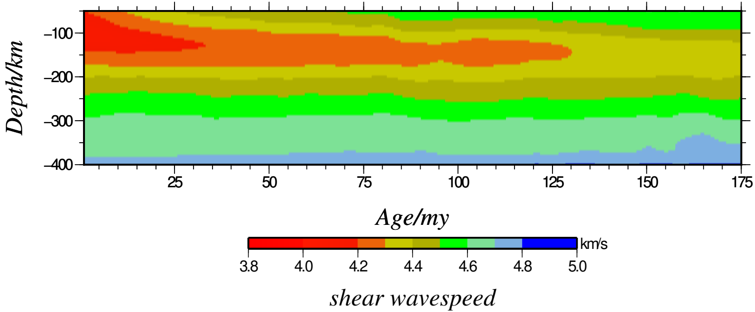
V_s vs T relationship

Laboratory measurements of μ for polycrystalline olivine at seismic frequencies show a strong decrease for temperatures considerably below the macroscopic solidus. They also show a strong dependence on grain size which is several orders of magnitude smaller in the laboratory experiments than the mantle grain size.



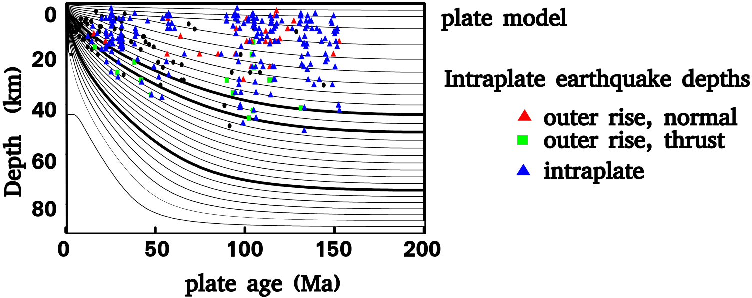
V_s vs T relationship

Shear wave velocity beneath the Pacific Ocean, obtained from surface wave tomography using the fundamental and first five higher modes, averaged as a function of age.



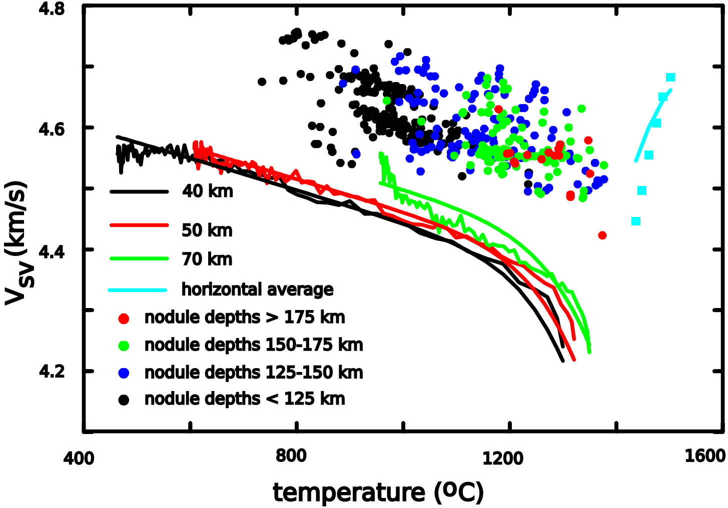
V_s vs T relationship

Plate model for the oceanic lithosphere.

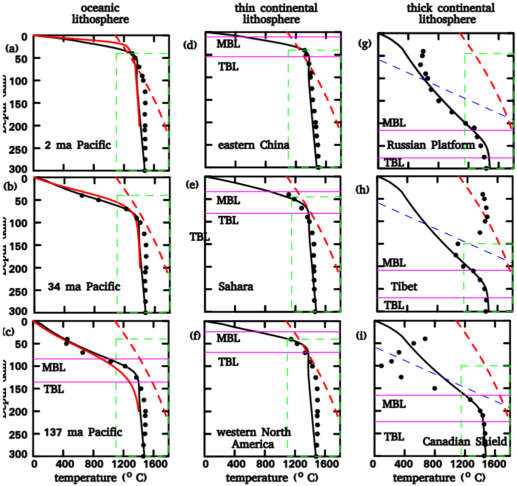


V_s vs T relationship

Comparisons of the geophysical and petrological values of $V_{sv}(T, z)$ used as constraints.



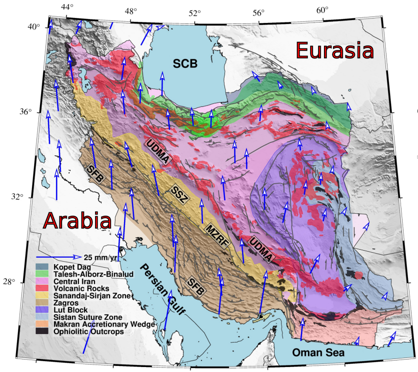
Geotherms calculated from $V_{s}v$ using the parameters from the fitting.

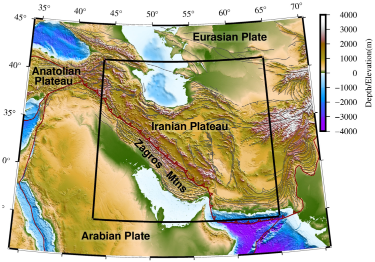


Summary

- Laboratory measurements of the μ for polycrystalline olivine at seismic frequencies show a strong decrease in μ with increasing T below the macroscopic solidus.
- This effect is probably due to stress relaxation on grain boundaries caused by diffusion.
- Laboratory experiments also show a strong dependence on grain size which is usually several orders of magnitude smaller in the laboratory experiments than in the mantle.
- This strong grain size-dependence makes it difficult to directly apply the laboratory results to mantle structure.
- Priestley & McKenzie took a simple empirical approach to relate seismic wave speed and temperature, allowing them to use the surface wave tomography results to map the upper mantle geotherm.

An application of surface wave tomography





SCB – South Caspian Basin

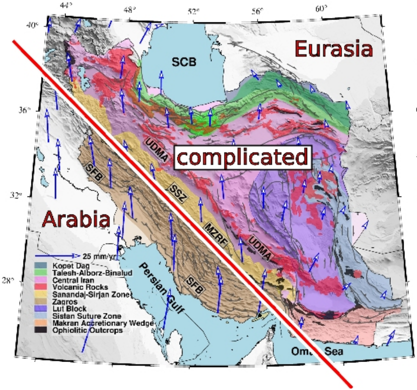
UDMA – Urumieh-Dokhtar magmatic assemblage

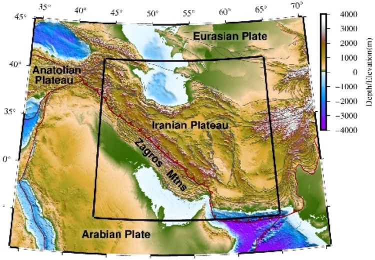
MZRF — Main Zagros Reverse Fault

SSZ – Sanandaj-Sirjan Zone

SFB – Simply Folded Belt

An application of surface wave tomography





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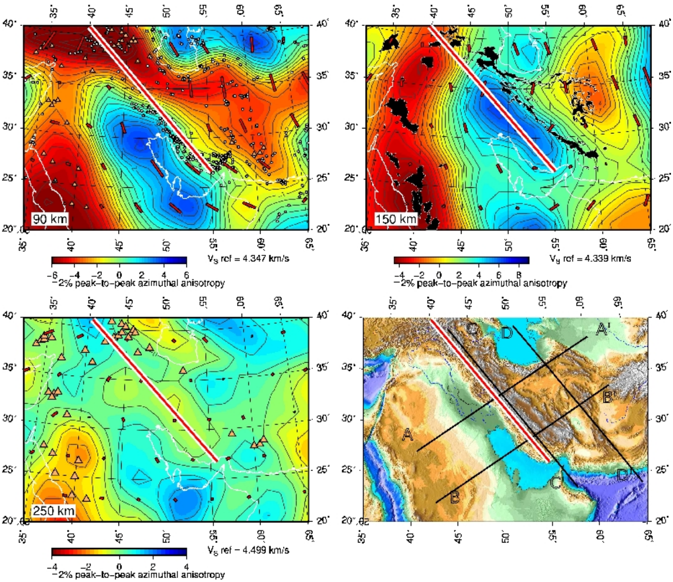

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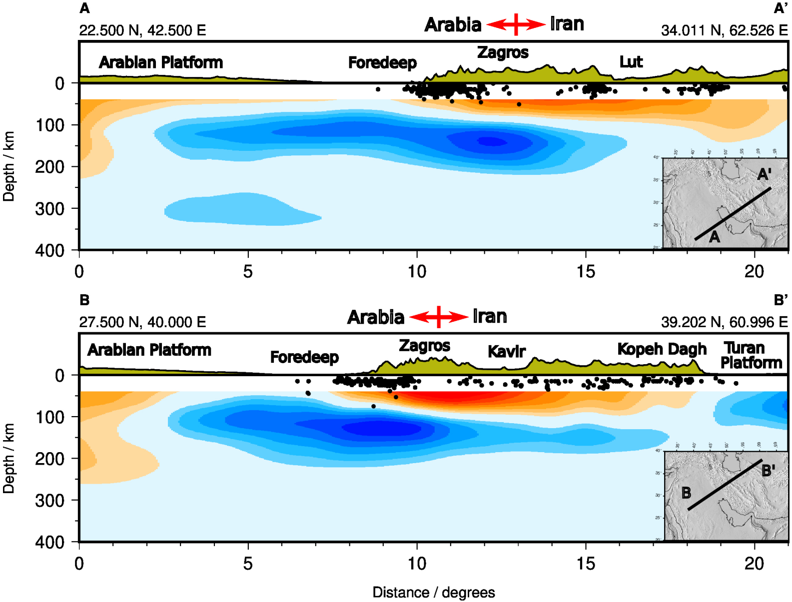

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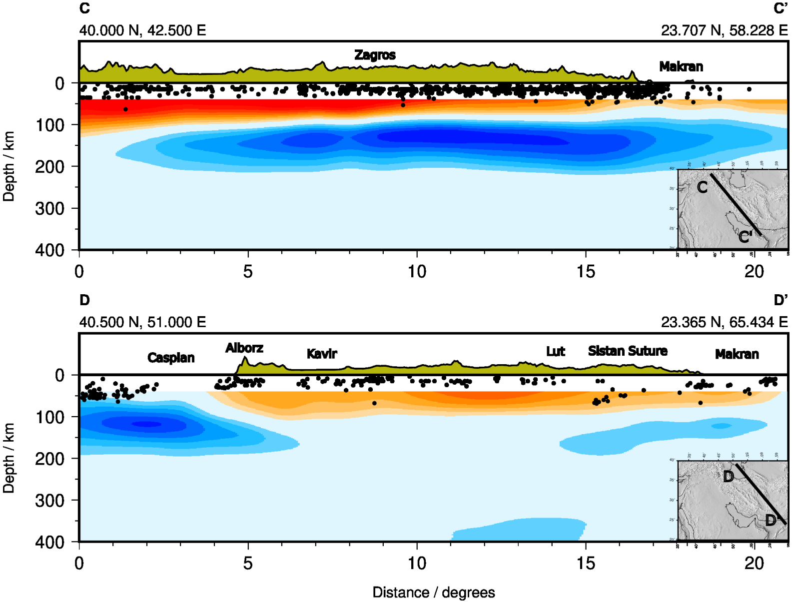
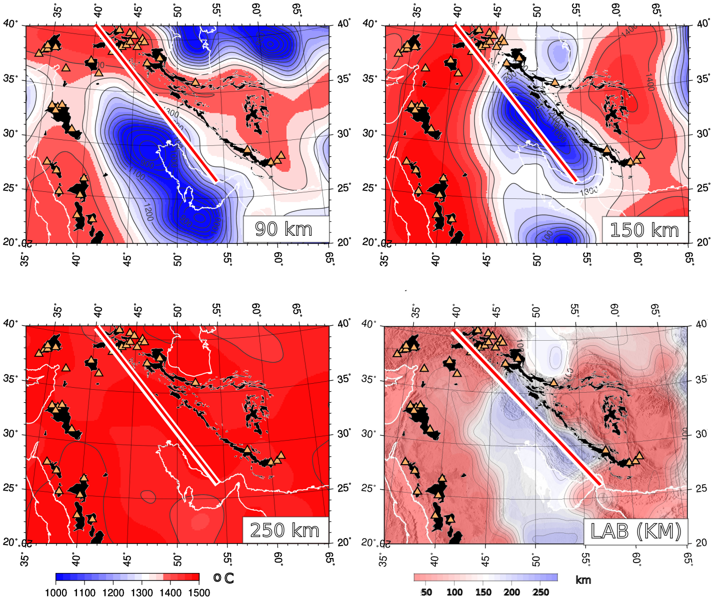


An application of surface wave tomography



An application of surface wave tomography